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TEACHING CHILDREN Mathematics

page 6

Connecting Parents, Students, and Mathematics

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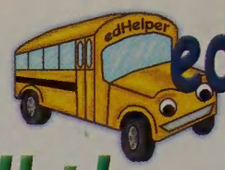

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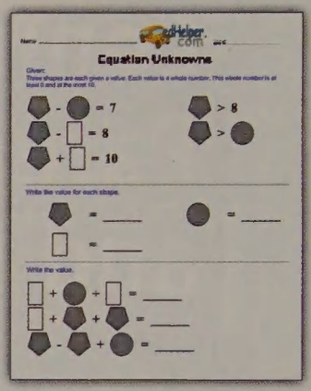

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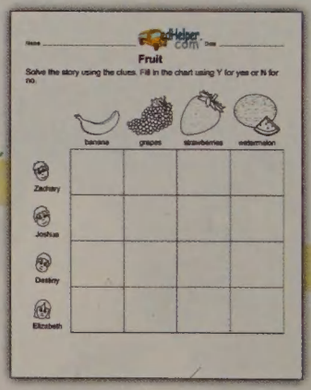

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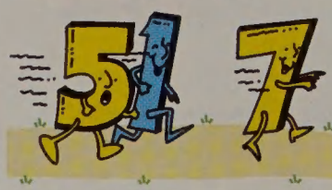

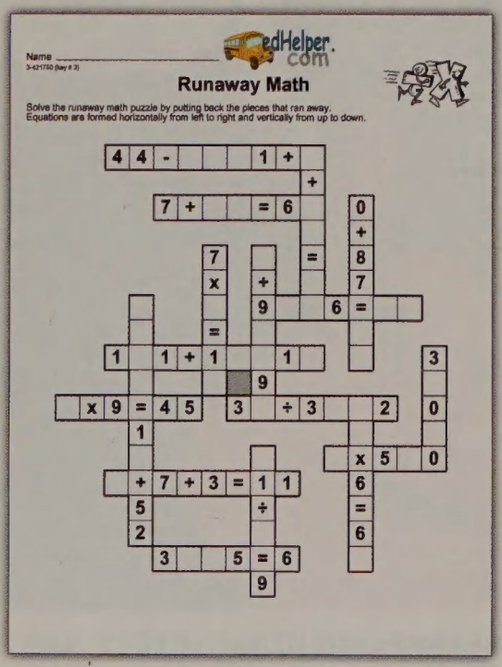
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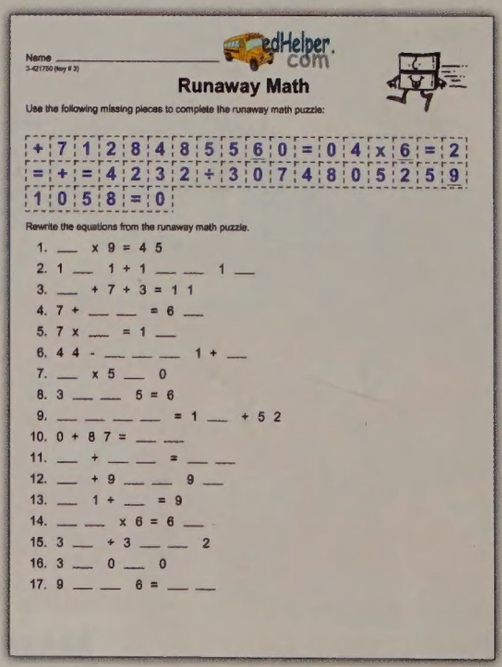
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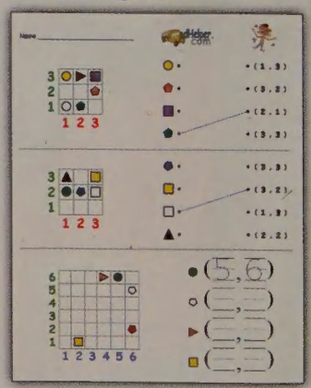

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ON THE COVER

Truth be told, mathematics teachers get discouraged with poor turnouts at events, especially when parents claim they want to know what and how their children are learning. "Identifying Opportunities to Connect Parents, Students, and Mathematics" does exactly what it claims to do as author Nancy M. Fagan suggests ways to successfully incorporate mathematics activities into established family-school events.

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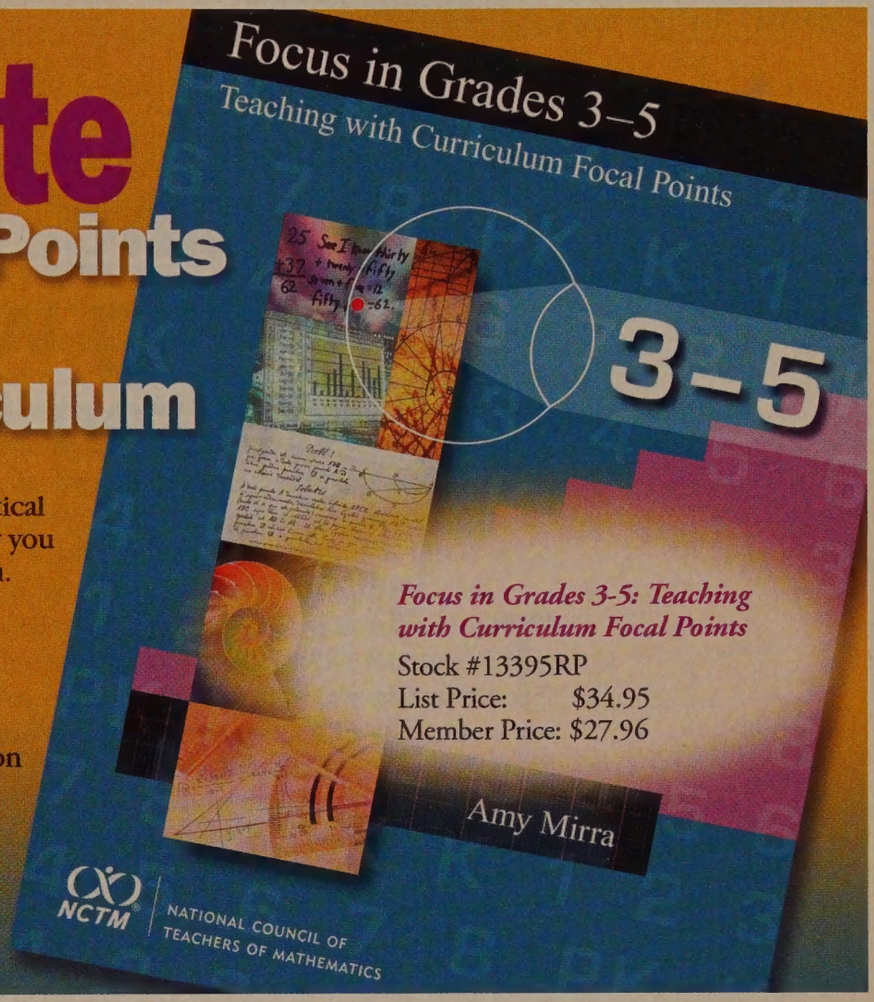

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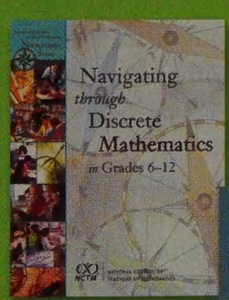

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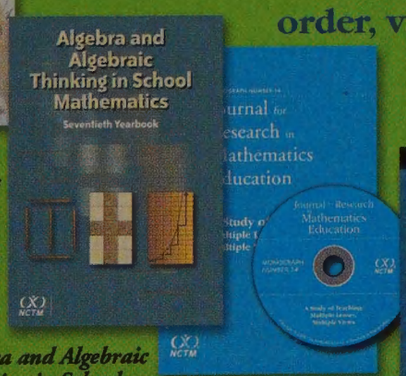

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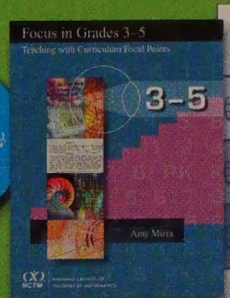

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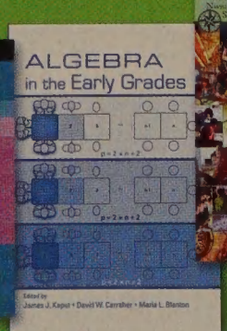

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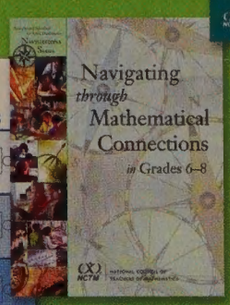

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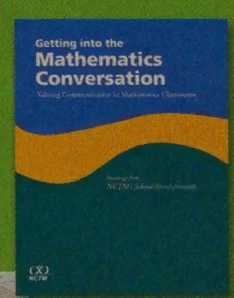

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Teaching, Learning, and Sharing with *Teaching Children Mathematics*

The Editorial Panel of *Teaching Children Mathematics (TCM)* welcomes you to Volume 15 and invites you to take an active role in our community of mathematics educators. We are committed to the goals and ideals of *TCM*'s mission to provide teacher resources that support the improvement of pre-K–6 mathematics education and ultimately provide more and better mathematics for *all* students.

TCM offers numerous opportunities for readers to grow professionally and exchange mathematical ideas, activities, and pedagogical strategies. Additionally, *TCM* offers a forum for sharing and interpreting research. NCTM's council-wide professional development focus of the year topic for 2008–2009 is "Learning and Equity." In Volume 15, you will find bountiful ideas to support learning and equity. In particular, the October 2008 focus issue, "Learning Environments that Support Mathematical Understanding," champions learning and equity with articles about learning environments that allow students to develop a robust understanding of mathematics and empower students from different backgrounds to build on their mathematical strengths.

The Editorial Panel of *Teaching Children Mathematics*. From left: Lynn McGarvey (seated); Jacqueline (Jacqui) Smith, NCTM Board Liaison (seated); Elizabeth (Beth) M. Skipper, NCTM Staff Liaison; Maryann Wickett; M. Patrice Tucker; Robert (Bob) Mann; Robert Q. Berry III, Chair. (Not pictured: Kate Kline)

Teaching and Learning with *TCM*

TCM's articles are designed to promote students' learning of mathematics by providing teachers with resources for professional development. Mathematics content and process goals are at the heart of everything you will read in *TCM*. In Volume 15, you will find a mix of article and department contributions that supply teachers with tools, ideas, effective processes, strategies for teaching mathematics, and ways to help students learn rigorous mathematics. You can look forward to articles related to problem posing, problem solving, and envisioning NCTM's Process Standards in mathematics classrooms.

TCM publishes several departments that offer innovative ideas for teaching and learning mathematics. The "Investigations," "Math by the Month," and "Problem Solvers" departments include problems and activities that are "ready to use" or that can be easily adapted by elementary teachers to engage students in problem solving. The "From the Classroom" department provides a forum in which teachers share classroom-tested teaching techniques, challenges, and ideas. The "Early Childhood Corner" department gives early childhood educators information and ideas relevant to the development of appropriate mathematics for young learners. The "Reviewing and Viewing" department critiques new instructional materials—books, products, and software—that may be used for teaching and learning mathematics.

Learning from Research with *TCM*

Research in mathematics education is practical only if it is read and used by classroom teachers. The National Mathematics Advisory Panel's report, *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (2008), recommends that instructional practice be informed by high-quality research. *TCM* provides a forum that helps readers interpret research findings and determine ways to incorporate them as part of their practice. The purpose of the "Research, Reflection, Practice" department is to acquaint teachers with research that is relevant to elementary classrooms and to suggest classroom strategies based on this research.

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TCM is a journal for sharing. Share your ideas and opinions with other readers by submitting a feature article, department manuscript, or opinion piece. The "In My Opinion" department allows readers to share their views about a current mathematics education issue. "Readers Exchange" enables readers to comment on *TCM* articles.

TCM is successful only when it serves the needs of its readers. We value your thoughts and opinions and work to incorporate them into our decision making and improve the journal's quality. We encourage readers to respond to the *TCM* surveys; a link to the current survey can be found at **nctm.org/tcm**. On the basis of feedback and overwhelming support from our readers to incorporate more articles that use children's literature in elementary mathematics classrooms, the Editorial Panel decided to revive the "Links to Literature" department, which last appeared in Volume 10 of *TCM*. This department provides suggestions and shows examples of how teachers can make connections between children's literature and mathematics to enhance mathematics instruction.

Reader survey responses also help us generate ideas for "Call for Manuscripts: Hot Topics," which address issues and challenges that mathematics educators face. The *TCM* Editorial Panel solicits manuscripts on these high-interest topics to keep the journal current.

TCM articles come from you, our readers. Your contributions make the journal a valuable resource for the mathematics education community. We encourage you to share your ideas, strategies, and insights. NCTM staff members can direct you to a department editor who will assist you in developing your ideas into polished manuscripts to share your expertise with your fellow *TCM* readers.

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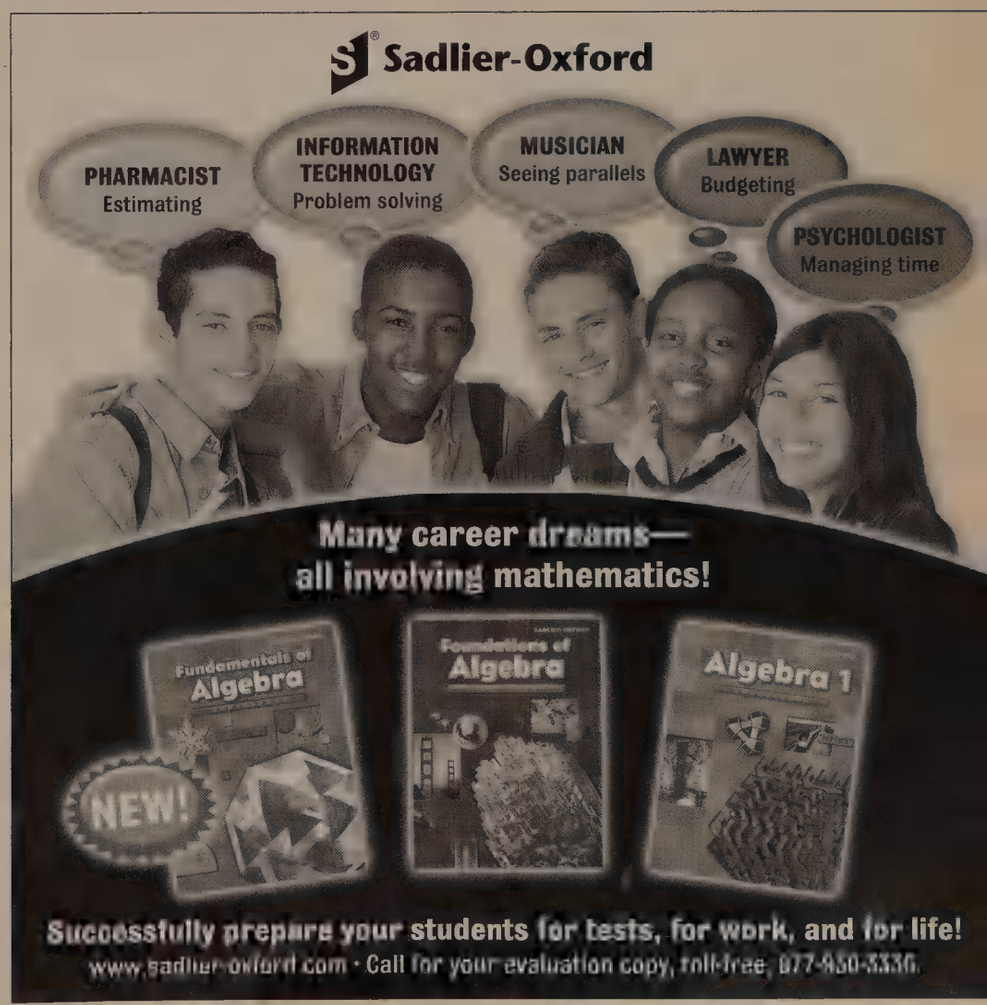
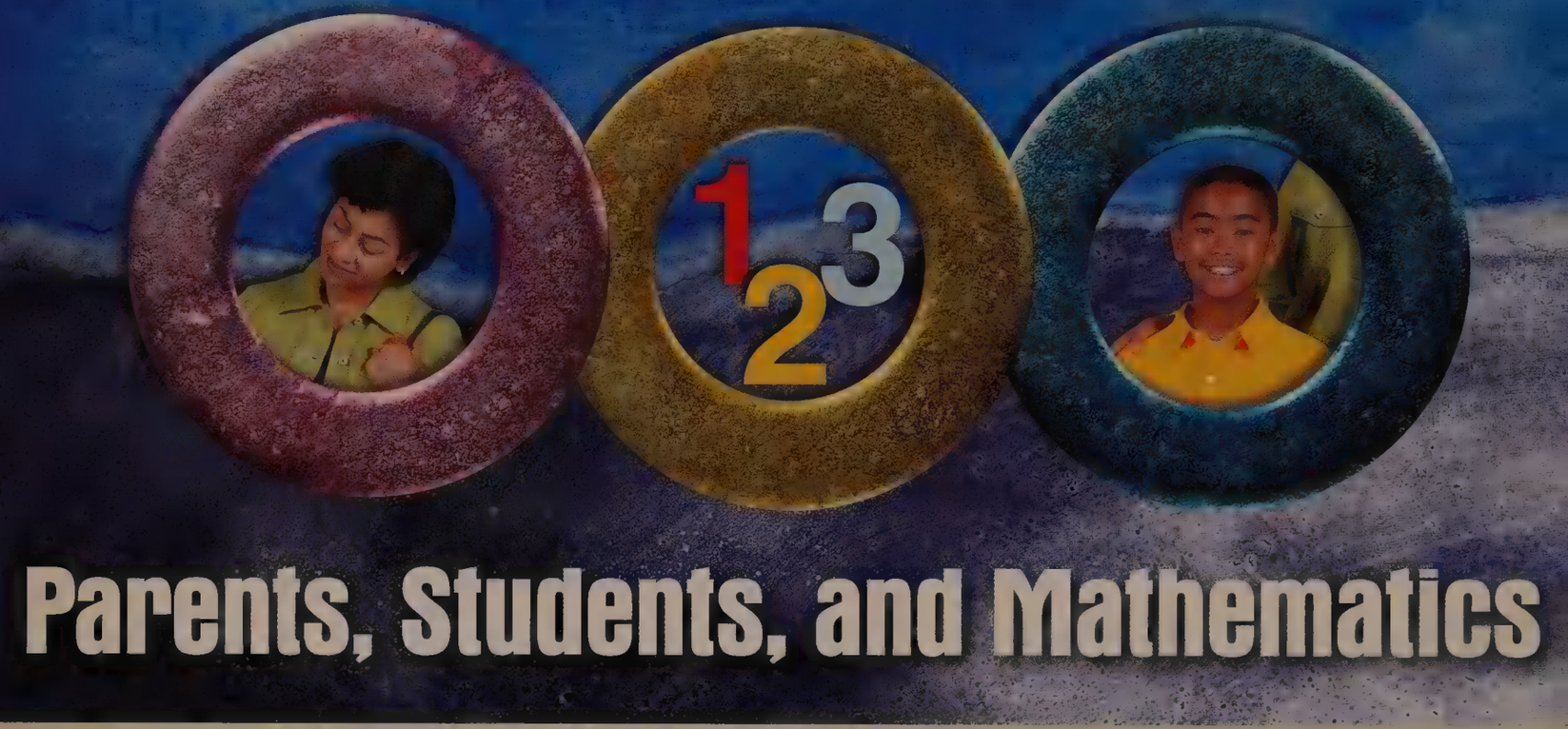


Identifying Opportunities to Connect



Parents, Students, and Mathematics

I ran down my to-do list for my first Parent Math Night: Invitations had been sent, handouts copied, overheads prepared, and take-home materials packaged. Everything was planned and organized, and I was excited about the event—"Math Night for Parents: What Does Computation Look Like These Days?" I thought my title was creative and certainly addressed concerns parents had about these "new-fangled" ways to do mathematics. Imagine my disappointment when a grand total of *three* parents came!

Principles and Standards for School Mathematics states, "Teachers and administrators should invite families, other caregivers, and community members to participate in examining and improving mathematics education. All partners in this enterprise need to understand the changing goals and priorities of school mathematics" (NCTM 2000, p. 378). With this in mind, I persevered and tried again a few weeks later. This time my title was "Math Night for Parents: Math Games." Again I thought my title would be inviting because parents also had questions about the games we used in our curriculum. The turnout was even lower than the first session—I had to find the night custodian to complete a partnership to play the games. This time, *frustration* outweighed disappointment.

By Nancy M. Fagan

Nancy M. Fagan, nfagan@columbia.k12.mo.us, is a mathematics coach for the Columbia Public Schools in Columbia, Missouri. She is interested in mathematics curriculum leadership.

I am well aware of the demands on parents' time—especially in single-parent families—but was I missing something? What was wrong? Why was attendance so low? Parents said they wanted to be involved in their children's education and understand our mathematics program better, but they did not follow through.

Back to the Drawing Board

I obviously needed a different approach. I began reflecting on my purposes and goals for holding parent events. Several quickly came to mind:

- To explain our mathematics program to parents and show how it differed from their own experiences in school (reform versus traditional curriculum)
- To allow parents the opportunity to experience the activities their children were doing in class
- To empower parents to help their children with homework
- To provide a forum for parents to ask questions and share concerns

Although I thought all these goals were valid, I also recognized that they were not necessarily realistic for short time periods. How much could we actually accomplish in two hours? I decided that my immediate goal was to successfully engage

families with mathematics. If I could achieve that, I knew I would open doors for future discussions and opportunities.

Link to Already Successful Events

I also thought about parents' perceptions of being asked to come to the school for yet another event. We already expect them to attend a curriculum night, two parent-teacher conferences, and various family events. Being mindful of my twofold considerations of engaging families with mathematics and valuing parents' time, I decided to use a different approach the next year. I decided to integrate a mathematics component into an existing family-school event.

My first attempt was to link mathematics to the fall festival, historically a successful, well-attended event. To the popular booths—lollipop pull, dunking booth, cake walk—we added my booth, "High Rollers." I patterned the game after a television game show from the 1970s. The children rolled number cubes and used their combinations to eliminate numbers from the game board (see **fig. 1**). See the **Appendix** for objectives and rules for the "High Rollers" game.

Children of various ages visited my booth. For the younger ones, the focus was on counting the pips on the number cubes while the parents helped decide which numbers to cross off the board. I talked with parents about how young children learn to recognize a dot pattern as a number representation without having to count all the dots and how they learn to count on from the larger number. While first and second graders played the game, I chatted with parents about number combinations to ten, doubles, and near doubles. I included examples of those strategies on the back of the miniature version of the game that I sent home with each family (see **fig. 2**).

For older children, the game took on a different dimension as they thought about strategy—especially when they played using numbers one to twelve. I talked with the students and the parents about the *probability* of rolling particular numbers. Through these informal conversations, I reached many more families than I had in all my other parent math sessions combined.

I teach in a Title I school that serves a high minority, low SES population. A large percentage of our students read below grade level, so most of our family events have a literacy focus. As a result, I knew I would have to be creative to attach mathematics to other existing school events, such as "Books and Boogie"—an event where students read with their families for thirty minutes to earn a ticket into a dance. In anticipation of this event, I decided to write a grant proposal to purchase

Figure 1

We used a laminated board and oversized dice to play the "High Rollers" game, but a special game board is unnecessary.

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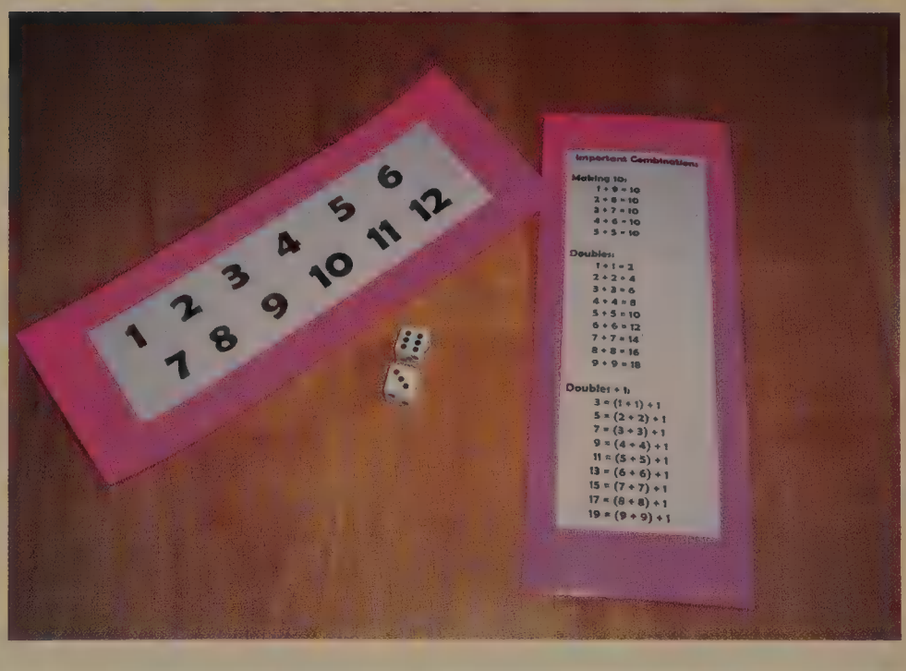
Figure 2

The miniature take-home version of a game included strategy examples.



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children's literature related to mathematics. After acquiring a grant from a local nonprofit organization, I purchased a variety of books appropriate for different grade levels and distributed them to teachers. (An excellent resource for selecting books is *New Visions for Linking Literature and Mathematics* [Whitin and Whitin 2004].) Teachers showcased the books and encouraged families to give them a try. Several parents commented that they were surprised to see "so much" mathematics in children's stories (see **fig. 3**).

On a smaller scale, each classroom teacher hosts a brown bag lunch a few times a year to bring parents into the school during the day. I took the opportunity to insert mathematics at these events by having the children teach their adult guests the mathematics games we had been learning. I circulated among the children and prompted them to talk about the mathematics in the games as well as the rules. Granted, not every parent can attend a school function during the day; however, those in attendance appreciated learning about the mathematics behind the games and watching their children's excitement in teaching and playing the games.

Success!

To an outsider, these three events may sound like minor accomplishments, but I was ecstatic. I witnessed parents playing mathematics games with their children, laughing together as they competed for better scores. I saw families curled up together enjoying books, such as *Alexander, Who Used to Be Rich Last Sunday* (Viorst 1987), and heard parents asking questions: "How much money does he have now?" I now realize the potential of integrating mathematics into a variety of events throughout the year to connect parents, students, and mathematics. Although I am still not completely satisfied, I know the work I have done is a good foundation that I will build on.

Figure 3

At the "Books and Boogie" event, students read books—with mathematics connections—to their family members for thirty minutes to earn a dance ticket.

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Recommendations

I base the following recommendations on my approach to, reflection on, and redesign of Parent Math Nights and the resulting success; consider them in the process of trying to boost parental involvement with mathematics at school events:

- Be creative. Integrate mathematics in *some* way if your school has a well-attended event. I made sure we had mathematics games in use during "Games and Giggles," an event that involved parents and children playing games together and then listening to a storyteller before adjourning.
- Showcase students in action. Have students make PowerPoint presentations showing what they do in mathematics class. Provide activities, puzzles, and games for students to teach their parents. Encourage students to write and present a readers' theater performance of a favorite book with a mathematics theme; parents love to see their children perform.
- Team up with colleagues. Work with one or more teachers on an interdisciplinary event. For example, I am working with the social studies and art teachers to develop an integrated project on Women in History. The students will research and write about a person and create a quilt to depict that person's life and achievements. My job is to teach the mathematics skills needed for quilting. The final products will be on display in a "museum" where the students serve as docents for parents and the public.
- Provide refreshments. Organize a potluck where each family brings something to share. Food is a big part of our social culture, but it does not have to be anything fancy—just an ice breaker.
- Provide childcare, if appropriate. Try to provide a special place and activity for toddlers that frees parents to join their school-aged children. We have many single-parent families at our school

and families with a wide range of children's ages. Some activities are inappropriate for all children. We contacted the local high school and found a service organization that was happy to help us at no charge.

- Be flexible with the time of your events. Vary the times of events to accommodate parents' different work schedules. It will ultimately allow more of them an opportunity to be involved. Schedule some events in the evening, some during brown-bag lunches, and some on Saturday mornings over doughnuts and coffee.
- Offer prizes. Visit businesses in your community for donations to your prize pool—coupons for free fast food, gift cards, and so forth. Use bonus points from book clubs to get free books to give away. Make a take-home version—with instructions and supplies—of the games you will play at the event (see **fig. 2**).
- Finally, do not be afraid to start small. Do not let other schools' elaborate Family Math Nights (see Lachance 2007; Schussheim 2004) intimidate you. Aim for one new event or activity a year until you build a repertoire. If something does not turn out as you expected, take time to reflect on it and identify what *did* go well. Chalk the rest up to experience and move on.

The "Math Lady"

The most satisfying part of designing and organizing these events is probably the follow-up validation I receive from the children. My students are so proud when their parents come to school, and they ask when we can do something like that again. Students regularly stop me in the hall to tell me they are still playing "that game with the numbers." My fifth graders enjoyed the mathematics literature so much that they even wrote their own mathematics picture books to share with younger students.

It would have been easy to give up on reaching the parents from my school after they did not respond to the initial invitations. It takes many hours of planning and preparation—as well as the devotion of personal time—to design and host a successful mathematics event for parents. However, when the invitation is delivered creatively, parents respond, and the effort is definitely worth it.

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Appendix

"High Rollers" game

Objective in a one-person game: Eliminate all ten numbers from the board.
Objective in a two-person game: Be the player who eliminates the final number from the board.
Materials: Game board with numerals one through ten, two number cubes, and a marking tool

Rules

1. The first player rolls the number cubes and totals the numbers.
2. From the numbers rolled, the player decides which numbers to eliminate from the board. The numeral total or any combination of the numerals may be eliminated. For example, if a ten is rolled, the player may choose to eliminate the ten; the nine and the one; the eight and the two—or even the one, two, three, and four!
3. If playing with a partner, alternate turns.
4. The game ends when all the numbers have been eliminated from the board or a roll produces a number with no available choices.

Notes

- A special game board is not necessary to play the "High Rollers" game; students could write the numerals one through ten on a piece of paper.
- The game could be played on an overhead projector to involve the entire class in a discussion about probability and strategy.
- To make the game more challenging, add the numerals eleven and twelve to the game board. ▲

Statistics in the Elementary Grades:

Exploring Distributions of Data

Youngsters and adults alike are confronted daily with situations involving statistical information. Making sense of data and dealing with uncertainty are skills essential to being a wise consumer, an enlightened citizen, and an effective worker or leader in our data-driven society. The importance of statistics education as an integral part of the mathematics curriculum was signaled by NCTM in its *Curriculum and Evaluation Standards for School Mathematics* (1989). Subsequent recommendations in state frameworks, NCTM's *Principles and Standards for School Mathematics* (2000), and most recently in the *College Board's Standards for College Success: Mathematics and Statistics* reflect consistent attention to data analysis, probability, and statistics across the grades (2006).

In 2007, the American Statistical Association released a report titled *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report*. That report provides learning trajectories for key ideas of statistics organized into three developmental levels—A, B, and C. Although these three levels may parallel the standard grade-level bands (elementary, middle, and secondary), they are based on students' prior statistical experiences rather than on grade level. The three companion articles in this month's issues of *Teaching Children Mathematics, Mathematics Teaching in the Middle School,* and the *Mathematics Teacher* illustrate how the GAISE report can be used to shape a coherent development of basic ideas related to the distribution of a data-based variable beginning with exploring distributions of data, then developing an understanding of center and spread, and finally building sound reasoning under uncertain conditions. For a perspective on K–12 statistics education, read all three articles in the series.

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"Contemporary Curriculum Issues" provides a forum to stimulate discussion on contemporary curricular issues across a K–12 audience. NCTM plans to publish sets of three articles, focused on a single curriculum issue. Each article will address the issue from the perspective of the audience of the journal in which it appears. Collectively, the articles are intended to increase communication and dialogue on issues of common interest related to curriculum. Manuscripts on any contemporary curriculum issues are welcome. Submissions can be for one article for one particular journal, or they can be for a series of three articles, one for each journal. Submit manuscripts at the appropriate Web site: tcm.msubmit.net, mtms.msubmit.net, *or* mt.msubmit.net *or contact editors Barbara Reys, reysb@missouri.edu, for* TCM*; Glenda Lappan, glappan@math.msu.edu, for* MTMS*; or Chris Hirsch, christian.hirsch@wmich.edu, for* MT*. Manuscripts should not exceed ten doubled-spaced typed pages.*

School teachers have long engaged elementary students in collecting and analyzing data but have often neglected to involve students in formulating the questions to be answered (so that the data are relevant and meaningful to students) and to provide opportunities for students to interpret data they have collected in light of their original question. The *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report* framework outlines a four-step statistical problem-solving process (Franklin et al. 2007) that should be at the forefront of all data analysis scenarios:

1. Formulate a question that can be addressed with data.
2. Collect data to address the question.
3. Analyze the data.
4. Interpret the results.

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Elementary school students (level A as described in the GAISE report) should be exposed to data analysis situations involving both *categorical* and *quantitative* data. Much of the data we collect in elementary schools is categorical. That is, students choose from various *categories* to respond to questions rather than giving numerical answers. For example, most questions such as "What is your favorite … (ice cream flavor, color, school lunch choice, season, etc.)?" elicit categorical responses. Questions that elicit numerical (or quantitative) responses should generally be limited to discrete situations at the K–5 level. For example, questions such as "How many chips are in a typical chocolate chip cookie?" or "How many candies are in a typical single-serving package?" can generally be answered by using whole numbers to count. In this article, we trace one example using categorical data and another using discrete quantitative data and follow each through the statistical problem-solving process.

Categorical Data—The Shoe Problem

1. Formulate questions. Most elementary school students are curious about their peers and, in particular, what their peers are wearing. One question that a group of students might ask is "What is the most popular type of shoe in our class today?" This question could have a variety of links to the students' lives. For example, how many students are prepared to go to physical education class without changing shoes? Other students might want to wear slip-on shoes (to avoid tying their shoes), but their parents do not think slip-ons are safe for school activities. Finding out how many students are wearing slip-on shoes provides data to answer the question, "Are slip-on shoes really as popular as we think they are?" A teacher might pose the question, "If you were to advise the local shoe store on the type of shoe the store should aim to have in stock, what would you recommend?"

2. Collect and represent the data. Multiple ways exist to classify shoes and to collect data on the basis of different classifications. One option is to have each student remove one shoe and place it in a pile in the middle of the room. Starting with the pile of shoes allows students to grapple with the question of what the categories might be. To determine if the categories will capture all shoes in the class, they can look at the range of shoes and the shoes still on their feet. Students may suggest two categories, such as "shoes that tie" and "shoes that do not tie." Discussing what is included in the "shoes that do not tie" category (e.g., buckle, Velcro, slip-ons) will help students understand that the two categories are mutually exclusive (nonoverlapping) and that the categories account for all possibilities. Students may also suggest more elaborate categories according to how the shoes fasten, their color or material, or other features. It is important to have students assess whether the categories cover all shoe types in the class (which can be done by asking students to look at their own shoes and determine into which category they would place them). The critical idea here is that whatever classification is used, the resulting data will *vary*. That is, not all students will be wearing the same type of shoes.

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3. Analyze the data. Once they have decided on the categories, students can sort the pile of shoes and make a graph on the floor. Taping a grid to the floor will help ensure that shoe size does not affect the height of the bar in each category. Students can then replicate the graph by placing sticky notes on a chart or by coloring boxes on graph paper. Each of these activities provides a representation of the *distribution* of the shoes. In this case, the distribution indicates different shoe categories and the number of shoes within each category. Using various representations (graphical and numerical) for summarizing data distribution is one of the most important concepts in statistics.

4. Interpret the results. After creating a data representation, students should focus on the shoe distribution. Elementary teachers often neglect to guide students to look beyond the "pictures" the students have created. The data picture is sometimes hung on the classroom or hallway wall with no discussion of what information students gained from collecting data and answering the originally formulated questions. After summarizing the data, encourage students to consider questions such as the following:

- Do all categories contain about the same number of shoes?
- Do some contain more shoes or fewer shoes than others?

At this stage of the problem-solving process, answering the original question should become the students' focus: "Which type of shoe is most popular in our class today, and what would you recommend to the local shoe store?" The answers may differ, depending on which categories were selected. For example, if "tie" and "don't tie" were the categories, shoes that do not tie may be the most popular. But if the categories were "tie," "buckle," "Velcro," and "slip-on," the answer may be shoes that tie.

Although students have now answered the original question, do not let the activity end here. More mathematical exploration can be done with the data! For instance, to solidify students' understanding of their distribution representation, ask them to locate themselves in the representation. If multiple representations have been created, focus on a representation (for example, a bar graph) where they cannot determine precisely which sticky note is theirs. In contrast, with the graph from the shoe data, students can easily identify which data point is theirs. If a category has only one data point in it (such as "shoes that buckle"), students may also be able to identify the owner of the data piece. Identifying an individual data piece is an important beginning point for young children as they transition from seeing individual data points to seeing a distribution as a whole. Now is also the time to push students' thinking with extension questions:

- Would we expect different results if we collected these data from another class at a different grade level in our school?
- How might the results look different? Why? (Is type of shoe related to age? Might we see fewer shoes that tie in kindergarten and more slip-on shoes in sixth grade?)
- What if we collected these data at a school in Hawaii? In Canada's Northwest Territories?
- What if we collected data in January instead of September?
- If we collected the data from workers who are building a new school, would we see differences? Why?

Here, the emphasis is on getting students to note reasons for *differences in distributions of data*, such as weather, grade level, or occupation.

Use this opportunity to encourage students to find other questions to answer from the collected data. For example, students might suggest that they can determine how many more people are wearing tie shoes than buckle shoes, or they can determine how many people in the class are wearing slip-on shoes or shoes that fasten with Velcro. Having students author these kinds of questions prepares them for the questions they will meet on standardized

tests and also helps them see the link between the question they have asked, the operations needed to answer the questions, and the data from the graph that must be used to answer the questions.

A note of caution: Teachers often encourage students to find the mean as a numerical summary with categorical data. As the next case demonstrates, the mean requires having numerical values for the variable of interest. With a categorical data set, we have categories (e.g., "tie," "slip-on," "Velcro") rather than numerical values. Finding the mean of the *frequency counts* for the different categories—for instance, computing the mean of the number of shoes in each category—is a typical mistake. This "mean" has no significance to the original question ("What is the most popular type of shoe in our class today?") or to the data and, thus, it is inappropriate to ask children to determine the mean of a set of categorical data. The following example illustrates a proper setting for using the mean as a numerical summary.

Numerical Data—The Soccer Problem

Level A statistical questions involving numerical data allow students to develop an interpretation of the mean and begin to explore quantifying variability in the data. Sports provide a convenient context to analyze numerical data.

1. Formulate questions. Soccer is a popular sport for both girls and boys; the number of goals scored by soccer teams on a particular weekend is a reasonable student curiosity that would lead to the question of how many total goals are scored in soccer games. (Note that for the remainder of the article "total goals" in a game will be referred to as the *score*.)

2. Collect and represent the data. Involving students in deciding what data will be collected is important. A class discussion might lead to the decision to have each student who plays on a soccer team report the score in the game they played over the weekend. Make the students aware that only one person per team reports the score to avoid duplicating data. Students should also discuss why it is a good idea to collect data on this past weekend's games rather than reporting the highest score per game of the season. (The latter could result in overestimating a typical soccer game score.) In this investigation, each game score will be represented with a tower of cubes. **Figure 1** shows this representation of data for the scores from nine games.

Figure 1

Cube representation for nine soccer games' scores

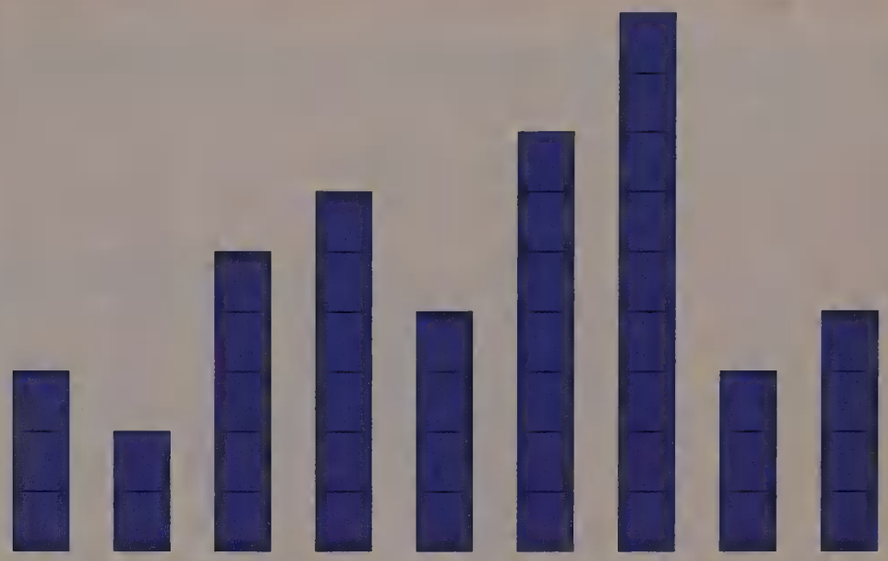
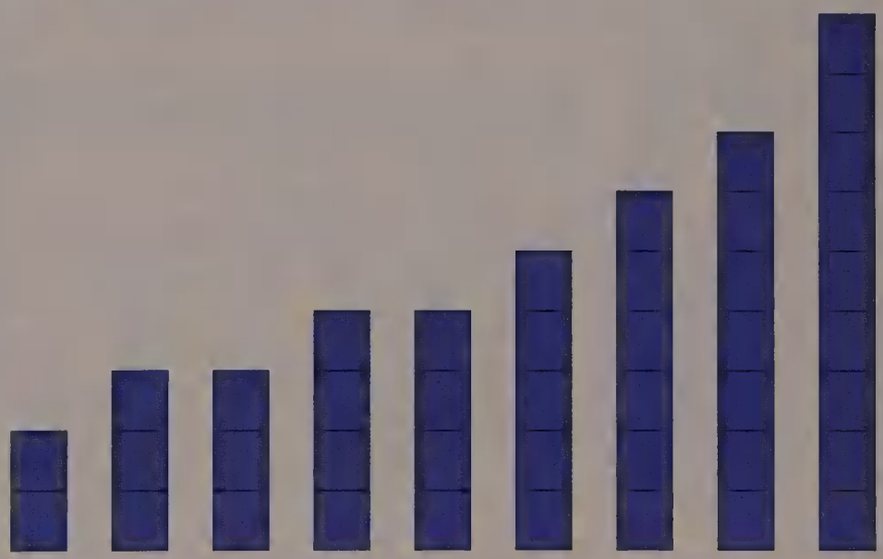


Figure 2

Ordered snap cubes



3. Analyze the data. Begin by asking students what they notice about the data. Students will likely report the lowest score of a game, the highest score of a game, and the scores that are the same. For ease of comparison, students may suggest arranging the towers from smallest to largest and recording the numerical value for the scores (see **fig. 2**).

Students should note that the scores are not all the same. That is, scores *vary* from one game to another. In this case, scores vary from 2 to 9. We might ask, "Based on all the goals scored from these nine games, what would be the game score if all games resulted in the same score?" This score is called the *fair*, or *equal*, *share* value for the data. A process for determining this value follows:

- The first step is to combine the scores from all games into one large group of individual goals.
- A total of 43 goals were scored in the nine games (see **fig. 3**).
- Remove nine cubes from the group. These nine cubes represent a single goal scored in each of the nine games. Thirty-four cubes remain in the group (see **fig. 4**).
- Next, remove another nine cubes from the group to represent an additional goal scored in each of the nine games (see **fig. 5**). Continue this process until no cubes are left (see **fig. 6**).

Figure 3

All 43 goals scored

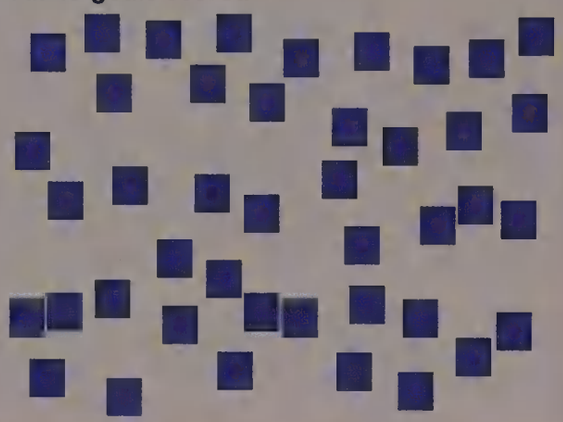


Figure 4

Step 1 of cubes

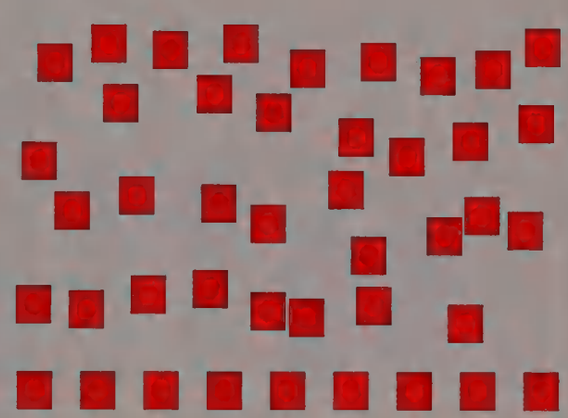
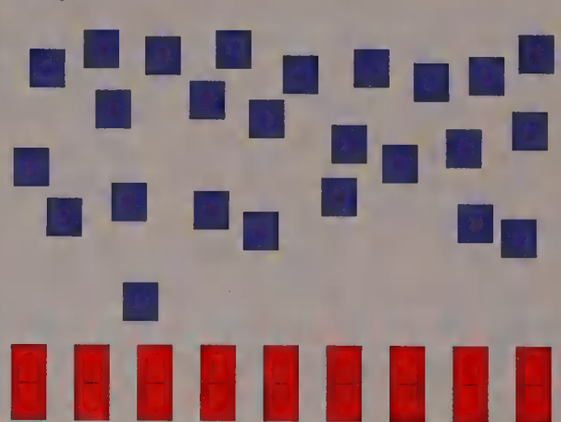


Figure 5

Step 2 of cubes



Thus, the fair share value for these data is 4 and 7/9. Because this quantity is not a whole number, interpreting its value in the context of this problem is somewhat difficult, especially in the early grades. One way to interpret this quantity is to say that for these data, the closest we can come to having a fair share value is for seven of the games to have a score of 5 and the two remaining games to have a score of 4.

This process of using cubes to determine the fair share value mirrors the algorithm for finding the mean: Combining all the goals at the outset maps to adding all the data points. Distributing the goals one at a time to each of the nine games until they are all accounted for maps to dividing by the number of data points. The cubes provide visual representations of both the fair share value and the process of finding the mean. Eventually, elementary students learn that the fair share value and the mean represent the same quantity for a collection of data.

4. Interpret the data. This is the part of the statistical problem-solving process when students reflect on their data-gathering procedure and interpret their results. The group might consider the following questions:

- "Would we expect the score from every game to be exactly the same next weekend? Why, or why not?"

Figure 6

Red cubes for each game with seven remainder cubes

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- "Do you think the fair share value would still be the same if we did this same activity next Monday?"
- "What if we collected data from games involving high school teams instead of our teams?"
- "What if we collected data early in the season or late in the season? Would we expect different results?"

All of these questions are intended to focus students' thinking on the issue of *difference in distributions of data* and what contributes to *variation* in the data distributions. Most of these questions do not have clear-cut answers; the objective is not to find *the* answer but for students to pose various factors that could influence the data.

Extensions

This activity can be extended by reversing the process for determining the fair share value. Suppose we know that the fair share value for nine games is 6. How many total goals might have been scored in each of the nine games if the fair share score is 6? To allow students to explore this independently, provide them with cubes and let them randomly make up the scores for nine games. You might put restrictions on some of the groups, such as "None of the games had a score of 6" or "Two of the games had a score of 10." The simplest distribution is for each of the nine teams to score 6 goals (see **fig. 7**). **Figure 8** shows two other possibilities.

A question could arise about the two sets of scores in **Figure 8**: "Which collection is 'closer' to being fair?" In statistical terms, the question is equivalent to asking, "Which of the two sets of total goals vary less when compared to the mean?" Our objective is to

Figure 7

Nine team scores, all size 6

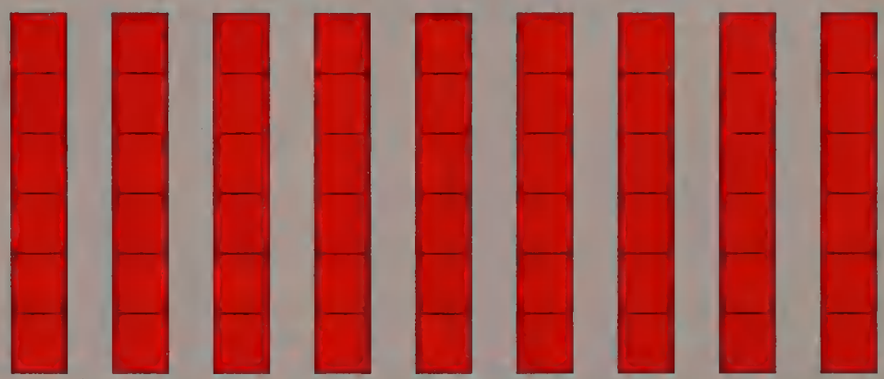
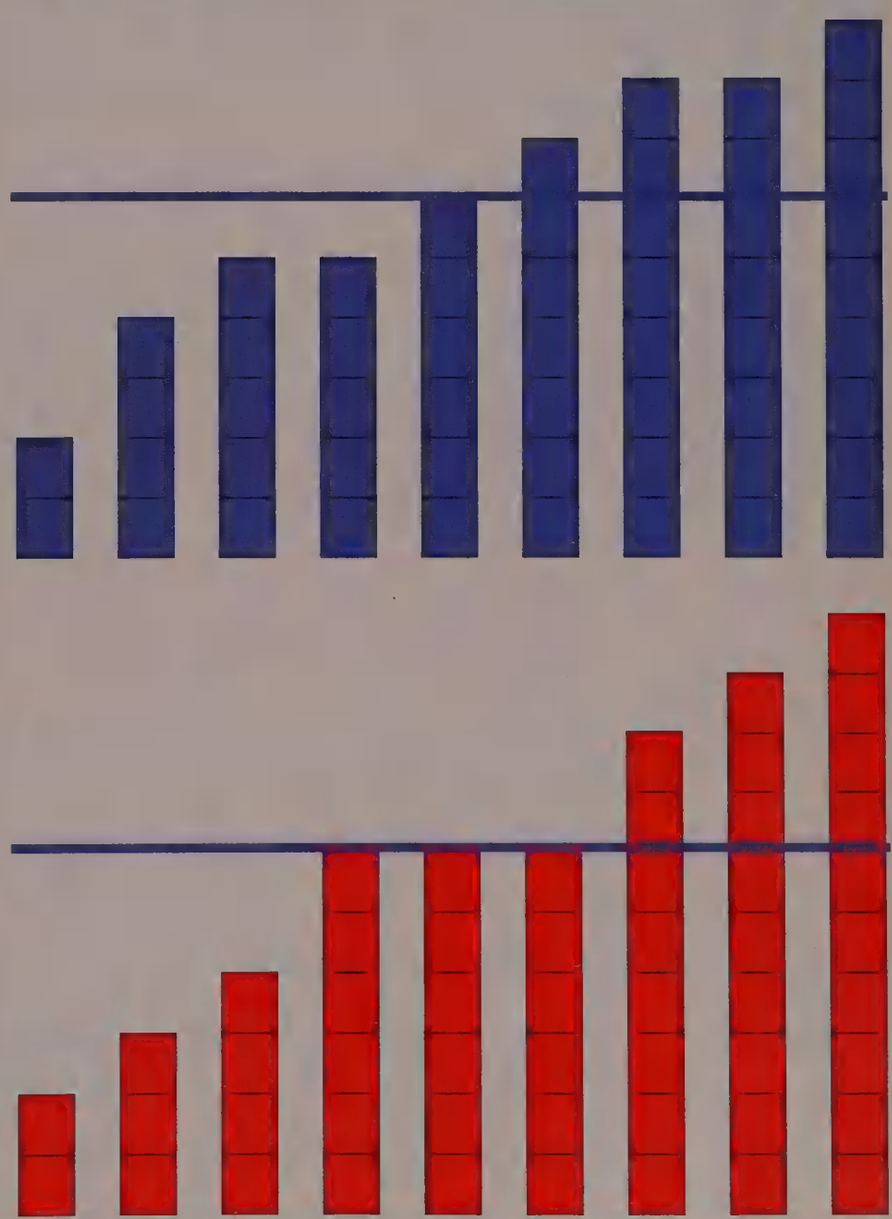


Figure 8

Snap distributions with a horizontal blue line representing 6 goals



develop a quantity that measures "how close" each of the cube distributions is to being fair. Many students will identify the red cube distribution as the one that is closer to fair because it has more 6s than the blue cube distribution. Another approach can be developed by examining how many snap cubes must be moved to level off the two distributions at the fair share value of 6. In general, "a step" occurs when a cube is moved from a tower *above* the fair share value of 6 to a tower *below* the fair share value of 6. To determine how close a snap cube representation for data is to being fair, we can count how many steps it takes to make it fair. The fewer steps it takes to make the distribution fair, the closer the distribution is to being fair. In the example in **Figure 8**, the blue cube distribution is closer to being fair because it takes eight steps to level the towers, whereas the red cube distribution requires nine steps to make it fair. (The "Contemporary Curriculum Issues" department article in the August 2008 issue of *Mathematics Teaching in the Middle School* [*MTMS*] takes this a step further by attaching values to each step.)

Conclusion

Elementary school students (or those working at level A as described in the GAISE report) should be exposed to both categorical and discrete quantitative data. Students should be actively involved in the statistical problem-solving process: designing the questions to answer, collecting and representing the data, analyzing the data, and interpreting the data. Teachers should plan in advance for questions that will help push students to interpret the collected data. "What if" questions help students begin to understand the nature of variability, a fundamentally important concept in data analysis. In summary, students completing level A should understand—

- the idea of the distribution for a set of data and how to represent and summarize the distribution (categorically or numerically);
- the concept of fair share as the mean value for a set of numeric data;
- the algorithm for finding the mean; and
- the notion of "number of steps" to obtain fairness as a measure of variability about the mean.

The fair share, or mean value, provides a basis for comparison between two groups of numerical data with different sizes (the group total is not an appropriate comparison when sample sizes differ). Students explore this at level B (typically in the middle grades). Also at level B, students transition from conceptually viewing the mean as a fair share value to the mean as a *balance point* of a distribution and extending the mean as a numerical summary for continuous numerical data. The analysis of categorical data is nicely extended at level B by incorporating a student's new understanding of proportional reasoning to easily compare groups (not necessarily of the same sample sizes). Kader and Mamer explore these and other related ideas in this month's *MTMS* companion article.

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Tracking Time

Representing Elapsed Time on an Open Timeline

Elapsed-time problems are notoriously difficult for children (Monroe, Orme, and Erickson 2002). Instruction on techniques for teaching and learning elapsed time is not emphasized in current mathematics education literature. Nor is it addressed in *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (NCTM 2006). This absence of instruction may be due to a position held by some mathematics educators that telling time and determining elapsed time are life skills rather than skills delegated to mathematics instruction. (See Editor's note.) However, time is addressed in the *Principles and Standards for School Mathematics* (NCTM 2000) Measurement Standards for both the pre-K–2 and 3–5 grade bands. And regardless of one's perspective on the delegation of time as a discrete content area, determining elapsed time is encountered during mathematics instruction by most children at some point and is often met with frustration. This is especially true when the start or end time falls between the hour and half hour. Children find it challenging to keep track of unit changes between hours and minutes. On a national assessment, only 58 percent of eighth-grade students were able to correctly identify that 150 minutes equals 2 1/2 hours (Jones and Arbaugh 2004). Elapsed-time instruction often focuses on converting units and keeping track of those conversions rather than on counting up or back from one time to another. This article examines how students are able to make sense of elapsed-time problems when instruction is connected to open number-line strategies. Adapting this technique—typically used for recording addition and subtraction counting strategies—provides a method for supporting students' thinking about elapsed-time problems.

By Juli K. Dixon

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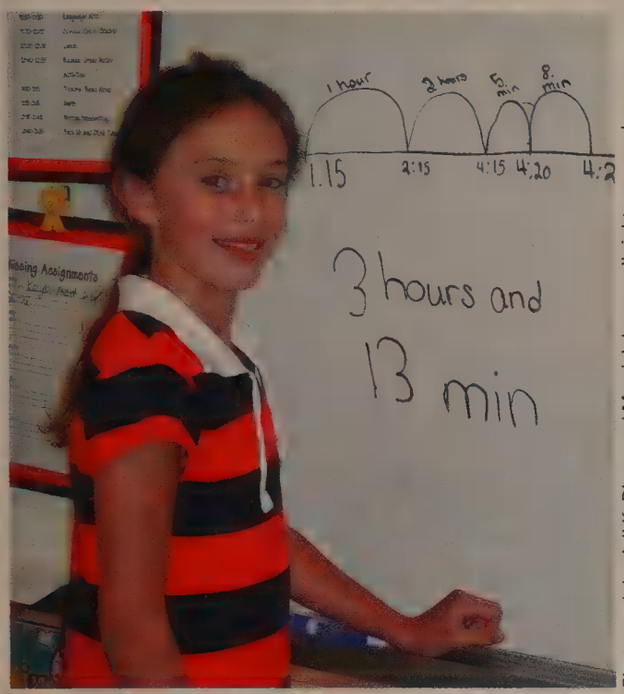
Open Number Lines

The open-number-line strategy is often used to solve whole-number addition and subtraction problems. This strategy is based on counting up, counting back, and finding distances between numbers—explorations consistent with those that can be used for determining elapsed time. Work samples from a third-grade class provide a window into how students can create and evaluate pathways to solve situations involving elapsed time.



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Addition

When students are presented with a problem such as 36 + 24 and asked to solve it using their own methods, they frequently use counting-on strategies. A common counting-on strategy is to start with one number and add the other number in manageable "chunks." The open number line provides a useful way of recording students' variations on this strategy. Consider the examples in **figure 1**. The first example uses two "jumps" of ten and then a "jump" of four to arrive at the sum of sixty. The second example shows a student adding four to get to forty and then adding another twenty to get to the sum. The third example illustrates first adding twenty in one jump and then adding the remaining four in the second jump. Notice that the equivalent distance between jumps is not necessarily maintained. Sometimes students will leave inconsistent distances between numbers on the number line for greater jumps because they do not focus on this aspect of the problem. In my work with children, I do not address the "length" of the jumps on the number line unless the student initiates it, and even then I allow the student to decide how to proceed.

When the open-number-line strategy is used for problems where the first addend is smaller than the second, such as 27 + 75 (see **fig. 2**), the ways students record their thought processes provide useful assessment data regarding the efficiency of students' strategies. Some students will start from the smaller number and add on the larger number, and others will begin with the larger addend, leaving less to add

Figure 1

Students generally do not focus on jump "length" in their solutions on the open number line.

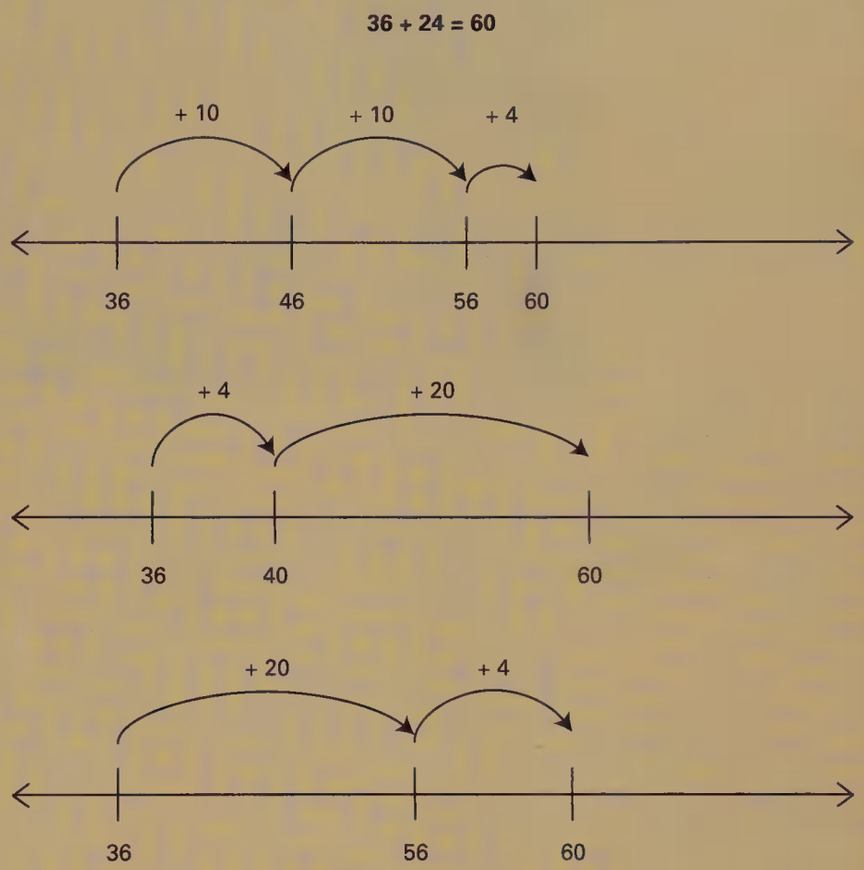
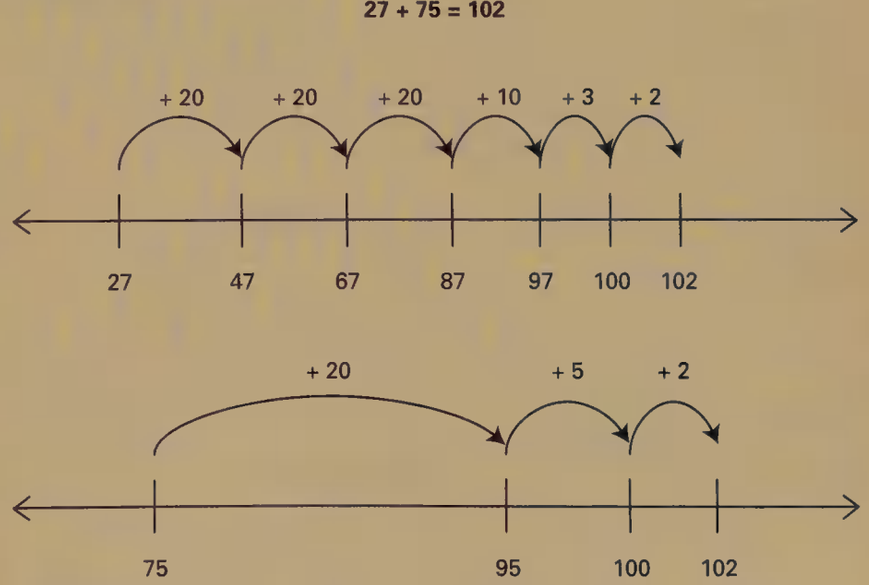


Figure 2

Two strategies to determine the sum when the first addend is smaller than the second



on to determine the sum. Students' thinking about their solutions to problems is made explicit as they record their solutions on the number line.

Subtraction

On the open number line, students' solution strategies for adding differ in sophistication and choice of number groupings. However, when subtracting on the open number line, students' solution strategies vary with respect to how they actually approach the problem. Three distinct methods exist for solving subtraction problems (see **fig. 3** for an illustration of each method using the example of 53–37):

- Method 1 is to count up from 37 to 53.
- Method 2 is to count back from 53 to 37.
- Method 3 is to "take away" 37 from 53.

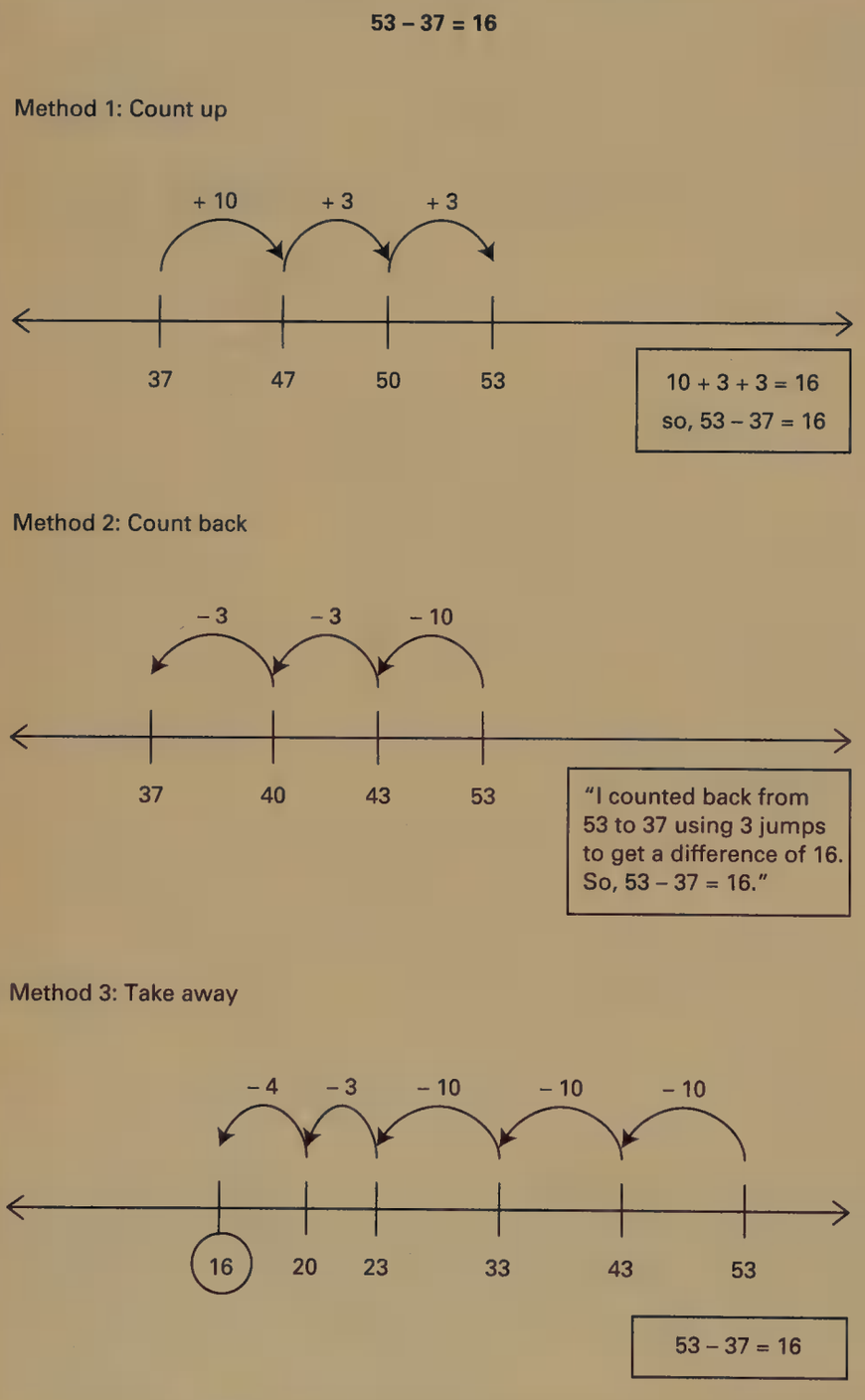
The goal when counting up or counting back is to find the distance between the sum and the given addend. So, for instance, when determining the solution for 53 – 37, students use counting up or counting back strategies to place both 37 and 53 on the number line and then find the distance between the two numbers by "jumping" from one to the other. Using the first method, a student might explain the process by saying—

> I started at thirty-seven and added ten to get to forty-seven. Then I added three so I could get to fifty, but I needed to get to fifty-three, so I added three more. Altogether, I added ten plus three plus three, which equals sixteen. So, fifty-three minus thirty-seven is equal to sixteen.

In the third method, the student begins by placing fifty-three on the number line and then subtracting thirty-seven jumps. The ending number on the number line determines the answer. Two of the three subtraction strategies, counting up and counting back—along with the counting on strategy for addition—describe common ways students can explore elapsed-time situations on the open number line.

Figure 3

Three methods for subtracting on the open number line

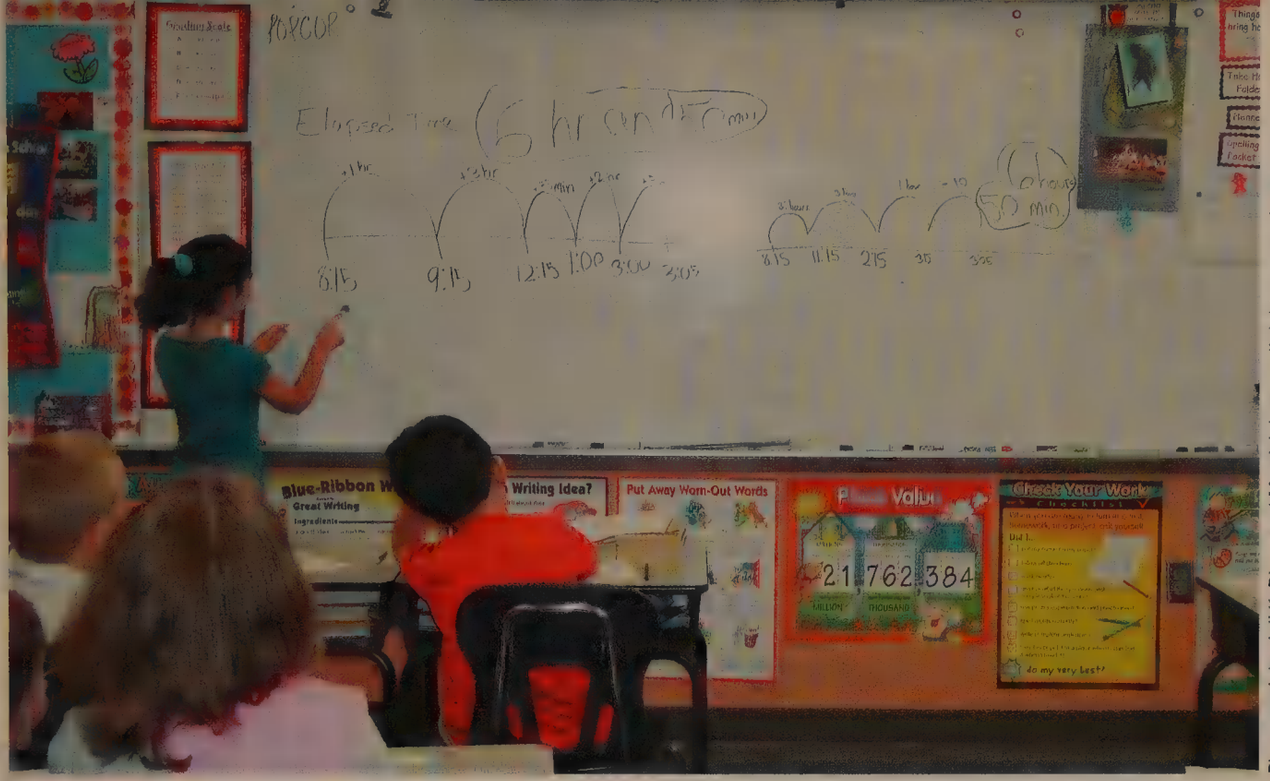


Determining Elapsed Time

When solving elapsed-time problems, students are often first asked to find the time that elapses between two given times. Students typically use the counting up strategy to determine elapsed time. The following word problem is a common type for introducing elapsed time:

> Marni started watching a movie at 3:30. It ended at 4:45. How long did Marni watch the movie?

Students can solve this problem by using counting strategies similar to those used with whole numbers, and their thought processes can be recorded



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on an open timeline (see **fig. 4**). A student might describe this process in the following way:

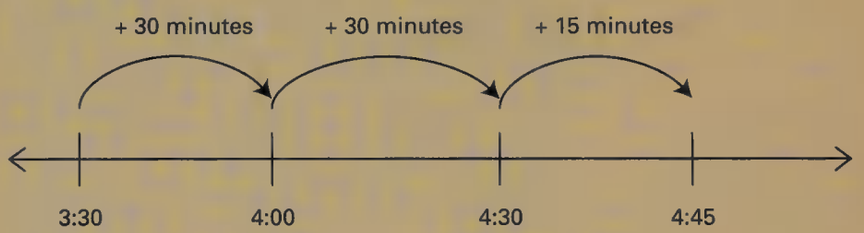
> I started my timeline at 3:30 and added thirty minutes to get to 4:00. I added another thirty minutes to get to 4:30 and fifteen more minutes to get to 4:45. Thirty minutes and thirty minutes is one hour, and then I have fifteen more minutes. Marni watched the movie for one hour and fifteen minutes.

Students are able to use common combinations of time to make hours such as thirty minutes plus thirty minutes or forty minutes plus twenty minutes, and they are also able to use benchmark times like hours, half hours, and quarter hours to help solve problems. Some students find it helpful to count up by hours first until there are no more fully elapsed hours and then count up by minutes to reach the end time. When finding the elapsed time between 1:20 and 4:40, a student might add on one hour three times to go from 1:20 to 2:20 to 3:20 to 4:20 and then add on twenty minutes to get to 4:40. Then students can see that three hours and twenty minutes elapsed between 1:20 and 4:40.

Representing their moves on the timeline facilitates keeping track of elapsed time so that students can make jumps that help them to solve the problem in sense-making ways. The example in **figure 5** provides an illustration of how benchmark times are incorporated in determining elapsed time between 3:40 and 8:15. The solution process involves an initial jump of twenty minutes to get to an hour benchmark. The next jump adds on the full hours remaining between the start and end times. The last jump closes the gap between 8:00 and 8:15. The student is then able to combine jumps in minutes to get

Figure 4

Counting up to determine elapsed time



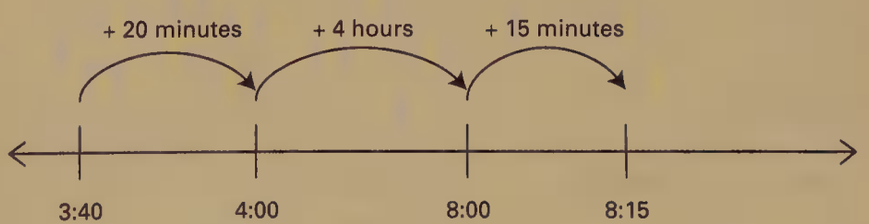
30 min + 30 min = 1 hour
1 hour + 15 min = 1 hour 15 min

Figure 5

Use of hour benchmarks in determining elapsed time

Start time: 3:40

End time: 8:15



20 min + 15 min = 35 min
Elapsed Time: 4 hours 35 minutes

Figure 6

Determining elapsed time that spans the twelve o'clock hour

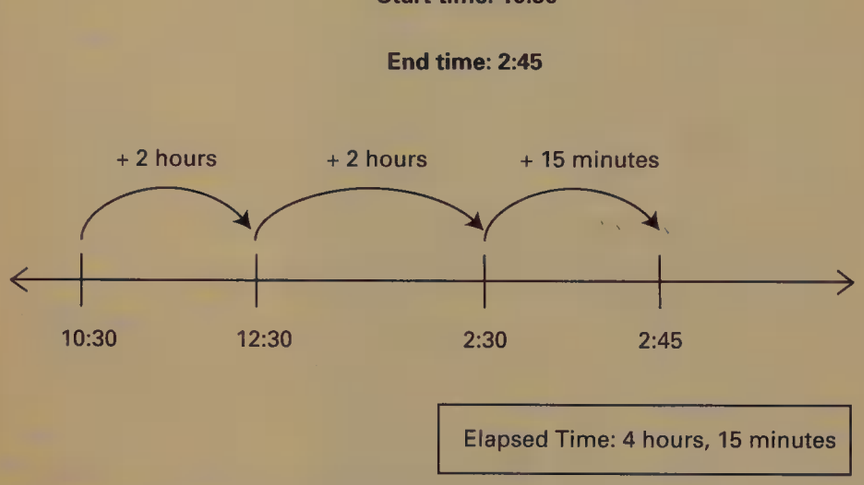


Figure 7

Determining end time by using a jump to a benchmark hour

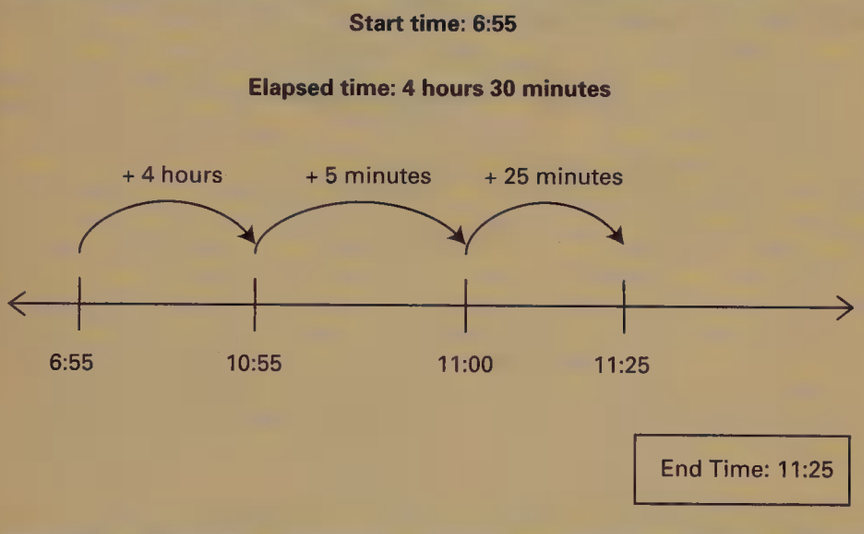
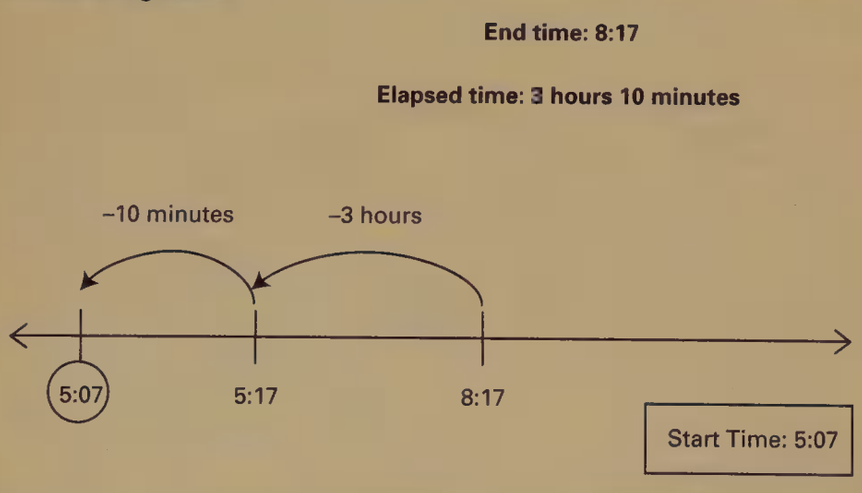


Figure 8

Counting back to determine the start time when the end and elapsed times are given



thirty-five minutes and add that to the four hours for an elapsed time of four hours, thirty-five minutes.

When students have difficulty determining elapsed time that spans over the twelve o'clock hour—such as between 10:30 and 2:45—they can use the timeline to record solutions. With such problems, students typically make a jump that ends in the twelve o'clock hour and continue with jumps to complete the gap in elapsed time (as in **fig. 6**). This process mirrors ways in which adults typically approach such problems in real-life situations.

Determining end time from start and elapsed times

When the start and elapsed times are given and students are asked to find the end time, they count on (as was shown with the examples under the Addition heading in this article). When determining end times, the answer is represented below the timeline, whereas the answer to elapsed time is recorded as jumps above the timeline. In **figure 7**, the jumps in elapsed time are recorded above the timeline and broken into sections compatible with using benchmark hours. The start time is 6:55, and the elapsed time is four hours, thirty minutes. The end time is determined by first counting on four hours, from 6:55 to 10:55. The remaining thirty minutes is counted on by breaking the remaining elapsed time into five-minute and twenty-five-minute increments. In this way, the student is able to jump to a benchmark hour of 11:00 before making the final twenty-five-minute jump to the end-time solution of 11:25.

Students generally have no difficulty solving such problems as long as they connect solving elapsed-time problems to working with whole numbers on the open number line. Once your students are successful in determining elapsed and end times using the open timeline, provide them with opportunities to determine start times. These are typically the most difficult elapsed-time problems to solve; however, using the open timeline may make such problems less complicated.

Determining start time from end and elapsed times

In **figure 8**, the end time is not on an interval of five. Examples like these are often avoided because of the perceived complexity involved. However, using the open timeline, students can solve such problems without difficulty. The following example demonstrates how counting back is used to determine the start time when the end time is 8:17 and the elapsed

time is three hours, ten minutes. In this example, a student might start with 8:17, count back three hours to 5:17, and then count back ten minutes to 5:07. The start time is represented under the timeline.

A Third-Grade Open Timeline

Following a brief introduction on how to record solutions to multidigit addition and subtraction problems on the open number line, a class of third graders was presented with a series of elapsed-time problems. Student work was collected and then grouped according to solution strategy (see **figs. 9**, **10**, and **11**).

Students were first asked to find the elapsed time between 8:15 and 3:05 (see **fig. 9**). They solved this problem in several different ways. All of the students counted up from 8:15 to 3:05 by using various jumps. One student jumped by hours to the twelve o'clock hour and then completed the hour by adding forty-five minutes to get to 1:00 (see **fig. 9a**). Several students made jumps to the twelve o'clock hour and then continued to jump the remaining elapsed time.

The work samples in **figures 9a** and **b** illustrate how students used both addition and subtraction of time within the same problem to determine elapsed

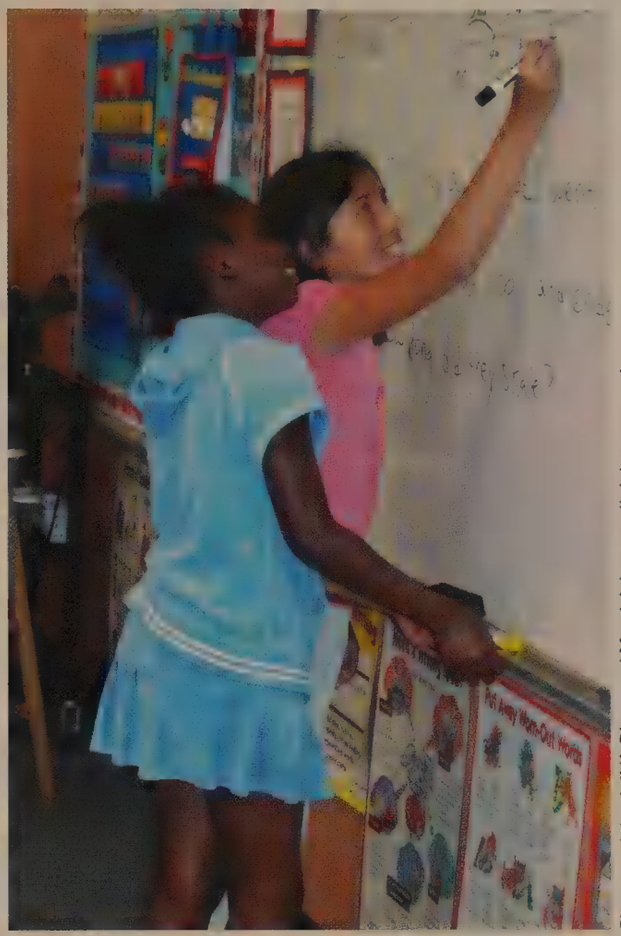


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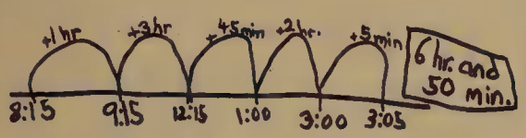
Figure 9

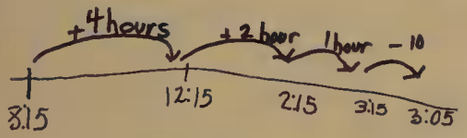
Some third-graders' elapsed-time solutions did not indicate jump directions.

On Monday's, Taylor's school starts at 8:15 and ends at 3:05. How long is she in school on Mondays?

(a)



(b)



6 hours 50 minutes

(c)

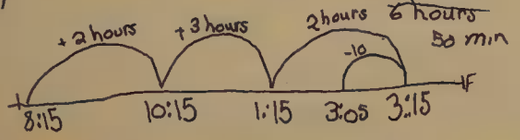
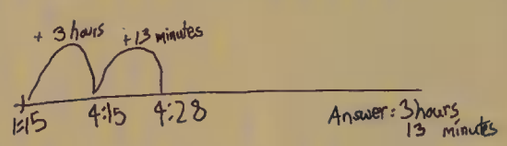


Figure 10

Student solutions to an elapsed-time problem where the end time is not an interval of 5

Alex and Jessica like to sail. On Saturday, they sailed away from the dock at 1:15 and returned at 4:28. How long were they out sailing?

(a)



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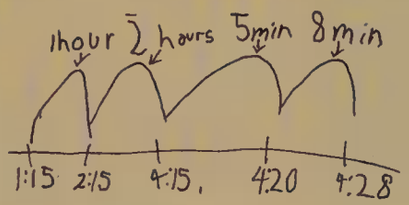

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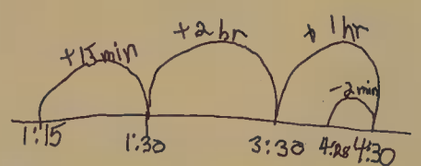
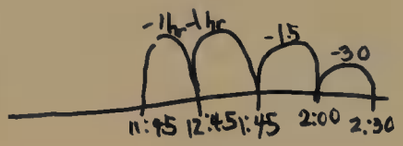


Figure 11

Student solutions to a problem where the start time is unknown

Jake got home from his hike at 2:30 after hiking for 2 hours and 45 minutes. At what time did Jake begin his hike?

(a)



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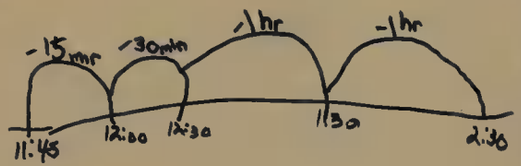

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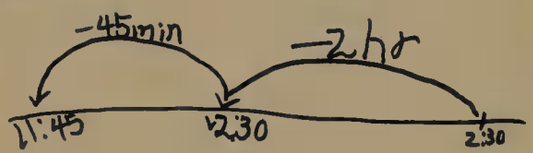

time. These students jumped using complete hours from 8:15 to 3:15 and then "jumped back" ten minutes to end at 3:05. A few students recorded both types of jumps without changing direction on the timeline. Those students were corrected by the students who said that the times you record need to stay in chronological order. Notice that one student did not use any arrows to record the directions of her jumps. Although we modeled and desire this type of notation, we made no attempt to correct students who recorded strategies that made sense to them.

Once students are able to use the open timeline as a tool to determine solutions to problems involving elapsed time, more challenging times and situations can be explored. Similar strategies were used in elapsed-time problems when the time did not end on a multiple of five (see **fig. 10**). When using the open timeline, these problems seemed no more difficult for students than those that started and ended on more common intervals.

Students had some difficulty interpreting problems where the start time was unknown; however, once students made sense of the context of such problems, they were able to solve them with relative ease. **Figure 11** provides students' solutions to a problem where the end time and elapsed time for a hike were given and students were to find the hike's start time. Once students understood that they were to find the hike's start time, they were able to determine that the start time was 11:45.

An Excellent Use of Time

Applying the open-number-line technique for adding and subtracting whole numbers to solve elapsed-time problems is a useful strategy. Students are able to make sense of the problems and record their thought processes as they seek solutions. One of the benefits of the open timeline is that students can use their own solution strategies to solve the problems. They are able to share their timelines with others in the class and provide justifications for their choices. Encourage them to do so. Such opportunities allow students to evaluate their own solution strategies as well as to see how others perceive their solutions. Students benefit from exposure to solution strategies that differ from their own and the chance to make sense of them. Students will see that many approaches to the same problem exist. Provide problems with and without context. Try the problems from this article with your students and see if they solve them in similar ways. You just might find it an excellent use of time.

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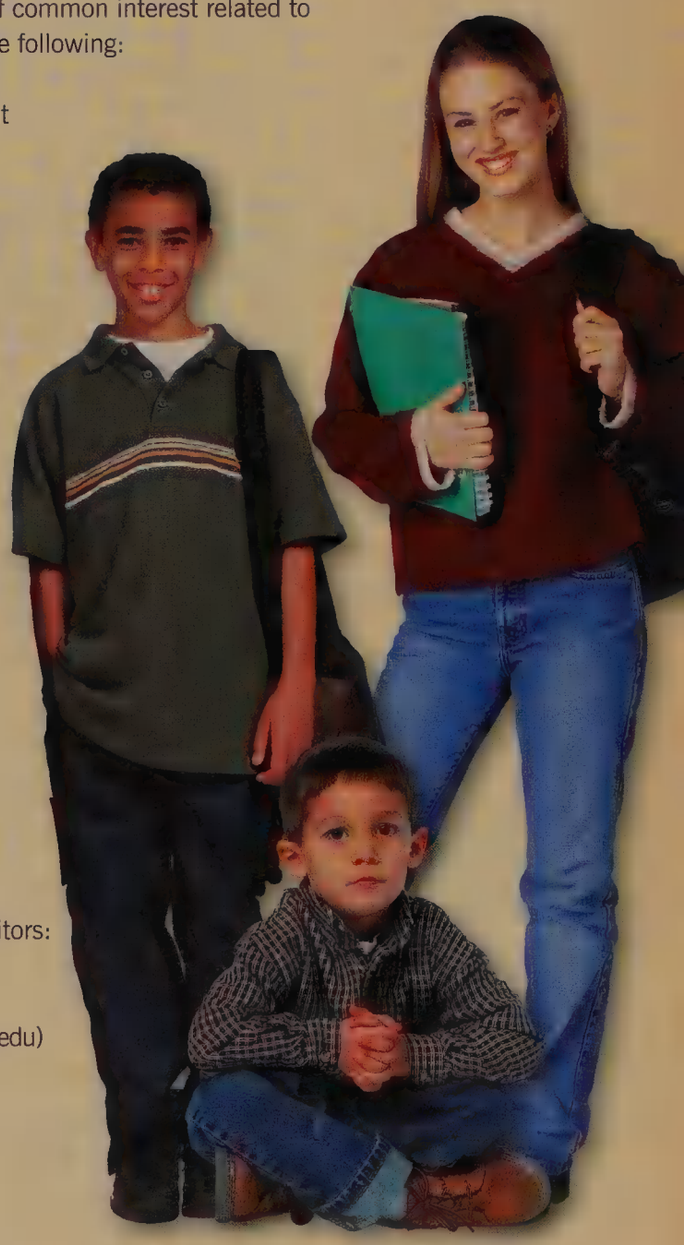
Editor's Note: Curriculum Focal Points *delineates the most important mathematical topics for each grade level (pre-K–8). Although* elapsed time *is frequently used to teach an important application of time, the authors of* Curriculum Focal Points *did not consider the topic to be one of the three Focal Points for elementary or middle school students. To learn more about the Focal Points, visit* www.nctm.org/focalpoints.aspx.

Contemporary Curriculum Issues

NCTM has provided strong leadership with regard to mathematics curriculum, most notably with the publication of *Curriculum and Evaluation Standards for School Mathematics* (1989); *Principles and Standards for School Mathematics* (2000); and, most recently, *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (2006). Curriculum goals and standards (the *intended curriculum*) and curriculum materials (the *written curriculum*) are powerful agents influencing teachers' instructional decisions and actions (the *implemented curriculum*).

The Editorial Panels of *Teaching Children Mathematics, Mathematics Teaching in the Middle School,* and *Mathematics Teacher* seek manuscripts that promote discussion and dialogue about important issues related to mathematics curriculum. A new department in each journal will provide a forum for stimulating discussion across a K–12 audience. The intent is to publish three articles, one in each journal, all focused on a single important curriculum issue. Each article will discuss the issue from the perspective of the audience of the particular journal (elementary, middle, or high school teachers). Collectively, the set of articles appearing in all three journals is intended to increase communication and dialogue on issues of common interest related to curriculum. Some possible topics for a curriculum-focused K–12 series include the following:

- What influence do state or national standards have on teaching and learning at the elementary, middle, and high school levels?
- What influence does, or should, technology have on mathematics curriculum?
- What are examples of learning trajectories for key ideas of statistics, and how do these ideas develop across the K–12 curriculum?
- What are the central ideas of algebra, and how do these ideas develop across the K–12 curriculum?
- How do teachers perceive, interact with, and use curriculum materials, and what impact do these materials have on teacher practice?
- What role do curriculum materials play in supporting student learning?



The "Contemporary Curriculum Issues" department will include three articles each year per journal. Manuscripts on any of the topics listed above or on other contemporary curriculum issues are welcome. The manuscript should be no longer than ten double-spaced pages, not counting figures and photographs. Submissions can be for one article for one particular journal, or they can be for a series of three articles, one for each journal. Manuscripts should be submitted at the appropriate Web site:

For *Teaching Children Mathematics,* go to **tcm.msubmit.net**
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- *Mathematics Teacher:* Chris Hirsch (christian.hirsch@wmich.edu) ▲

Touring Washington, D.C.

Problem

Davion and Hannah's family have arrived in Washington, D.C. As they look around the National Mall, they see many historical landmarks that they would like to visit, including the Washington Monument, the Lincoln Memorial, and the U.S. Capitol Building.

Wanting to see as many landmarks as possible, but having only one day to tour the city, they purchase a city map showing the locations of major landmarks. To help them efficiently plan their day, they sketch a simplified map (see **fig. 1**) of the sites they hope to visit and the distance between each site. As they study their map, they notice that multiple roads and sidewalks connect the sites. (Distances between landmarks are measured using a nonstandard unit.)

In what order should they visit the landmarks? Because their time is limited, they want to walk the shortest distance possible and still see everything. It does not matter if they walk along the same road or sidewalk more than once, and they do not need to walk along every road or sidewalk. The family may start anywhere on the map, but they must finish at the same location where they started. What is the shortest total distance they can walk?

Variations

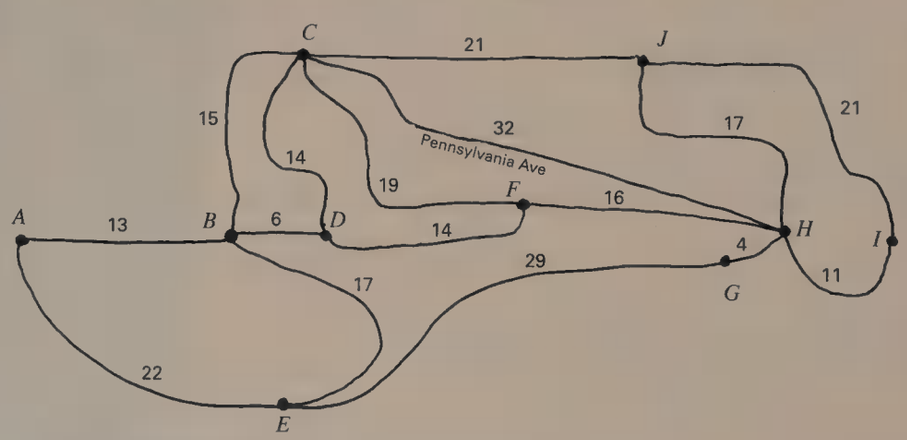
Challenge older students to determine the most efficient route to visit all of the landmarks labeled on the map. For younger students, modify the problem by limiting the number of landmarks the family visits to a manageable number for their specific grade level.

Extensions

- How do the problem and answers change if it is no longer necessary to end where a person began?
- Create your own maps and then determine the shortest possible distance so that all of the places can be visited.

Figure 1

Simplified map of selected Washington, D.C., landmarks



Map Legend

A	Lincoln Memorial
B	World War II Memorial
C	The White House
D	The Washington Monument
E	Jefferson Memorial
F	National Museum of Natural History
G	National Museum of the American Indian
H	Capitol Reflecting Pool
I	U.S. Capitol Building
J	Martin Luther King Jr. Memorial Library

Distances between sites are in unknown units.
Map is not to scale.

Edited by **Brian Schad**, *schad@aaps.k12.mi.us, a fifth-grade teacher at Lawton Elementary School in Ann Arbor, Michigan;* **Joseph Georgeson**, *jgeorgeson@usmk12.org, middle school mathematics department chair and teacher of eighth graders at the University School of Milwaukee in Wisconsin; and* **Sarah Slater**, *sslater@gmail.chelsea.k12.mi.us, a third-grade teacher at Pierce Lake Elementary School in Chelsea, Michigan. Each month this section of the "Problem Solvers" department features a new problem for students. Readers are encouraged to submit problems to the editors to be considered for future "Problem Solvers" columns. Receipt of problems will not be acknowledged; however, problems selected for publication will be credited to the author.*

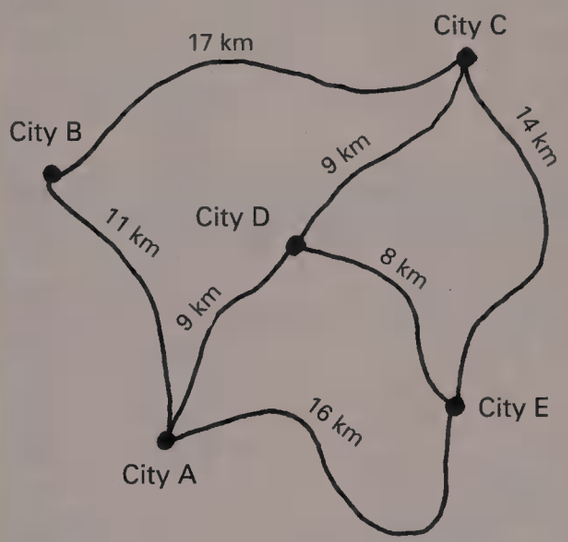
Classroom Setup

Begin the investigation by discussing the importance of a having a map when visiting an area that you are unfamiliar with. Introduce the problem by drawing a simple map on the board, such as the one in **figure 2**. Point out that the map includes towns, roads, and distances in kilometers between towns. Ask, "If you could begin a journey in any town that is drawn on the board and you wanted to visit every town, where would you start, and what route would you take?"

As you present the Touring Washington, D.C. problem to your students, be careful to avoid divulging too much information about how the problem can be solved. You want the students to do the thinking and problem solving. Guide them to explore different problem-solving strategies, such as guess and check and looking for patterns. Follow up with another question: "Suppose you want your journey to be the shortest distance possible and you want to end at the same point where you began your journey; what route would you take?"

Figure 2

Simple circuit map



Distances between points are in kilometers.

Where's the Math?

The Touring Washington, D.C., problem incorporates graph theory, arithmetic, and various problem-solving strategies. The study of graph theory is thought to have begun in 1736, when Leonard Euler posed the now-famous Bridges of Königsberg problem. As Königsberg citizens walked around their city on Sunday afternoons, they supposedly considered these questions: Is it possible to visit all four city sections, cross all seven bridges only once, and end at your starting point? How many different paths would allow you to visit the four sections of the city without retracing your path?

The Traveling Salesman problem is another famous example of a similar mathematics problem. A salesman must visit some towns, avoid retracing his path, and end up where he started. Travel costs are a consideration, so finding the optimal path is important. Towns are represented by dots called *vertices*. Connections, called *edges*, between the cities are represented by curved or straight lines. The edges in the Traveling Salesman problem are roads or highways. In the Touring Washington, D.C., problem, vertices could represent the landmarks to be visited, and the edges could be the streets or sidewalks connecting them.

Books about mathematics contain these and other similar problems, and books about games, puzzles, and recreational mathematics often include them, too. Here are more examples of such problems:

- Which letters of the alphabet can you trace without going over any path more than once?
- Can you trace all the diagonals of a pentagon without retracing any edges?
- The shapes that you draw are called *networks*. Under what conditions is it possible to trace a network?
- What kinds of networks have a tracing that ends up at the starting point?

By analyzing the number of vertices and the number of edges connecting them, students can discover some general patterns. These ideas are accessible for students of any age to explore. Students can establish their own rules and make conjectures about networks they draw. Studying these problems gives students fun, enjoyable ways to learn mathematics.

Sample student-created map for a fictitious nation

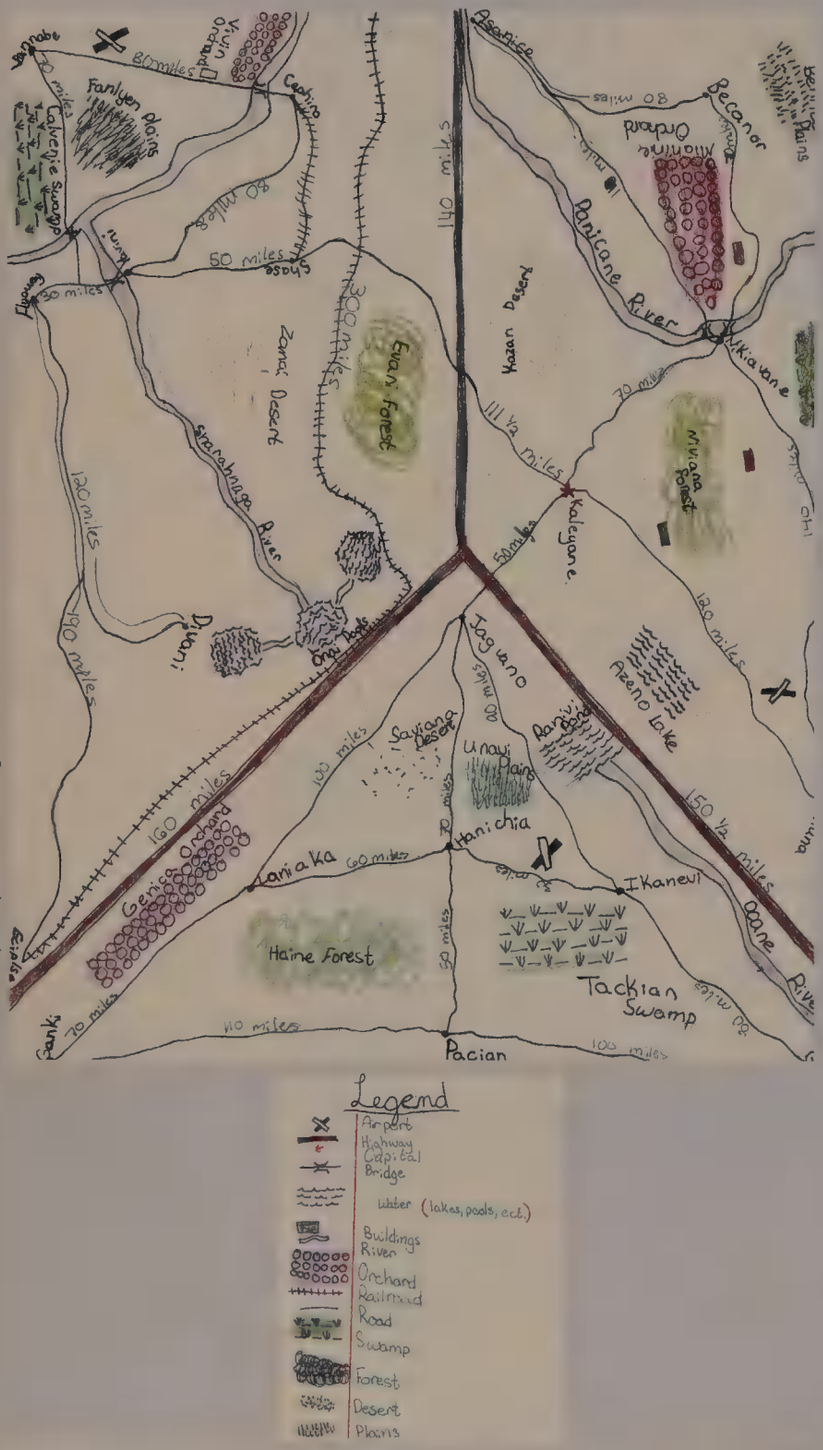


Illustration by Emily Caldwell; all rights reserved

Each individual teacher must decide whether to allow calculators for computing the distances in this activity. Monitor the students as they work the problem; listen carefully as they discuss their ideas. Interject only if a student is having difficulty. They will probably arrive at many different answers. Remember that the process for finding the answer in problem solving is sometimes more important than the answer.

Give students a later opportunity to review their problem solving. As they do, keep their focus on the different strategies they used rather than on their answers.

Real-Life Applications

In addition to recreational uses, this type of problem solving can be applied to mail and package delivery routes. One delivery company considers only routes that allow drivers to turn to the right. In some cities, employees read meters for natural gas or electricity consumption, and determining the employees' paths presents this type of problem. Sight-seeing companies in cities such as Washington require such problem solving to acquire and maintain efficient itineraries.

Whether problem solving for fun or practical use, using graphs and graph theory gives students interesting, accessible ways to discover and explore mathematics. Conjectures can easily be checked, and discoveries can range from very simple to extremely complicated. Next time you walk one route or another, think of the possibilities.

Share Your Students' Work

Please try this problem in your classroom. We are interested in how your students responded to the problem, what problem-solving strategies they used, and how they explained or justified their reasoning. Include information about how you posed the problem, samples of student work, and photographs. E-mail your thoughts, reflections, scanned student work, and photographs to **Brian Schad** at schad @aaps.k12.mi.us by **October 1, 2008**, or send them to his attention at Lawton Elementary School, 2250 S. Seventh, Ann Arbor, MI 48103. Include your name, grade level, and school with your submission.

Additional Resources

Did you know that NCTM has published a collection of some past "Problem Solvers" columns? Visit **nctm.org/catalog** for more on this and other NCTM resources, including professional development offerings, other publications, and online resources.

Sakshaug, Lynae E., Melfried Olson, and Judith Olson. *Children Are Mathematical Problem Solvers*. Reston, VA: National Council of Teachers of Mathematics, 2002. ▲

(Solutions to a previous problem begin on the next page.)

Solutions to the From Leaks to Liters Problem

This problem required students to collect and organize data within the context of measurement and proportional reasoning. It appeared in the August 2007 "Problem Solvers" section and was stated as follows:

Problem

Dripping taps in rich [industrialized] countries lose more water than is available each day to more than 1 billion people.—United Nations Development Programme, 2006

Veronica notices that the faucet in her bathroom is dripping. She begins to wonder how much water is being wasted drip by drip. She decides to let the water drip into a small paper cup to see how long it will take before the water starts to spill over. On the basis of this information, she says she can figure out how much water will drip away in 1 minute, 2 minutes, 5 minutes, 10 minutes, 1 hour, and 1 day. What method or methods could Veronica use to calculate the total amount of water for each time period? Explain your thinking.

Variation: This problem can be modified for younger students by allowing them to use a measuring tool, such as a graduated cylinder, to determine the amount of water that drips in 1 minute. Then have the students use this value to determine the volume of water that drips in 2 minutes, 5 minutes, 10 minutes, 1 hour, and 1 day. The calculations will be simpler if the students start with a given volume per minute.

Extension: Students can compare the amount of water they use daily to the 10 liters that an average person in rural Africa uses daily for washing, cleaning, cooking, and drinking (see www.africanwellfund.org/water stats.html). To provide some perspective, 10 liters is about the same amount we use when we flush the toilet once.

Ask students to *estimate* how much water they use in a day and then record the *actual* amount on a water use chart like the one to the right. Have them list additional activities and estimate, in liters, the amount of water used for each task. In class the next day, students can work with a partner to calculate the total amount of water used by the whole class.

Ask the students the following questions:

- How does your daily water use compare with that of people in rural Africa? With that of your peers? With that of your teacher?
- Which activity uses the most amount of water?
- Which activity uses the least?
- How much water could you save by changing some of your daily habits? Explain your reasoning.

Water use chart

Activity	Average Volume of Water Used	Tally for Each Use
Wash hands or face (1 minute)	6 liters	
Brush teeth (2 minutes)	1 liter	
Shower (10 minutes)	80 liters	
Bathe (20 minutes)	110 liters	
Flush toilet (1 time)	10 liters	
Drink water (1 bottle)	1/2 liter	

Record of calculations for the amount of water that drips at different time intervals

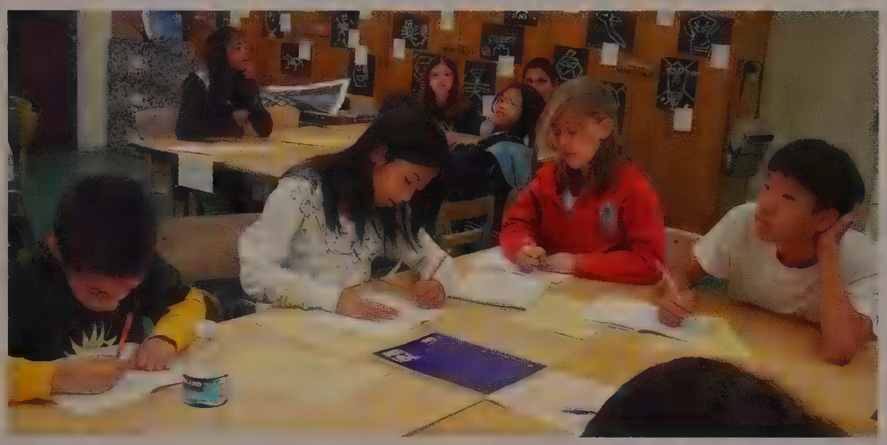
It took ___ seconds to fill the cup. This represents about 10 milliliters (mL) of water.
Use this information to make a prediction:
In 1 minute, ___ mL of water will have dripped away.

Time	Calculation	Amount of Water Dripped (mL)
1 minute		
2 minutes		
5 minutes		
10 minutes		
1 hour		
1 day		

Edited by **Mark Ellis**, *mellis@fullerton.edu, an assistant professor in the College of Education at California State University–Fullerton, where he prepares teachers of middle school mathematics and works with local classroom teachers in creating learning environments that support sense-making in mathematics; and* **Cathery Yeh**, *catyeh@aol.com, who teaches third grade at Arnold Elementary School in Cypress, California. Each month this section of the "Problem Solvers" department discusses the classroom results of using problems presented in previous issues of* Teaching Children Mathematics.

Figure 1

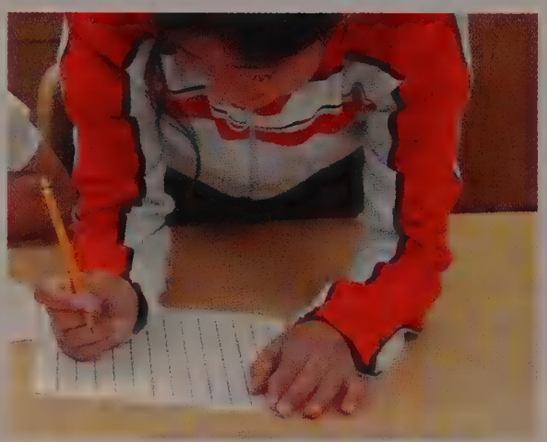
Sandy Kremenic's fifth graders initially worked on the problem independently.



Photograph by Sandy Kremenic; all rights reserved

Figure 2

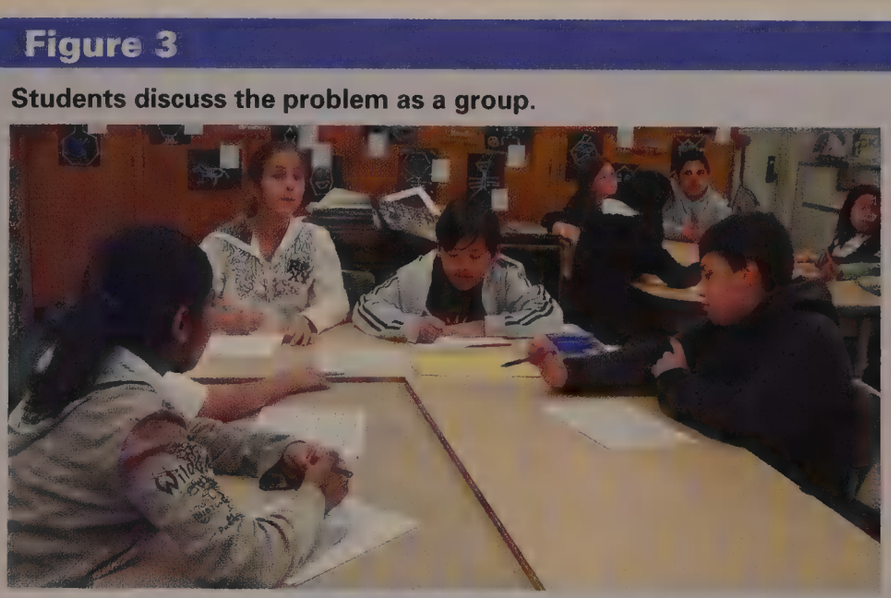
A student uses graph paper to help her solve the problem.



Photograph by Sandy Kremenic; all rights reserved

Figure 3

Students discuss the problem as a group.



Photograph by Sandy Kremenic; all rights reserved

Sandy Kremenic's fifth-grade class at Plainview Elementary School in Cahuenga, California, was studying the water cycle and water conservation when the teacher noticed the From Leaks to Liters problem in the August 2007 issue of *Teaching Children Mathematics*. She realized that the problem would serve as a perfect connection to their current science unit and also provide an opportunity for her students to engage in cooperative problem solving.

Before presenting the lesson, Kremenic gathered a copy of the problem for each student, along with graph paper and plain white paper. On the overhead screen, she showed the class the problem's first paragraph about the leaky faucet. The students read the problem carefully to be sure they understood what they needed to solve the problem. They also discussed the possible methods to figure out Veronica's calculation. As a class, they observed the length of time needed for their leaky faucet—a large paper cup punctured with a needle—to fill a 10 mL cup. Kremenic's account of what followed is below:

> We discovered that the small cup filled in only 34 seconds. However, from that point, the students had a difficult time figuring out how many milliliters were wasted in one minute, one hour, and one day. This led to a discussion of what possible operations may be used, and why it would work.

After this whole-class discussion, Kremenic walked around to observe and listen as students worked on the problem. Most students first did some thinking on their own before discussing strategies with their groups (see **fig. 1**). Many students drew a picture to model the situation and better understand the problem (see **fig. 2**). Then students began working in groups of five or six to discuss their ideas (see **fig. 3**). When each group had arrived at a solution, Kremenic had them share with the whole class.

Students discovered that a range of answers had been generated, which led to a rich discussion of how this could have happened. They discovered that the range of answers was caused by errors in the conversion of minutes to one hour and hours to one day. Kremenic describes how the students figured out the amount of water wasted in 2 minutes, 5 minutes, 10 minutes, 1 hour, and 1 day:

> After my students figured out how many milliliters were wasted in 1 minute, they multiplied that time by 2 to figure out what was wasted

at 2 minutes, and then multipli[ed] by 5 for 10 minutes. To get the amount wasted in 1 hour, students knew they could multiply their answer for 10 minutes by 6 since there are 60 minutes in an hour and $6 \times 10 = 60$. When they got to 24 hours, some students multiplied their original per minute rate by 24 instead of converting the 24 hours to minutes (or using their per hour rate). However, I just had to ask them if hours are the same unit of measure as what we were working in. Then, they realized they had to change their unit of measure to minutes.

The class discovered that their leaky faucet would waste 18 mL of water in 1 minute, 36 mL in 2 minutes, 180 mL of water in 10 minutes, 1,080 mL of water in 1 hour, and 25,920 mL of water in one day. The students found this very surprising, and it gave them a powerful example of how a lot of water—almost 26 liters—can be easily wasted from such a small drip.

This activity served as a perfect correlation to Kremenic's science unit on the water cycle and how water is a limited resource.

Overall, the From Leaks to Liters task gave students an opportunity to reflect on something they, as well as adults, often take for granted—the precious value of clean water. While conducting the experiment, students had opportunities to collect, organize, and analyze data. In the process, they practiced and developed measurement skills (both time and metric units of volume) and proportional reasoning. The initial task of determining how many seconds it would take for water dripping from a leaky faucet to fill a small paper cup and transferring this information to the amount of water wasted in 1 minute, 10 minutes, 1 hour, and 1 day helped strengthen students' understanding of the relationships between units of time. Finally, students deepened their knowledge of the concept of volume through linking a concrete experience (measuring the dripping water) to an abstract concept (the amount of water that would be wasted in one day).

Special thanks to Sandy Kremenic and her fifth-grade students at Plainview Elementary School in Cahuenga, California, for their contributions to this article. ▲

CALL FOR MANUSCRIPTS FOR THE SEVENTY-THIRD NCTM YEARBOOK (2011)

Motivation and Disposition: Pathways to Learning Mathematics

Motivation and disposition are important factors in developing students who are successful in mathematics. Motivation involves the art of generating interest and curiosity in the study of mathematics. Motivating tasks not only challenge students but also contribute to the development of confidence in their ability to understand mathematics and to solve mathematical problems. Dispositions include factors such as flexibility, persistence, self-efficacy, an inclination to reflect on one's thinking, and the valuing of mathematics as it applies to the study of other subject areas and in real-life situations.

This yearbook will help readers to understand the importance of motivating students to learn and developing positive dispositions toward mathematics in the classroom—affective outcomes that transcend skill and conceptual development and that can affect or shape a student's attitudes toward the study and use of mathematics for a lifetime. The Editorial Panel envisions a yearbook that will inform and inspire readers and will serve as a practical resource for teachers, school leaders, teacher educators, curriculum developers, and researchers.

The Seventy-third Yearbook Editorial Panel invites the submission of articles for the NCTM Yearbook *Motivation and Disposition: Pathways to Learning Mathematics.* Prospective authors should submit manuscripts for review by **December 1, 2008.** Contributions from classroom teachers, curriculum supervisors, teacher educators, mathematicians, and researchers are encouraged. Editorial decisions will be made by the members of the Editorial Panel: Daniel J. Brahier (issue editor), William R. Speer (general editor), M. Lynn Breyfogle, Melfried Olson, Marian Small, Marilyn E. Strutchens, and Denisse R. Thompson.

For complete details, please visit the NCTM's Web site at www.nctm.org/2011yearbook.aspx. Address questions to Daniel Brahier (brahier@bgsu.edu) or William Speer (william.speer@unlv.edu).

Jaye Downing, Christine Gallego, Sam Luna, Karolyn Williams, Dora Saldamando, and Grace Tapia-Beltrán

All about Me

"Math by the Month" activities are designed to engage students to think as mathematicians do. Students may work on the activities individually or in small groups, or the whole class may use them as problems of the week. No solutions are suggested, so students look to themselves for mathematical justification and develop confidence to validate their work.

August means getting ready for a new school year, catching up with old friends, and making new acquaintances. The following activities are designed to help you and your students get to know one another through mathematical investigations involving number sense and operations, collecting and representing data, geometry, measurement, logic, communication, and representation.

WEEKLY ACTIVITIES

ALL ABOUT ME: K–2 — AUGUST 2008

4 **Your family ...** Who lives at your house? Count how many of the people are adults. How many are children? How many brothers and sisters do you have? Are any of them older than you are? After you collect and record your data, decide how to represent it to others. Then collect data from your entire class and order it from the fewest number of people living in a house to the greatest number of people living in a house. **Challenge:** Find the *difference* between the largest number and the smallest.

11 **Your pets ...** Find out who in your class has at least one pet. Represent the data with numbers, drawings, or words. What else would you like to know about your friends' pets? What kinds of pets do your classmates have? Make another representation of this additional data. If you do not have a pet, what pet would you like to have? What kind of pet do most people in the class have? What pet do the fewest number of people have? What else did you discover about your classmates' pets?

18 **How many?** How many feet are in your classroom? Use words, drawings, or a physical model to show your findings. How many feet are in other classrooms? Compare the number of feet in your classroom to the number of feet in other classrooms. Order the number of feet in the different classes from the smallest number of feet to the largest number of feet. Using the data you have, figure out how many students are in each class.

25 **Let's make a quilt pattern!** All of the quilt patterns will have 5 quilt squares in each column (down). First, count your family members _____ and your pets _____. Circle the larger number. This tells you how many 1-inch squares to put in each row (across) your quilt.

- How old are you? _____ Write this numeral on the center square.
- On each square of row 1, draw 1 straight line for each letter of your first name.
- On each square of row 2, draw 1 green triangle for each family member who lives at home. Fill extra squares in row 2 with a yellow hexagon.
- On each square of row 3, create a pattern using your 2 favorite colors.
- On each square of row 4, draw 1 blue rhombus for each pet that you have. Fill extra squares in row 4 with yellow hexagons.
- Count how many letters are in your last name. _____ On each square of row 5, draw 1 squiggly line for each letter.

When you finish your quilt, write an autobiography including your age, how many people are in your family, your two favorite colors, and how many pets you have. Display everyone's autobiography and quilt. Using the clues from the autobiographies, can you figure out who made each quilt pattern?

All members of the writing team teach at Pueblo Gardens Elementary School in Tucson, Arizona. Jaye Downing teaches second grade, Christine Gallego teaches first grade, Sam Luna teaches fifth grade, Karolyn Williams teaches a third- and fourth-grade multiage level, Dora Saldamando teaches fourth grade, and Grace Tapia-Beltrán teaches fifth-grade mathematics.

Edited by **Dana Islas**, *who teaches kindergarten at Pueblo Gardens Elementary School. Readers are encouraged to submit problems to be considered for future "Math by the Month" columns to the department editor at* **dana.islas@tusd1.org**. *Receipt of the problems will not be acknowledged; however, those selected for publication will be credited to the author. Readers are also invited to submit creative solutions and adapted problems to* **tcm@nctm.org** *for potential publication in "Reader's Exchange." Please include "Reader's Exchange" in the subject line. Because of space limitations, responses beyond the 250-word limit are subject to abridgement. Submissions are also edited for style and content.*

WEEKLY ACTIVITIES

ALL ABOUT ME: 3–4

AUGUST 2008

What's in a name? Discover how much your first name is worth if every consonant has a value of $5 and vowels are $10. Compare the value of your name with those of your friends. Graph the class results to see which names have the greatest and least value. Find the *range* of numbers (the difference between the greatest and least value). What do you notice on the graph about the names with the greatest and least value?

Birthday bliss ... How many months are in a year? Use this information to estimate how many months old you are today. Write that number. Now, start with the month you were born and count forward to (and include) the present month. Was your estimate too high or too low? Did the actual result surprise you? If each month had exactly four weeks, estimate and then count how many weeks old you are. **Challenge:** Can you also determine your age in days? Hours? Minutes?

You are square! Use paper pattern block shapes to represent your family members. You are the square. If you are a girl, add one blue rhombus to the square. If you are a boy, add one tan rhombus to the square. Represent each adult with a yellow hexagon. Represent each of your siblings with a trapezoid and each pet with a triangle. Use all of the shapes to create a polygon, making sure that your blocks are connected along matching sides. Glue the polygon to a sheet of paper. Use the back of your paper to write additional family details. Share your creation with a classmate.

How tall are you? Work with a partner to measure your height. Be sure that everyone uses the same unit of measure (inches or centimeters). Collect data for your class and create a variety of graphs to compare your results. Why is using a consistent unit of measure important? Total everyone's height. What else is about this long or high?

WEEKLY ACTIVITIES

ALL ABOUT ME: 5–6

AUGUST 2008

Your favorite foods ... Multiply or divide to complete each function table below. Substitute the letter next to each solution above the corresponding number in the puzzle to the right. What is a favorite food? Now create your own function tables and a puzzle for your favorite food. Exchange with a friend and solve each other's puzzle.

___	___	___	___
90	40	1	4

IN	OUT
8	64
7	56
6	48
5	____ (A)

IN	OUT
48	12
44	11
40	10
16	____ (O)

IN	OUT
100	10
200	20
300	30
10	____ (C)

IN	OUT
20	180
30	270
40	360
10	____ (T)

Your space ... Draw a scale map of your bedroom: Gather centimeter grid paper, measurements of your bedroom, and a list of objects in your bedroom with their measurements. Include items that make your room unique to you, such as furniture, an area rug, posters, a closet, and doors. Create a key for your scale drawing. Share and post your drawing.

What do you want to know? Make a list of questions that would help a classmate get to know you better. Now, work with a partner to choose a question to learn more about your classmates. Here are some possible ideas:

- Did you travel during the summer?
- What is your favorite sport? Favorite book?
- Do you play a musical instrument? Which one?
- Which languages do you speak?

Once you agree on one question, collect your data. Organize and represent your findings to present to the class. You may use a Venn diagram, bar graph, or pie chart. Use fractions and percentages to report your data.

How do you measure up? Each person measures and records his or her arm span. Calculate the average length of an arm span in your class. Using the average arm span as a unit, how many units wide is the room? Measure the perimeter of the room, the distance from the classroom to the cafeteria, and the basketball court's perimeter.

Christine Balderas/iStockphoto.com

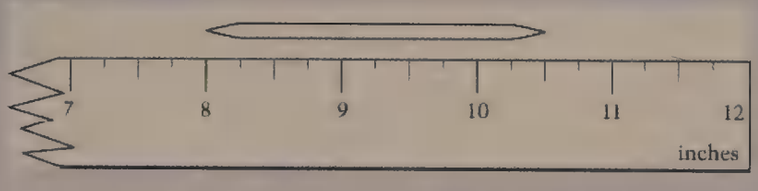
Andrew M. Tyminski, Monica Weilbacher, Nicole Lenburg, and Cindy Brown

Ladybug Lengths: Beginning Measurement

Measurement is one of five Content Standards promoted by the National Council of Teachers of Mathematics as crucial facets of students' mathematical knowledge (NCTM 2000). Skills and reasoning that develop in association with measurement are applicable in everyday life as well as in many career choices. A significant topic within elementary school curricula, measurement is addressed through the use of investigative activities. Unfortunately, students traditionally struggle with measurement tasks, which points to limited understandings of concepts involved in such tasks (Kamii 2006; Thompson and Preston 2004). Therefore, examining the manner in which measurement is approached within the early elementary classroom is a germane pursuit.

Figure 1

A lack of conceptual understanding of measurement can result in answers such as 10 1/2 when students measure the object below (Thompson and Preston 2004).



Andrew M. Tyminski, atyminsk@purdue.edu, is an assistant professor of mathematics education at Purdue University, West Lafayette, Indiana. His interests include the professional development of mathematics teachers and preservice and in-service mathematics teachers' pedagogical decisions in the moment of teaching. Monica Weilbacher, mweilbac@purdue.edu, and Nicole Lenburg, nlenburg@purdue.edu, are elementary education majors at Purdue University. They both enjoy working with children, especially in the area of mathematics. Cindy Brown, cbrown@tsc.k12.in.us, teaches kindergarten at Burnett Creek Elementary School in West Lafayette, Indiana.

Edited by **Signe E. Kastberg**, *skastber@iupui.edu, an assistant professor of mathematics education at Indiana University-Purdue University Indianapolis.* *"Early Childhood Corner" addresses the early childhood teacher's need to support young children's emerging mathematics understandings and skills in a context that conforms with current knowledge about the way that children in prekindergarten and kindergarten learn mathematics. Readers are encouraged to send submissions to this department by accessing* **tcm.msubmit.net**. *Manuscripts should be double-spaced and must not exceed eight double-spaced typed pages.*

Students are typically exposed to tasks that focus only on developing *skills* for measuring length, area, volume or capacity, and weight. Opportunities to develop conceptual understandings associated with measuring each of these attributes (Van de Walle 2007) are often missing from the tasks. For example, when children measure length by counting along a ruler, they may not be aware of what they are actually counting. Children may count numbers on the ruler but may not necessarily understand that they are actually counting *the spaces between* the numbers (see **fig. 1**).

Third graders consistently have difficulty with measurement tasks such as the one in **Figure 1** (Lindquist and Kouba 1989), suggesting that learning measurement is more complex than learning how to use a ruler. Primary grade teachers can play an integral role in planning and implementing lessons that support students' growing understanding of measurement. Developing a concept of measurement and strategies for approaching measurement tasks is important, especially to the development of children in grades pre-K–2. Young children need opportunities to engage in activities that allow them to construct their own measurement concepts, such as what it means when we say something is five inches long. At an early age, measurement activities should involve use of concrete materials and follow a learning trajectory that—

- allows for explorations using iterations of nonstandard units of measurement;
- provides opportunities for children to develop their own measurement instruments; and
- transitions toward the use of tools involving standard units of measure.

This article describes a series of activities used to support children's development of the concept of measuring length. Monica Weilbacher and Nicole Lenburg developed the activities as part of their elementary mathematics field experience. Input and feedback from their cooperating kindergarten

teacher, Cindy Brown, and their university professor, Andrew M. Tyminski, were used to extend the activity series and provide opportunities for kindergarten students to begin developing a sense of length measurement.

The Lesson

The first activities were designed to give children the opportunity to use iterations of nonstandard units to determine the length of various objects. Because this was one of the first exposures to the concept of measuring length for Brown's kindergartners, Weilbacher and Lenburg decided to read a story to help introduce the concept. Using literature within mathematics lessons can help students make text-to-self connections, which are especially important in the primary grades. A story also serves as an appropriate context to introduce and promote a discussion of length. The teachers chose *Ladybug on the Move* (Fowler 1993) because of the engaging ladybug character and its interesting travel paths. In the story, the ladybug travels from various locations, such as from the leaf to the door and from the watering can to the stones. The teachers recognized the book as an excellent starting point for investigating length, in this case, the length of the ladybug's path.

Weilbacher and Lenburg began by reading the story aloud to the students. Using a white board at the front of the room, they replicated the insect's travels with a ladybug model and pictures of various landmarks from the story. Straight lines drawn between pairs of landmarks represented each portion of the ladybug's path. To begin the investigation, they drew two different ladybug paths (lines) on the board. One line was approximately six inches long; the second was closer to twenty inches long. Students were to choose the path on which the ladybug would travel the farthest. Kamii suggests that beginning with this type of question helps children engage and provides a reason for them to answer the more specific questions that will follow (2006).

The children were quick to choose the longer line as their answer. When asked how they knew the ladybug traveled farther on the longer line, they replied that it "looked bigger." The students' ability to use perceptual judgment in this case was expected, as the teachers had purposefully drawn lines of clearly different lengths. However, wanting to motivate the children to gather information beyond observation, they asked the students how they could "be sure" that one line actually was longer than the other. Because this was an introductory

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lesson on length and measurement, students were understandably confused by the question. Unsure of how to proceed, Weilbacher and Lenburg decided to introduce the idea of iterating a series of units as a way to measure and compare two line lengths. They began to line up cutouts of the ladybug along the shorter path, at which point a student shouted, "You, like, measure!" The students began making connections between the term *measure* and the use of units.

Before using the ladybugs to measure the lines, students estimated how many ladybugs, lined up end to end, would match the length of the two lines. Estimation should play a role in children's development of measurement concepts; it allows students to develop a familiarity with the unit and to focus on both the measuring process and the attribute being measured (Van de Walle 2007):

- What does it mean to estimate how long this line is?
- How many of these units will I need to match the given length?
- What number seems reasonable?

The shorter line was a little longer than one ladybug unit. When students made their estimates, they gave reasonable answers, such as three or four ladybugs. For the longer line, most students predicted inappropriately large numbers, such as seventy-five or one hundred ladybugs. A few students made more accurate predictions with numbers in the twenties. Students' answers gave some information on their current estimation abilities as well as their developing number sense. Students who predicted

that a large number of ladybugs would be necessary to match the length of the line likely viewed the line as "very" long. For most kindergarten students, one hundred is not just *a* big number, but *the* big number. All responses were recorded on the board, and the process of using ladybugs to measure the lines began. As children helped place ladybugs on the line, they voiced concerns with the placements: Should the ladybugs face the same way? Should the ladybugs be right next to each other? The teachers' goal was to encourage the children to develop a consistent method of lining up units, in this case, ladybugs. Students themselves would have an opportunity to practice measuring with nonstandard units in the next stage of the lesson.

Using lima beans as another nonstandard unit of measure and their own activity booklets containing pages with ladybug paths of various lengths, students were to estimate and record the number of units needed to match line lengths. Then they used lima beans to actually measure the lines. For the most part, the students worked alone, but some discussion among peers took place. The teachers walked around and observed interesting problem-solving methods. Some students had no difficulty completing the task, but others ran into trouble using the lima beans as a unit of measure. Modeling the procedure for lining up the objects on the line (see **fig. 2**) seemed to eliminate the problem of students overlapping beans. Weilbacher and Lenburg had anticipated teachable moments when students would furnish slightly different answers for the same line, but unanticipated problems also arose. Several students arranged their beans so that they almost resembled a wave: some beans placed slightly above the line, some slightly below, and some right on top of the line. Students also oriented their beans differently on the lines. Some placed the lima beans side by side lengthwise, and others lined them up widthwise. The differences in lengths measured in beans provided an opportunity to talk about orientation and consistency in how the children used the beans (units) to measure the lines. The primary goal of the task was to provide students with an opportunity to develop the concept that measurement is a number that represents "a comparison between the attribute of the object being measured and the same attribute of a given unit of measure" (Van de Walle 2007, p. 375).

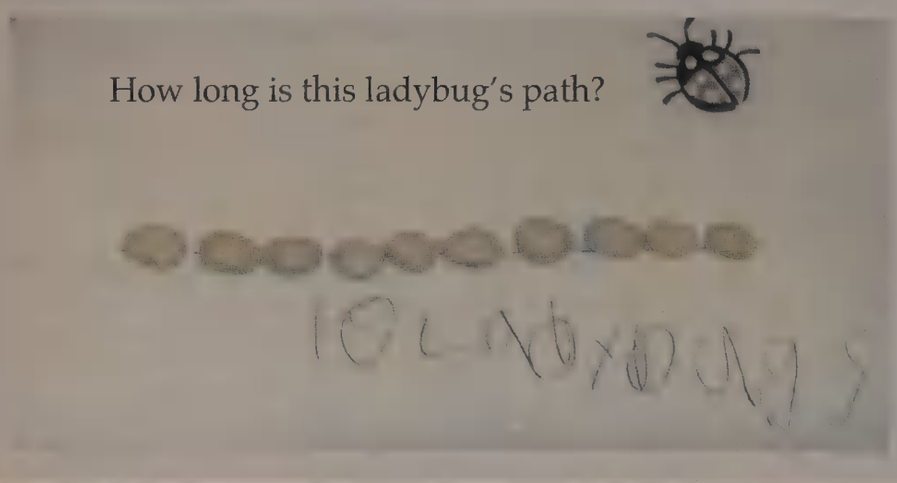
The activity also encouraged students to ask, "What if you measure short of the line or over the line?" As the teaching team had planned the task, they had predicted that some students would not completely cover the line, leaving a portion of the line less than one bean length uncovered and resulting in an answer of one less bean. Other students added an extra bean to their line to make sure that no part of the line was showing, although the beans then extended past the line. One student explained his reasoning: "Well, there still was line showing, so I knew I could fit more and measure it better." This approach provided additional opportunities to discuss estimation and precision, different bean sizes, and different ways beans could be placed on the line. Weilbacher and Lenburg concluded the discussion by explaining that each student should decide how to orient the beans and whether the line should be completely covered with beans or "overflowing" with beans. Most students accepted this explanation, but a few were still very concerned about getting the "right" answer.

Using an iteration of nonstandard units to measure an object's length is the first stage in a learning trajectory toward a conceptual understanding of length measurement. Children should have multiple opportunities to use iteration of many units as a means of determining length. These experiences should also include the opportunity to measure the same object with different-sized units. Tasks such as those used in the ladybug activity help children begin to develop an understanding of the relationship between the size of the unit and the number of units needed to match a given object's length.

The kindergartners reexamined their line lengths with a new unit of measure—paper clips. Again, the students were to make and record their estimates and then measure their lines with paper clips. The teachers asked, "Would it take more or fewer paper

Figure 2

The teachers modeled the procedure for lining up the lima beans to measure the lines.



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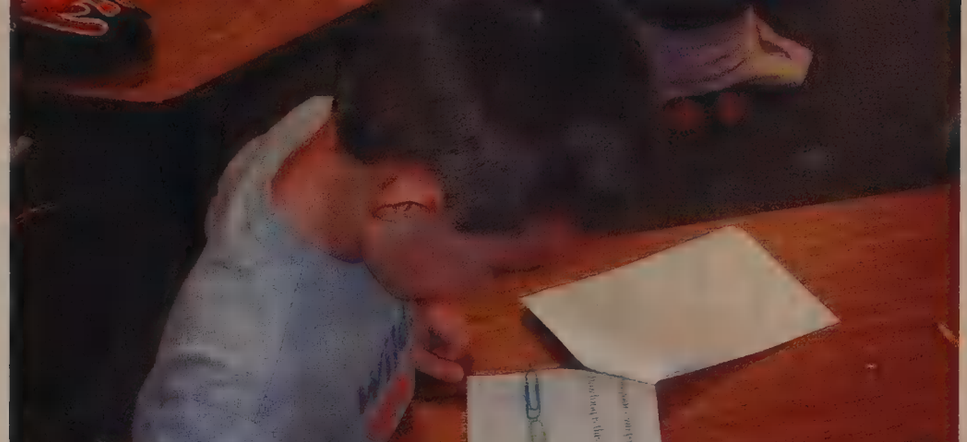
clips than it did lima beans to measure a certain line?" Almost all the students seemed to understand that fewer paper clips would cover the line because the paper clips were larger than the lima beans.

Overall, students seemed excited and engaged during the lesson. Weilbacher and Lenburg concluded that their use of nonstandard units helped students begin developing a concept of what it means to measure length. Further, the opportunities to estimate and measure with two different-sized units were beneficial in introducing important mathematical relationships.



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What Comes Next

After Weilbacher and Lenburg implemented the activities described, it was up to Brown to continue the learning trajectory initiated by this lesson. Brown's plans for the next few lessons on length were designed to give her kindergartners more practice with iterating units. The children were engaged in measuring various objects using Unifix cubes and learning links. In subsequent lessons, Brown asked her students to answer questions, such as, "Who has the longest shoe, the widest hand, or the shortest hair?" These question types are especially motivating for kindergarten students, and Brown's were eager to engage in measurement activities in order to find the answers.

Brown chose Unifix cubes and learning links as nonstandard measuring units because they can be connected and used as "first rulers." The development of students' own measurement instruments is the next logical stage in developing meaning for measurement. Brown's use of Unifix cubes helped students make the cognitive connection between measurements as counts of objects and the use of a ruler. After giving students some experiences lining up Unifix cubes along given objects, Brown asked them to make a rod of ten cubes to measure classroom objects.

The last stage of her unit on measuring length involved supporting her students' transition from the use of nonstandard to standard units of measure. To accomplish this goal, Brown used inchworm manipulatives. Students can use iterations of inchworm manipulatives to directly compare lengths of objects, while engaging in comparisons of the standard unit. Manipulatives such as Unifix cubes can also be connected to form rudimentary inch rulers.

Engaging tasks, however, are not always enough to ensure that students are constructing the knowledge we hope they are. Talking with the students is important—during and after the measurement tasks—to see what they have gleaned from the activities. Following each measurement activity, Brown led a community circle discussion in which the students talked about their discoveries. She found that her students became more precise in their measuring as the instructional unit progressed, and she was pleased with their developing understanding of units of measurement and rulers.

Conclusion

Measurement is an important life skill beyond mathematics classrooms, and foundations for such skills begin in the early primary grades. Teaching length measurement as solely a formula often leaves students with misconceptions. On the other hand, a series of well-planned experiences helps children on their way to a meaningful understanding of length, unit, rulers, and measurement.

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INTRODUCING COMPLEX SYSTEMS into the Mathematics Curriculum

Our children live in a highly sophisticated world composed of interlocking complex systems. An appreciation and understanding of such systems is critical for making effective decisions about our lives as individuals and as community members. The activities presented in this article provide rich opportunities for children to experience an introduction to complex systems during which they think mathematically about relevant relationships, patterns, and regularities in dealing with authentic problems (English 2006; Lesh and Doerr 2003).

The study of *complex systems* is a new approach to science that is concerned with how relationships between parts give rise to the collective behaviors of a system, and how the system interacts and builds relationships with its environment (Bar-Yam 2004). Complex systems are not confined to science, however. They cut across a range of disciplines including mathematics, engineering, management, medicine, history, and literature. Political parties, business organizations, team sports, the human body, families, and the World Wide Web are all examples of complex systems.

In recent years, educators have emphasized the need to incorporate a basic understanding of complex systems within the K–16 curriculum (e.g., Jacobson and Wilensky 2006; Lesh 2006). Although students in secondary schools receive some exposure to complex systems, mainly within science courses (e.g., the study of ecosystems), they typically develop limited understanding of such systems (Jacobson and Wilensky 2006). Students learn the names of system parts by rote, but complex phenomena themselves are often ignored in the curricula.

We must find effective ways of introducing meaningful experiences with complex systems, beginning in the elementary grades. This article explores one way of doing so, namely, through *mathematical modeling* activities. Prior to investigating these activities, it is worthwhile to delve into some fundamental features of complex systems.

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Complex Systems

Researchers who study complex systems across a range of disciplines are increasingly reluctant to provide a hard-and-fast definition of such systems. Nevertheless, they do identify many key features that these systems share. For elementary school children, addressing the features of *parts*, *wholes*, *and their relationships* is an appropriate way to introduce complex systems.

As Bar-Yam (2004) explains, everything is made up of parts. One of the main difficulties we face in solving problems is our tendency to think a problem lies in the parts, when it is actually in the relationships among them. For example, the way a society functions—including inevitable conflicts—depends on how people relate to and interact with one another, not how people behave separately.

When we explore complex systems, we cannot focus only on intricate details nor can we focus solely on the large-scale view. We need to move back and forth between seeing the parts and the whole. In this way, we can judge which aspects of the parts are relevant to describing the whole. This dynamic relationship between the parts and the whole is referred to as *emergence* (Bar-Yam 2004). A team sport is a good example. In selecting team members, coaches must consider not only the particular attributes of individual players but also how these attributes might function collectively to yield a successful team. In turn, reviewing a team's performance requires a consideration of both how the team functioned as a whole and how each player contributed as a team member.

Another key feature of the relationships between parts and wholes is *interdependence* (Bar-Yam 2004). In dealing with complex systems, we need to consider how the parts of a system affect one another. That is, we must ask how strong the dependencies are between each part; and how manipulating or removing a particular part will affect the rest of the system. Sometimes the effect is minor, and at other times, it is significant. Sometimes many parts of the system are affected; other times, only a few are. These different kinds of interdependence are major considerations in our efforts to solve problems ranging from family issues to major global concerns. Therefore, giv-

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ing children opportunities to investigate situations involving relatedness and relationships is important.

The two activities presented in this article were designed to create such opportunities. The first activity involves selecting a swimming team for the 2004 Olympic Games (see **fig. 1**) and the second involves choosing workers for a new gardening business (see **fig. 2**).

Mathematical Modeling

Before we introduce the activities, we first need to explain what we mean by *mathematical models* and *modeling*. These terms have been used in various ways in mathematics education research. In my work, I view models as *conceptual tools that are used to construct, interpret, explain, and mathematically describe a situation* (Doerr and English 2003; Lesh and Doerr 2003). The process itself is often described as building a link or bridge between mathematics as a way of making sense of our physical and social world and mathematics as a set of abstract, formal structures (Greer 1997).

When children create models in dealing with complex systems, they engage in important mathematical experiences that are often under-represented in elementary mathematics curricula. These experiences include the following:

- Constructing, in contrast to being given, important mathematical ideas
- Describing and explaining concepts and processes
- Debating and justifying mathematical arguments
- Using mathematical creations to make predictions
- Representing mathematical ideas in a format of choice
- Working productively with data that require quantifying, coordinating, organizing, and representing

In contrast to typical school problems, modeling activities do not present key mathematical ideas "up front" for children. Instead, mathematical concepts appear within the models children create, not in the problem itself. To use an analogy, modeling problems are akin to design specifications given to engineers who are building a space shuttle. The problem provides constraints and resources, but it does not say exactly what the final product should be (English and Lesh 2003).

The modeling activities I have implemented in elementary classrooms ask children to construct models that enable a particular client to make a decision in a real-world problem situation. The activities allow for multiple approaches to solutions and call for multifaceted products. They can be solved at different levels of sophistication, enabling all children to access the powerful mathematical ideas and relationships embedded in the problem. Furthermore, because modeling is inherently a social experience designed for small-group work, children develop effective communication and teamwork skills.

Implementing Modeling Activities

The modeling activities displayed in **figures 1** and **2** were implemented recently in fourth- and fifth-grade classrooms respectively. The activities were

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part of multiyear modeling programs conducted in several Brisbane (Australia) elementary schools.

At the start of the programs, the children engaged with preparatory activities that included (a) interpreting mathematical and scientific information presented in text and diagram; (b) reading simple data tables; (c) collecting, analyzing, and representing data; (d) preparing written reports from data analysis; (e) working collaboratively in group situations, and (f) sharing end products with class peers by means of verbal and written reports. For example, one preparatory activity involving the study of native animals required the children to read and respond to written text on "The Lifestyle of Our

Figure 1

Selecting an Olympic swim team

Let the Selections Begin

Freestyle and breaststroke were the only strokes used for the swimming events during the earlier modern Olympic Games. Backstroke was added in the 1904 Olympic Games. In the 1940s, breaststrokers discovered they could go much faster by bringing both arms overhead together, which later became the butterfly stroke—the fourth stroke used in competitive swimming.

Winning an Olympic Gold Medal is a "cat and mouse" game with competitors striving to improve their performance while watching closely the performance of their rivals. They have to train so that their peak performance will occur when the games are on, not months before.

While the Aussies were battling with the powerful USA team in Los Angeles in 1984, some of their swim team took on Europe's best in the Mare Nostrum Series. At the swim meets held in Europe and the USA, the Olympic champions, world record holders, aspiring young novices, and seasoned swimmers all set strong times. Over the years, world records have been broken, setting the benchmark even higher. Look at **table 1**, which shows the number of times the men's 100 meter world freestyle record has been broken at non-Olympic events between 1956 and 2000. The 2000 Sydney Olympics led to a new men's 100 meter world freestyle record of 47.87 seconds. Australian swimmers were among those athletes setting the highest standards as the 2004 Athens Olympic Games approached.

The Olympic Swimming Committee works very hard to select the most suitable swimmers for the Olympic Games. If Australia wants a chance at dominating the swimming pool arena, they must make sure the athletes selected for the team are in their prime condition.

Australians were able to watch live coverage of the Athens Olympics for up to sixteen hours a day, but because the time zone in Athens is seven hours behind Australia's, Aussies had to stay up through the night if they wanted to catch the main Olympic events.

Readiness Questions

1. When was backstroke included in the Olympics Games?
2. How was the fourth swimming stroke developed?
3. Why is it important for the Australian Swimming Committee to select the most suitable swimmers?
4. During the earlier Olympic Games, what were the original swimming strokes?
5. If Australians have to watch the men's 100 meter freestyle final at midnight in Brisbane, what time would the race be scheduled in Athens?

Table 1. Men's 100 Meter World Freestyle Records (Not Olympic)

Country	Time (secs)	Date
Netherlands	47.84	19 September, 2000
Australia	48.18	16 September, 2000
Russia	48.21	18 June, 1994
USA	48.42	10 August, 1988
USA	48.74	24 June, 1986
USA	48.95	06 August, 1985
USA	49.24	06 August, 1985
USA	49.36	03 April, 1981
USA	49.44	14 August, 1976
USA	49.99	25 July, 1976
USA	50.39	24 July, 1976
USA	50.59	23 August, 1975
USA	51.11	03 August, 1975
USA	51.12	21 June, 1975
USA	51.22	05 August, 1972
USA	51.47	03 September, 1972
USA	51.94	23 August, 1970
Australia	52.20	19 October, 1968
USA	52.60	27 July, 1967
France	52.90	13 September, 1964
Brazil	53.60	20 September, 1961
USA	54.40	18 August, 1961
Australia	54.60	28 January, 1957
Australia	55.20	19 January, 1957
Australia	55.40	11 November, 1956

Bilby," which included tables of data displaying the size, tail length, and weight of the two types of bilbies (big-eared, rabbit-like Australian marsupials).

Four 40–45-minute sessions are usually devoted to a modeling activity, with the sessions spread across one—or perhaps two—weeks. Both activities begin

(Continued on p. 44)

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Information

The Olympic Swimming Committee has already selected the women's swim team. However, they are having difficulty selecting the most suitable swimmers to compete in the men's 100 meter freestyle. The committee has collected data on the top seven (7) male swimmers for the 100 meter freestyle event. The data collected (see **table 2**) show each of the swimmer's times over the last ten (10) competitions.

The Olympic Swimming Committee has decided to make you part of their selection team. They need your help in selecting two (2) male swimmers for the 100 meter men's freestyle event. The committee members are hoping that the information they collected (see **table 2**) will make it easier for you to decide. You must make sure to select the best swimmers.

Problem

As selectors for the Olympic Swimming Committee, you need to use the data in **table 2** to develop a method to select the two most suitable swimmers for the men's 100 meter freestyle event.

Write a report to the Olympic Swimming Committee telling them who you selected and why. You also need to explain the method you used in selecting these swimmers. The selectors will then be able to use your method in selecting the best swimmers for the other swimming events.

Table 2. Swimming Times for the Top Seven Swimmers in the Last Ten Men's 100 Meter Freestyle Events

Key: PB = personal best DNC = did not compete Competition	**Ashley CALLUS (PB: 48.43 secs)**	**Michael KLIM (PB: 47.98 secs)**	**Eamon SULLIVAN (PB 50.06 secs)**	**Ian THORPE (PB: 48.11 secs)**	**Todd PEARSON (PB: 48.45 secs)**	**Grant HACKETT (PB: 48.67 secs)**	**Adam PINE (PB: 48.68 secs)**
2004 Telstra Swimming Grand Prix	51.50 secs	50.44 secs	50.35 secs	49.23 secs	50.19 secs	51.42 secs	51.87 secs
2004 Telstra Olympic Team Swimming Trials	49.31 secs	49.78 secs	50.06 secs	48.83 secs	49.78 secs	50.40 secs	50.24 secs
2003 Telstra FINA World Cup	48.06 secs	50.12 secs	50.24 secs	48.94 secs	48.83 secs	DNC	49.67 secs
2003 Telstra Australian Championships	49.07 secs	DNC	51.86 secs	49.07 secs	49.29 secs	50.32 secs	50.69 secs
2002 Pan Pacs	DNC	48.44 secs	52.43 secs	48.98 secs	49.64 secs	48.67 secs	48.93 secs
2002 Telstra Swimming Grand Prix	51.12 secs	48.58 secs	51.32 secs	48.11 secs	48.45 secs	51.90 secs	51.62 secs
2001 Telstra FINA World Cup	48.43 secs	48.43 secs	51.74 secs	48.81 secs	50.80 secs	DNC	48.90 secs
2001 Pan Pacs	47.81 secs	49.13 secs	53.73 secs	50.79 secs	50.30 secs	51.93 secs	49.46 secs
2001 Telstra Australian Championships	49.46 secs	49.53 secs	55.12 secs	49.05 secs	49.67 secs	51.69 secs	50.27 secs
2000 Telstra FINA World Cup	49.62 secs	47.98 secs	DNC	49.99 secs	48.98 secs	51.42 secs	48.68 secs
2000 Olympic Games	DNC	48.56 secs	DNC	DNC	DNC	DNC	DNC

Figure 2

The Green Thumb Gardens problem (adapted from Hjalmarson 2000*)

Green Thumb Gardens to Open Soon

Brisbane, Queensland.—When it rains, it pours. Throw some sunshine into the mix, and you get luscious, thick, and tall green lawns across town.

While James Sullivan loves a nice green lawn, he also wants to take care of it.

James is the owner of Indooroopilly's latest landscaping business, Green Thumb Gardens. Green Thumb will open on October 1 for the busy lawn and garden season.

James is no stranger to lawns. After keeping landscapes beautiful by working with a large Queensland landscaping company for 15 years, he has decided to branch out on his own. "I've enjoyed spending time outdoors all my life, and I love nature," said James. "There's nothing like the smell of flowers blooming in the spring and the smell of freshly-mowed lawns."

James is returning to his roots by returning to Indooroopilly. He is a graduate of the University of Queensland's landscape architecture department. In fact, James has been mowing lawns since he was big enough to start one, and he's hoping that his business will be similar to his customers' yards: continually growing.

"The best thing about my job is that I get paid to be outside working with Mother Nature and taking care of people's yards."

Green Thumb has already signed agreements with local businesses to help them present the best image to their customers.

"The landscaping outside is often a customer's first impression of a business, so it's very important to have a professional, attractive, well-kept area before the client walks through the door," said James.

James's small staff and green and white trucks are ready for the spring season and will be busy well into the summer and autumn months with regular lawn maintenance and leaf removal.

James is hoping for great spring weather that causes the grass to grow at a rapid rate. "Most lawns are mowed once a week or once every two weeks depending on the weather and the type of grass," said James. "We'll take care of any size lawn and cut it how the homeowner normally would."

For more information or a free landscaping estimate, contact Green Thumb Gardens at their office at Harts Road, Indooroopilly.

Readiness Questions

1. Who is the owner of Green Thumb Gardens?
2. What kind of services does Green Thumb Gardens provide to customers?
3. How often do lawns need to be mowed?
4. When are the busiest months for Green Thumb Gardens? Why?

Information

At Green Thumb Gardens, James Sullivan will provide lawn-mowing service for his customers. Another local landscaping service has closed, so James has offered to hire four of their former employees in addition to taking on some of their former clients. He has received information from the other landscaping business about the employee schedules during June, July, and August of last year. The employees were responsible for mowing lawns and selling other yard products like fertilizer, weed killer, and bug spray.

The other business recorded how many hours each employee worked each month, the number of lawns each employee mowed, and how much money they made selling other products. The lawns are divided into big, medium, and small jobs. Big jobs may have larger lawns or more work than medium or small jobs. Some lawns may be small but may have many obstacles for the mower to get around or different kinds of edging or trimming to be done, which determines the "size" of the job. The other company also recorded the number of kilometers one of the company trucks was driven to client sites during each month.

Problem

James needs to decide which four employees he wants to hire from the old business. Using information in **tables 3**, **4**, **5**, and **6**, help him decide which four people he should hire for this summer. Write him a letter explaining the method you used to make your decision so he can use your method for hiring new employees each summer.

---

*Hjalmarson, M. *Green Thumb Gardens to Open Soon.* Unpublished modeling problem. Purdue University: 2000.

Tables 3, 4, 5, and 6
Information Provided to Green Thumb Gardens

Table 3. Hours Worked

Employee	June	July	August
Jonathan	80	80	80
Cynthia	75	65	70
Jack	66	64	63
Kayla	45	50	55
Tim	67	70	79
Aaron	65	70	78
Matthew	80	79	78
Julie	40	42	46
Travis	80	75	80
Kim	78	76	79

Table 4. Kilometers Driven

Employee	June	July	August
Jonathan	198	200	201
Cynthia	199	201	198
Jack	197	199	198
Kayla	201	203	199
Tim	200	199	200
Aaron	198	196	195
Matthew	200	204	202
Julie	196	198	197
Travis	201	203	204
Kim	195	199	198

Table 5. Total Number of Lawns Mowed

	June			July			August		
Employee	Big	Medium	Small	Big	Medium	Small	Big	Medium	Small
Jonathan	15	12	30	16	14	34	16	15	35
Cynthia	18	10	35	19	12	35	14	16	36
Jack	14	16	22	15	16	22	13	16	22
Kayla	15	13	15	14	13	17	15	12	18
Tim	20	12	14	22	14	16	20	13	25
Aaron	16	27	32	14	18	33	15	19	42
Matthew	32	12	9	30	11	10	30	10	13
Julie	9	22	12	12	15	16	8	10	12
Travis	13	34	32	13	33	31	15	35	12
Kim	12	11	25	11	10	26	13	14	30

Table 6. Average Income per Week from Products Sold

Employee	June	July	August
Jonathan	$150	$175	$170
Cynthia	$ 75	$ 80	$ 80
Jack	$125	$150	$150
Kayla	$ 80	$ 72	$ 65
Tim	$135	$130	$125
Aaron	$127	$153	$165
Matthew	$110	$115	$120
Julie	$ 55	$ 54	$ 60
Travis	$300	$255	$275
Kim	$200	$250	$265

(Continued from p. 41)
by providing children with background information on the complex system being explored (team sports and business organization). This is followed by some "readiness questions" to assist the children in interpreting the given information and data. The children complete these questions in the first session, either as a teacher-supported whole-class activity or as a small-group activity (3–4 children per group).

The problem scenario is presented next. For the Olympic activity (see **fig. 1**), the children assist the Olympic Swimming Committee in selecting teams for the 2004 Olympic Games. For the Green Thumb activity (see **fig. 2**), they act as advisors on hiring workers for a new gardening business. In both cases, the children are asked to construct a model to meet the needs of the clients in dealing with their particular complex systems. The children work the problem in small groups of 3–4 in the next two sessions, with the final session reserved for group reports to the class.

During this small-group work, the teacher observes and facilitates—but does not direct—the children's mathematical learning. That is, when necessary, the teacher provides scaffolding, such as questioning children for explanation and justification; challenging children; querying an inappropriate action; and providing overall encouragement, support, and motivation. At the same time, the teacher avoids providing comments or questions that specifically steer the children toward a particular (teacher-determined) solution. In the final session, when the children present group reports on their models, the teacher performs several important functions: encouraging constructive peer questioning and feedback, helping children identify similarities and differences between the various models they have created, leading a discussion on the key mathematical ideas generated, and advancing children's mathematical thinking as they explore and extend these new ideas.

Designing Modeling Activities

How can we design activities that involve modeling with complex systems? Creating such activities is a challenge in itself; nevertheless, it is rewarding for both teachers and students. A number of important attributes must be taken into consideration when designing these activities. Richard Lesh and his colleagues (e.g., see Lesh and Doerr 2003) refer to these attributes as principles for designing productive activities. Drawing upon these principles, I highlight the following attributes as critical to task design:

- The complex system should be meaningful.
- The activity should elicit models that comprise important mathematical ideas and relationships.

Figure 3

Summary of learning cycles in one class for the Green Thumb problem

Cycle 1: Focusing on Problem Parts
- Scanned the tables of data to find employees who scored highly in one or more of the data components (hours worked, number of lawns mowed, etc.).
- Displayed limited consideration of whether some problem parts (e.g., kilometers driven) might have less impact on employee choice than others.
- Selected employees "with different specialties" (one group only).
- Overall, displayed an unsystematic approach, with limited mathematical thinking→ frequent arguments→ need to mathematize employee selection.

Cycle 2: Using Mathematization Processes
- Applied two main mathematical operations to aggregate the data for each employee:
  (a) simply totaled the amounts in two or more data components (hours worked, lawns mowed [treating "big," "medium," and "small" separately], kilometers driven, etc.)
  (b) found the average for each component ("How about we work out their average .... It's kind of difficult working out how much they worked each month. Sometimes they worked less and sometimes more.")
- Some groups used totals or average scores to select their employees, either focusing on all data components or selected components.

Cycle 3: Identifying Trends and Relationships
- Some groups transitioned to a more sophisticated level as they looked for trends and relationships to help them choose their employees. One group explored trends within data components: "Kim is always gaining ... 200, 250, 256 [in the money category]." This led the group to compare trends across components: "So Travis should be our first guy. He may have done 5 hours less than Jonathan, but he did more jobs."

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- Children should explicitly document their thinking in various ways.
- Criteria should be provided for children's self-assessment.
- The models children construct should be shareable and reusable.

Meaningful complex systems

For children to relate to and make sense of the complex system being presented, the system should be one that reflects a real-life situation and builds on children's existing knowledge and experiences. The modeling activities I developed with classroom teachers have been designed around their curriculum themes, such as the Commonwealth Games, the Olympics, early gold discoveries, classroom gardening, major weather events, river ecology, and early colonization.

Integrate the modeling activities within the existing classroom program; do not treat them as "add-ons" in an already-crowded curriculum. The activities enrich the problem-solving component of the mathematics curriculum and help children link their learning meaningfully across disciplines. For example, one modeling activity implemented in a fifth-grade classroom explored the water quality of a large Brisbane creek and provided the basis for children's actual exploration of their local school creek, enriching the children's scientific studies of their local environment.

Important mathematical ideas and relationships

A modeling activity should require children to develop an explicit mathematical construction, description, explanation, or prediction. The models children create should be mathematically significant by focusing on underlying structural characteristics (key ideas and their relationships) of the complex system being addressed. As one child reminded her group when they were making subjective decisions in working the Green Thumbs activity, "So, just on that information, you can't say, 'OK, we want him.' This is pulling people out of the hat again … [but] we need some structurable thing."

In completing the modeling activities, children independently elicit a range of powerful ideas and processes that rarely appear in existing elementary mathematics curricula. These include making comparisons involving trends across data; ranking and aggregating different forms of data; using informal understanding of proportional reasoning and rate; determining means; operationalizing data (e.g., formulating an operational definition of what it means to be a "productive worker"); and creating and working with weighted scores.

Children usually show cycles of mathematical development in creating their models. In working the Green Thumbs problem, one class of children displayed three main cycles, summarized in **figure 3**. In the first cycle, they focused unsystematically on the various parts of the problem (hours worked, number of lawns mowed, etc.) and displayed limited mathematical thinking. Their ad-hoc approach led to arguments about how they were selecting the employees. As a result, the children advanced to the next cycle, which involved the use of mathematization processes. Some children moved into the third cycle when they identified trends and relationships in their data.

Because children explicitly document their mathematical understandings as they progress through these cycles, they provide us with valuable insights into their mathematical developments.

Documenting group thinking

Modeling activities should encourage children to externalize their thinking and reasoning as much as possible and in a variety of ways. The activity should promote representations such as lists, tables, graphs, diagrams, and drawings. Children ought to construct models that involve more than a brief answer. They should include descriptions and explanations of the steps they took to construct their models and an indication of how their models could be applied to related complex systems. For example, in working the Olympics activity, James's group created **table 7**. The children's model comprised the following components:

- For the Olympic Games activity "Let the Selections Begin" (see **fig. 1**), examine **table 2**, which shows swimming times for the top seven swimmers in the last ten men's 100 meter freestyle events. Examine one row (or competition event) at a time and compare the swimmers' times in each column.
- Award a score of 2 to the swimmer with the lowest time and a score of 1 to the swimmer with the second lowest time in that event.

- Add the score you gave each swimmer and record them in a table.
- Consider the personal best times (PBs) of each swimmer and award 2 points to the lowest PB and 1 point to the second lowest.
- Aggregate each swimmer's scores to find the total ("total rank").

The two swimmers with the highest totals were chosen for the Olympic team. This particular model was only one of many constructed by the children; they also—

- tallied the number of winning races for each swimmer in each race and compared the totals;
- aggregated the two or three lowest times of each swimmer and compared the totals;
- assigned scores (and weighted scores) to the two lowest times of each swimmer and aggregated the scores (as in James's group's model but without assigning weighted scores to the two lowest PBs);
- found the average of each swimmer's times and chose the swimmers with the lowest averages; and
- examined each swimmer's PB only, together with the extent of variance between the PB and the swimmer's different race times.

Some children's attention to one part of the problem (e.g., focusing on PBs) at the exclusion of other parts led to interesting debates as the children shared their models with their peers. One child asked a group, "How come you just compared their personal best? They don't do that all the time." Other children commented that the swimmers' most recent times should be considered "because they could have been slow when they first started, and they could have got stronger … and now they're going faster and faster." In response, other children stated that the swimmers also "could get slower and slower," or "they could just stay on their PB" or "they could go faster and then slower and then fast."

Self-assessment

Modeling activities should provide children with sufficient criteria for determining whether their final model is an effective one: Does the model meet the client's needs in dealing with the given complex system as well as with related systems? Such criteria also enable children to progressively assess and revise their creations as they work the problem.

The Green Thumbs activity provides information on the duties of the employees, the nature of the lawns to be mowed, the employee schedules over the busy summer period, and the value of the products sold. After repeatedly assessing whether their model would determine the most productive workers, some children ended up discarding one or two components in their final models (e.g., discarding the kilometers driven). Others viewed the different criteria as a way of choosing employees "with different specialties." Either way, the activities should enable children to judge for themselves when their model is acceptable for solving the problem at hand. In this way, children do not resort to seeking the teacher's approval on whether their solution is "good enough."

Shareable, reusable models

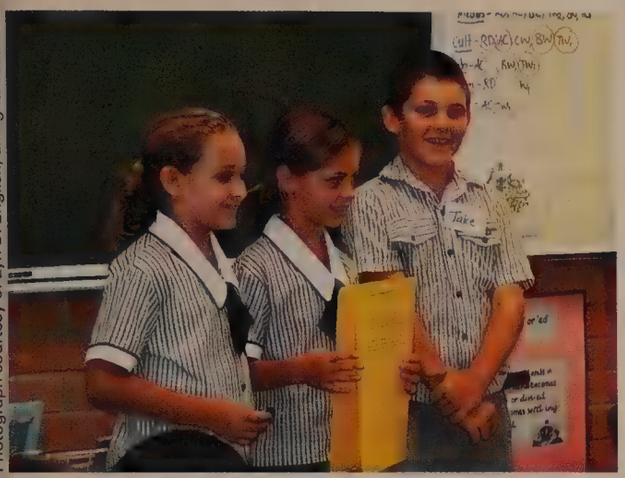
Other important aspects of the activities are that children's models be shared with others and also be applicable to related complex systems. On completing the problem, children use session 4 to present group reports to their class peers, who subsequently ask questions about the models and provide constructive feedback. Each group member thus has the shared responsibility to ensure that their end-products are informative; "user-friendly"; and clearly, persuasively convey the intended ideas, relationships, and processes.

To remind children that their final models must be reusable, the activities direct them to construct a model that can be applied to related situations (e.g., selecting other teams for the Olympics or other sporting events; establishing a business that employs

Table 7

The Olympics Problem Created by James's Group

	Ashley Callus	Michael Klim	Eamon Sullivan	Ian Thorpe	Todd Pearson	Grant Hackett	Adam Pine
Rank	10	10	0	11	4	1	1
Personal Best	0	2	0	1	0	0	0
Total Rank	10	12	0	12	4	1	1



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part-time and full-time staff at an amusement park). In one modeling activity that I implemented in a sixth-grade classroom involving the construction of a consumer guide, the children coined the term *universal* to refer to models that could be generalized. They used this notion as one yardstick in providing constructive feedback to their peers.

Assessing Learning

How might we assess children's learning during these modeling activities? Such assessment depends, in part, on a teacher's existing approaches to determining children's progress. However, the very nature of these activities provides rich insights into children's thinking as teachers observe each group's responses. The Web site www.edci.purdue.edu/casestudiesforkids/index.htm has resources for assessing children's learning when they are working with modeling activities. One of the resources is an observation form to record the mathematical concepts and skills the children developed, the nature of the group interactions, how the children organized and operated on the given data, the ways in which they represented their models, how they justified their assumptions and findings, and the nature of the critical feedback they provided on the group reports (e.g., did the children address their peers' mathematical ideas, rather than simply stating that they "liked" a group's presentation).

Concluding Points

We need to revisit our elementary mathematics curriculum in light of the rapidly changing demands of our society. The pervasive nature of complex systems in today's society means we can no longer ignore their impact on the lives of our children. This article has addressed one approach to introducing children of all achievement levels to introductory complex systems, namely, through mathematical modeling.

Five main attributes of effective modeling activities have been highlighted here. These include the exploration of a meaningful complex system, the elicitation of models comprising important mathematical ideas, the explicit documentation of children's thinking, the provision of criteria for self-assessment, and the construction of shareable and reusable models. With their multifaceted and multidisciplinary nature, these activities also serve as valuable learning experiences that link several disciplines in the school curriculum.

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The "Case Studies for Kids" Web site is maintained by Betsy Berry in the Mathematical Sciences Department of Indiana University—Purdue University, Fort Wayne (berrys@ipfw.com). ▲

MODELING STUDENTS' MATHEMATICS USING STEFFE'S FRACTION SCHEMES

By Anderson H. Norton and Andrea V. McCloskey

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Each year, more teachers learn about the successful intervention program known as Math Recovery (USMRC 2008; Wright 2003). The program uses Steffe's whole-number schemes to model, understand, and support children's development of whole-number reasoning. Readers are probably less familiar with Steffe's fraction schemes, which have proven similarly useful in supporting children's development of fractional reasoning. The purpose of this article is to introduce some of these schemes. We provide examples of student work accompanied by discussions of how fraction schemes can be used as tools for insight into student reasoning. We hope that teachers will find the schemes useful in understanding their students as mathematicians.

In their Fractions project, Steffe (2002) and Olive (1999) conducted teaching experiments with six pairs of students over a three-year period—from their third-grade year through their fifth-grade year. Steffe and Olive hypothesized that the students' fractional schemes could be constructed through reorganizations of their whole-number schemes, which Steffe and colleagues identified in previous research (e.g., Steffe, Cobb, and Glasersfeld 1988). As a former student of Steffe, Bob Wright built on this research of students' whole-number schemes by engaging teachers in teaching experiments with students in the early grades (2000; 2003). Wright's increasingly popular Math Recovery program uses teaching experiments as interventions for struggling students, during which teachers challenge students at the "cutting edge" of their knowledge and strategies (2000, p. 142).

We have extended the work of Steffe, Olive, and Wright by engaging elementary school teachers in conducting their own teaching experiments with pairs of students. The teachers posed fraction tasks to the students and observed their problem-solving strategies in an effort to understand how students think. As part of the project, we administered pretests and posttests to all of the students in the participating teachers' classrooms. The test tasks were similar to tasks the teachers posed in the teaching experiment. We use student work from the pretests to provide examples of responses that indicate the fraction schemes described here.

Operations and Schemes

Schemes are teacher constructs used to model students' cognitive structures. They consist of three components: a template for recognizing situations

in which the scheme applies, mental actions (operations) that are activated when such a situation is recognized, and expected results of operating (see **fig. 1**). Operations constitute the key component of a scheme because schemes are, in essence, ways of operating. For example, a student might be asked to add 8 plus 5, activating the mental action of counting on from 8, which might be represented by the student saying, "8; 9 is one; 10 is two; 11 is three; 12 is four; 13 is five; *13*." Teachers might attribute to this student a scheme for counting on. Situations perceived as involving addition, or joining objects, fit the recognition template and trigger the scheme. This activates the operations of iterating a unit of one and double counting, represented in the student's verbalization, "9 is one, 10 is two … " Finally, the student's actions fit the expected result of the scheme by reaching the addend, "five." A student might make a mistake in her counting sequence, reaching, for instance, "14 is five," and her actions could still fit the expected result of reaching "five."

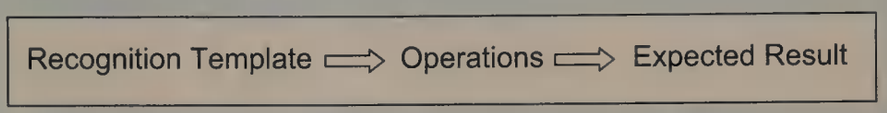
Schemes differ from strategies in several ways:

- Schemes describe ways of operating that usually occur outside of the student's awareness.
- Schemes are activated at once, rather than in a procedural manner.
- Schemes are teacher constructs; teachers attribute schemes to students in order to explain students' actions (including verbalizations).

A scheme fits a teacher's observations of a student in the same way that a scientific theory fits observations of natural phenomena. The teacher cannot say that the scheme actually exists within the student's mind any more than Kepler could say that his laws of planetary motion existed within the planets. The teacher attributes the scheme to the student only because it is useful in explaining and predicting the student's actions.

Figure 1

Components of an operational scheme



As mentioned before, operations are mental actions, abstracted from experience to become available for use in various situations. Rotation is an operation that you might have abstracted from experiences of turning your head, spinning your body, or twisting a screwdriver. People use such an operation whenever they recognize a common shape that is turned at an uncommon angle, as often occurs when studying geometry. (Young students often fail to recognize a square when it is not oriented as they expect; such students are not using a rotation operation.) Likewise, we can identify several key operations that students use to understand fractions (Olive 1999; Steffe 2002). We outline these in the next section.

Fraction operations

Here we describe several important fraction operations, which we summarize in **Table 1**:

- *Unitizing* produces a unit, or whole, for a student. Anything can be taken as a unit by establishing it as a separate entity, such as extracting foreground from background. Students may also unite a collection of objects into a whole, producing a *composite unit*.
- *Partitioning* a continuous unit (e.g., a candy bar), a continuous composite unit (e.g., three-fifths of

Table 1

Fractional Operations

Operation	Description	Example of using the operation
Unitizing	Treating an object or collection of objects as a unit, or a whole	Treating two trapezoids as one whole (as with pattern blocks)
Partitioning	Separating the unit, or the whole, into equal parts	Equally sharing a pizza among four people
Disembedding	Imaginatively pulling out a fraction from the whole, while keeping the whole intact	After the pizza has been sliced in fourths, imagining what three-fourths of the pizza would look like
Iterating	Repeating a part to produce identical copies of it	Using a one-fifth piece to identify a three-fifths piece (as with fraction rods)

Fractional Schemes

Scheme	Operations	Example task
Simultaneous partitioning scheme	*Unitizing* the whole, *partitioning* the continuous whole using a composite unit as a template	Share this candy bar equally among you and two friends (also see **fig. 2**).
Part-whole scheme	*Unitizing*, *partitioning*, and then *disembedding* a part from the partitioned whole	Show me two-thirds of the candy bar (also see **fig. 3**).
Equi-partitioning scheme	*Unitizing*, *partitioning*, and then mentally *iterating* any part to determine its identity with the other parts	If you share this candy bar equally among you and two friends, draw what your piece would look like.
Partitive unit fractional scheme	Given an unpartitioned whole and a unit fractional piece of it, *iterating* the fractional piece to produce a continuous *partitioned* whole and to determine the size of the unit fraction relative to the whole	If I give you this much [*show a one-third piece and unpartitioned whole*], what fraction of the candy bar would you have? (Also see **fig. 4a** and **b**.)
Partitive fractional scheme	Given an unpartitioned whole and a proper fractional piece of it, *partitioning* the piece to produce unit fractional pieces, *iterating* the unit fractional pieces to reproduce the proper fraction and the whole, *coordinating* unit fractions within the proper fraction and the whole (units coordinating at two levels) to determine the size of the proper fraction relative to the whole	If I give you this much [*show an unpartitioned two-thirds piece and unpartitioned whole*], what fraction of the candy bar would you have? (Also see **fig. 5**.)

a candy bar), or a discrete composite unit (e.g., a collection of twelve marbles) produces equal parts by marking separations within the unit.

STUDENTS' WAYS OF OPERATING ARE DEEPLY CONNECTED TO THEIR MEANING AND SENSE MAKING

One aspect of partitions that can confuse students is the idea that partitioning a unit into n parts requires only $n - 1$ marks, leading some students to claim they have produced fifths, for instance, instead of sixths. We have also observed students counting partitions, including the marks at either end of a partitioned whole, instead of pieces, which leads them to conclude they have produced sevenths, say, instead of sixths.

- *Disembedding* occurs when a student imagines pulling out copies of some number of parts within the partitioned whole, while leaving the whole intact. So, a student might produce three-fourths by partitioning the whole into four parts and disembedding three of those parts. "Disembedding is the fundamental mental operation on which part-whole comparisons are based" (Steffe and Olive 1996, p. 118). We will discuss part-whole schemes and other critical fraction schemes in the next section.
- *Iterating* involves repeating a part to produce identical copies of the original. These copies might be connected to form a new continuous fraction. For example, a student might imagine iterating one-fifth of a candy bar three times to produce three-fifths of that candy bar. In other cases, a student might iterate a discrete composite unit to form a new discrete collection, constituting some fraction of the original collection. In any case, students using iteration understand that iteration of the original part produces parts identical to the original in that any one of the parts can be used to substitute for any other.

Fraction schemes

In this section, we describe several of Steffe's fraction schemes, along with analyzed samples of student work. The samples also serve as examples of tasks teachers might pose to students to test for corresponding schemes and to better understand students' reasoning with fractions. We intend for the analysis to serve as a guide for interpreting student responses to such tasks. We summarize the schemes, along with similar tasks, in **table 2**.

Partitioning schemes

Partitioning is the key operation of a *simultaneous partitioning scheme*, in which a student mentally projects a composite unit into a continuous whole, using the composite unit as a "partitioning template" (Steffe 2003, p. 239) with the goal of producing equal parts. For example, a student might produce five equal parts within a whole by imagining five equally spaced units within the whole. If a student further understands that the parts are identical, so that any one of them can be iterated enough times to reproduce the whole, the student is said to have an *equi-partitioning scheme*; understanding that the resulting pieces are identical to one another is implicit, but there is no explicit goal of reestablishing the whole by iterating any of these pieces. Consider the response illustrated in **figure 2**.

The work of Student 1 indicates he used a partitioning operation to successfully resolve his task. He apparently used this operation in a manner consistent with either a simultaneous partitioning scheme (projecting a composite unit of six parts into the continuous whole) or an equi-partitioning scheme. It appears the student first made some marks on the top of the bar but was unsatisfied with them. The student then apparently erased the original marks and adjusted them accordingly. We might infer that this student iterated his initial guess to test whether it would equally partition the whole into six parts. Such a use of iterating indicates an equi-partitioning scheme, but we warn against trying to directly teach such ways of operating. Direct instruction might lead to students' constructions of new procedures or strategies but not to meaningful ways of operating—in this case, iterating a part to check whether it can be used to create six equal parts. Students must recognize the need to iterate, and teachers can facilitate such development by, instead, asking students questions: "How do you know that is 1/6? If six people were sharing this (candy) bar, would that be a fair share? Show me that it is fair."

Partitioning schemes provide students with a basis for developing their first fraction schemes, such as a part-whole scheme. In fact, the next two schemes we discuss can be considered reorganizations of partitioning schemes.

Part-whole scheme

Students attributed with a part-whole scheme conceive of partitioned fractions as so many pieces in the partitioned fraction out of so many pieces in the partitioned whole. This scheme relies on operations of identifying (unitizing) a whole, partitioning the whole into equal pieces, and disembedding some number of pieces from the partitioned whole. Although the pieces in the fraction and the whole are all the same size, they may not be identical, in the sense that students may not be able to substitute one piece for another. This is because part-whole schemes are based on reorganizing a partitioning scheme, such as a simultaneous partitioning scheme, but are not based on an equi-partitioning scheme. Consider the task illustrated in **figure 3**.

Figure 2

Student 1's partitioning task

8. Shade in 1/6 of the bar.

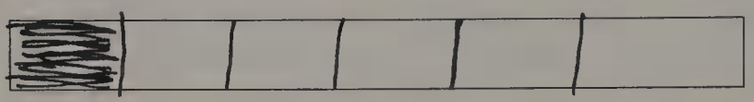
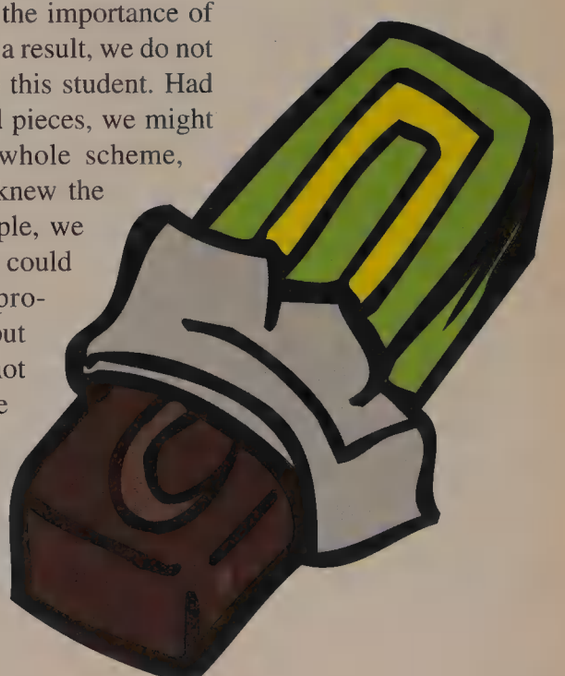


Figure 3

Student 2's part-whole task

7. Shade in 3/5 of the bar.

Examining the work of Student 2, we infer that the student can relate fractional language to the task: He knew that the denominator indicated the number of pieces in the whole, and the numerator indicated the number of those pieces in the fraction. However, the pieces he produced are not equal, and so we wonder if he understands the importance of producing equal-sized pieces. As a result, we do not attribute a part-whole scheme to this student. Had the student produced equal-sized pieces, we might infer that he had used a part-whole scheme, but we might not infer that he knew the pieces were identical. For example, we have worked with students who could use part-whole reasoning to produce the appropriate picture; but many of these students could not identify one-fifth of the whole within that picture, implying that the students could produce equal partitions (perhaps using a simultaneous partitioning scheme), but they did not have an equi-partitioning scheme.



Partitive unit fractional scheme

Students limited to a part-whole scheme rely on a partitioned whole and cannot identify the size of a given fractional part by iterating it within an unpartitioned whole. The transition to such treatment of the composite whole yields a *partitive unit fractional scheme* (Steffe 2003, p. 242). Most of the students we have worked with in grades 5 and 6 have developed part-whole schemes but have not developed partitive unit fractional schemes. Data from the 2003 National Assessment of Educational Progress supports a generalization of this conclusion (Kastberg and Norton 2007). We attribute this, in part, to an almost singular focus on part-whole tasks in fractions curricula.

The partitive unit fractional scheme is based on a reorganization of an equi-partitioning scheme (partitioning that produces identical parts). In utilizing a partitive unit fractional scheme, a student understands that any unit fractional part can be iterated so many times to reproduce the whole and that this number of iterations determines the *size* of the fraction relative to the whole. Furthermore, the partitive unit fractional scheme "establishes a one-to-many relation between the part and the partitioned whole" and involves "explicit use of fractional language to refer to that relation" (Steffe 2002, p. 292). In addition to introducing fractional language, reorganizing an equi-partitioning scheme into a partitive unit fractional scheme introduces a new goal. As a partitioning scheme, the primary goal for an equi-partitioning scheme was to produce equal (and identical) parts; as a fractions scheme, the primary goal for a partitive unit fractional scheme is to determine fractional size relative to the whole, through iteration.

Both a part-whole scheme and a partitive unit fractional scheme generate fractional language, but the difference between the powers of the schemes is evident in resolving the task illustrated in **figures 4a** and **4b**. Students with only a part-whole scheme cannot determine the fraction, because the whole is unpartitioned. Students with a partitive unit fractional scheme can determine the unit fraction by iterating it to reproduce the whole and naming the fraction as the reciprocal of the number of iterations. On the other hand, a partitive unit fractional scheme cannot be used to determine the fractional size of a nonunit fraction, because the iterations will not reproduce the whole (unless, of course, the fraction in question simplifies to a unit fraction; e.g., two-sixths).

The work of Student 3 (see **fig. 4a**) indicates that he used an iterating operation in a way that fits our model of the partitive unit fractional scheme. This student drew a line to indicate where the smaller bar coincided with the whole bar, and then produced identical pieces until he reproduced the whole. The student further understood how to relate this iterating process to fractional language: He knew that because he used the piece four times to reproduce the whole, the piece was one-fourth of that whole. We can reasonably attribute a partitive unit fractional scheme to Student 3. We may wonder if his understanding generalizes to nonunit fractions.

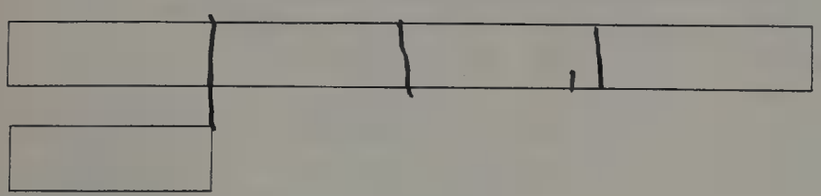
Student 4 left fewer residual markings for us to analyze (see **fig. 4b**). It seems her first step was similar to that of Student 3: She marked where the small piece coincided with the whole piece. However, her work does not clearly indicate how she went on to use this mark to arrive at the correct answer. We are left wondering whether the student used an iterating operation. One alternate hypothesis a teacher may propose is that Student 4 repeatedly partitioned the whole into various pieces (halves, thirds, fourths, etc.) until she arrived at a partitioning that coincided with the mark she made.

Figure 4

Partitive unit fractional tasks

(a) Student 3's work fits our model.

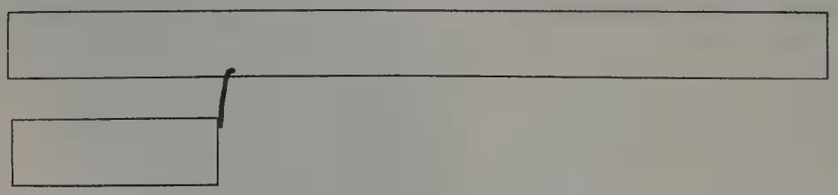
7. If the longer bar is a whole bar, what fraction is the shorter bar?



Answer: $\frac{1}{4}$

(b) Observing Student 4's actions would give us more insight than a paper "snapshot" does.

7. If the longer bar is a whole bar, what fraction is the shorter bar?



Answer: $\frac{1}{4}$

This conjecture may seem far-fetched, because few teachers would solve the problem using this tedious approach, but we have witnessed students using this more basic partitioning strategy to do just that.

Student 4 did not draw the marks we would expect to support the hypothesis described above. It seems more reasonable that she performed some sort of iterating operation with a physical motion instead of her pencil, such as by moving her finger along the whole bar, repeating the size of that first piece, until she realized that four of those pieces would constitute the whole. Because this conjecture seems more likely, given the single mark she left, we conclude that she used a partitive unit fractional scheme. This is an example in which her teacher would probably want to give this student a similar task and observe the student solving it. Seeing the student's actions would yield insight that we lack from this paper "snapshot."

Partitive fractional scheme

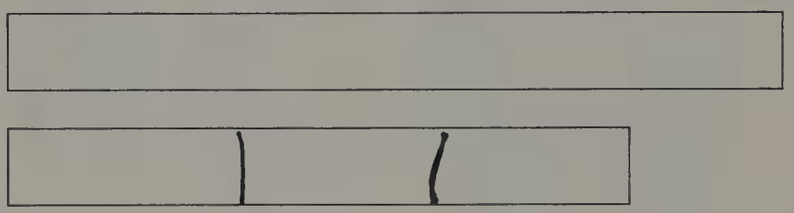
The *partitive fractional scheme* is a generalization of the partitive unit fractional scheme. Students can use the more general scheme to conceive of a proper fraction, such as three-fifths, as three of one-fifth of the whole. This involves producing composite fractions from unit fractions through iteration, while maintaining the relation between the unit fraction and the whole. It also involves *units coordination* at two levels, because the student must coordinate three-fifths as three iterations of the fractional unit and the whole as five iterations of the fractional unit. In other words, three-fifths is a unit of three fractional units, and the whole is a unit of five fractional units. Consider the coordination indicated by the student response in **figure 5**.

If we were only interested in correctness, then we could quickly assess the response of Student 5 as incorrect and move on. However, we are more interested in how students reason. Her answer is not far from the correct one, and the few marks she made warrant further consideration. It is not clear how she decided that the shorter bar should be partitioned into three pieces, but we can see that these three pieces are reasonably equivalent in size. More interestingly, Student 5 seemed to know that three pieces in the shorter bar are actually fourths of the whole bar. She was reasonably successful in simultaneously dealing with thirds and fourths, concluding that the shorter piece is three-fourths of the whole bar. This indicates she was coordinating units of units. Although three-fourths is not correct, the work of Student 5 indicates she was operating in a way consistent with our understanding of the partitive fractional scheme.

Figure 5

Student 5's partitive fractional task

9. If the longer bar is a whole bar, what fraction is the shorter bar?



Answer: $\frac{3}{4}$

Using Schemes

We have introduced some of Steffe's fraction schemes in hopes that teachers will find them useful in assessing and understanding their students' ways of operating with fractions. But we have warned against trying to teach these schemes directly. Students' ways of operating are deeply connected to their meaning and sense making, and attempts to circumvent or replace those ways of operating can only yield disconnected knowledge. Rather, teachers should find ways to work within their students' operating methods while generating a need to develop more powerful ways of operating. For example, tasks such as the one illustrated in **figure 4** can be presented with manipulatives, such as Cuisenaire rods.

SCHEMES—WHICH USUALLY OCCUR OUTSIDE OF THE STUDENT'S AWARENESS—ARE TEACHER CONSTRUCTS, ATTRIBUTED TO STUDENTS TO EXPLAIN THEIR ACTIONS

Students with partitive unit fractional schemes should be able to estimate the size of the smaller rod relative to the longer (whole) rod. This ability to estimate might generalize to nonunit fractions and contribute to proportional reasoning (Nabors 2003). Students limited to a part-whole understanding of fractions might also engage in such tasks by using the smaller rod to *segment* the larger rod (i.e., place copies of the smaller rod side by side until they equal the length of the larger rod). With

the larger rod partitioned by the smaller rods, these students can use their part-whole schemes to determine the fraction. Segmenting can be a pathway to developing the iterative operation of an equipartitioning scheme and a partitive unit fractional scheme. The key is for students to have opportunities to carry out such experimental actions, generating new ways of operating from their existing ways of operating.

Thanks to the teachers, students, and administrators at Templeton Elementary School for their work with us. We especially appreciate the dedication of teachers like Valerie Gliessman, Myra Hogan, Regina Tippmann, and Marilyn Gingerich.

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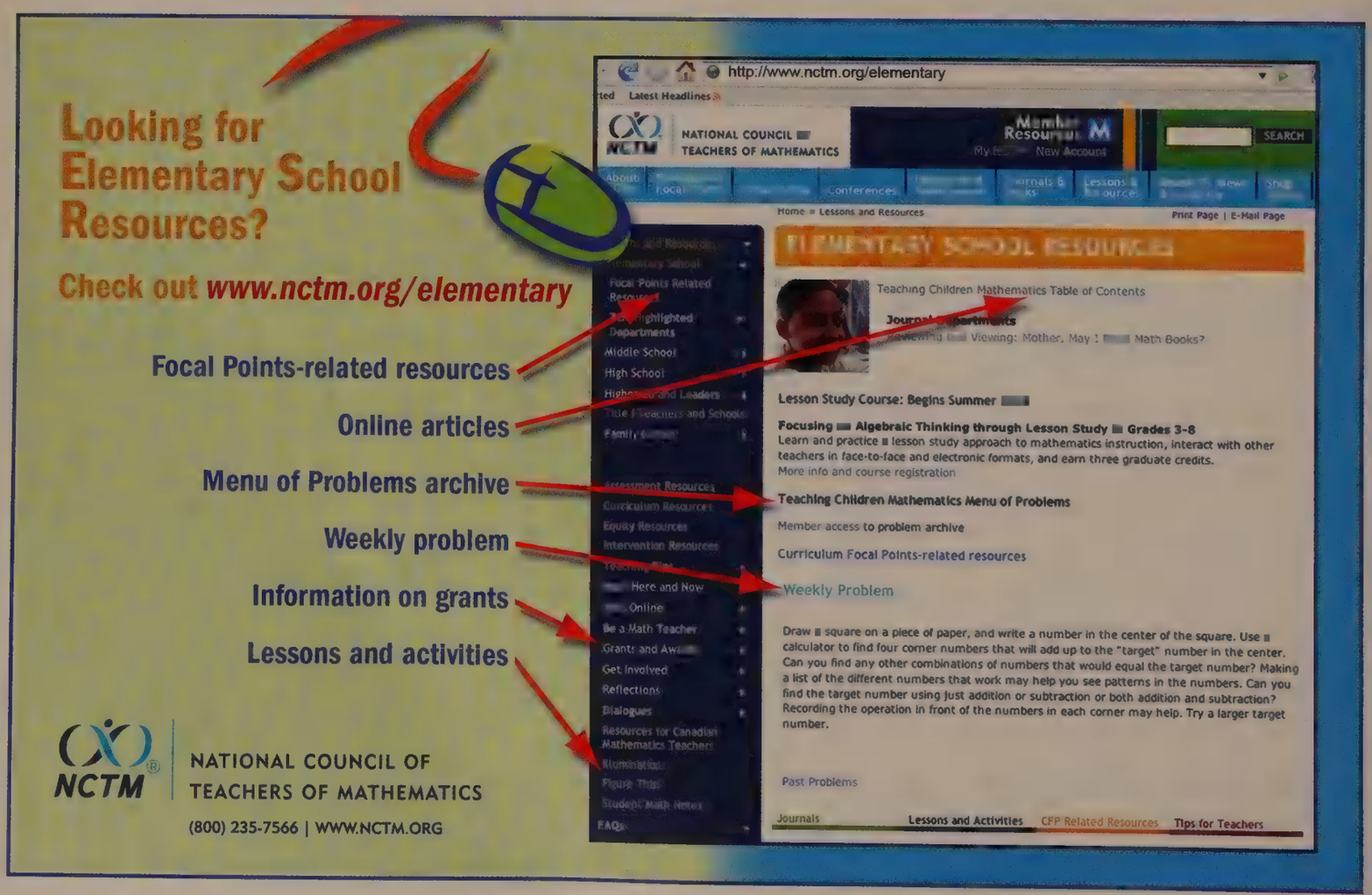
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Teaching Children Mathematics Table of Contents
Lesson Study Course: Begins Summer
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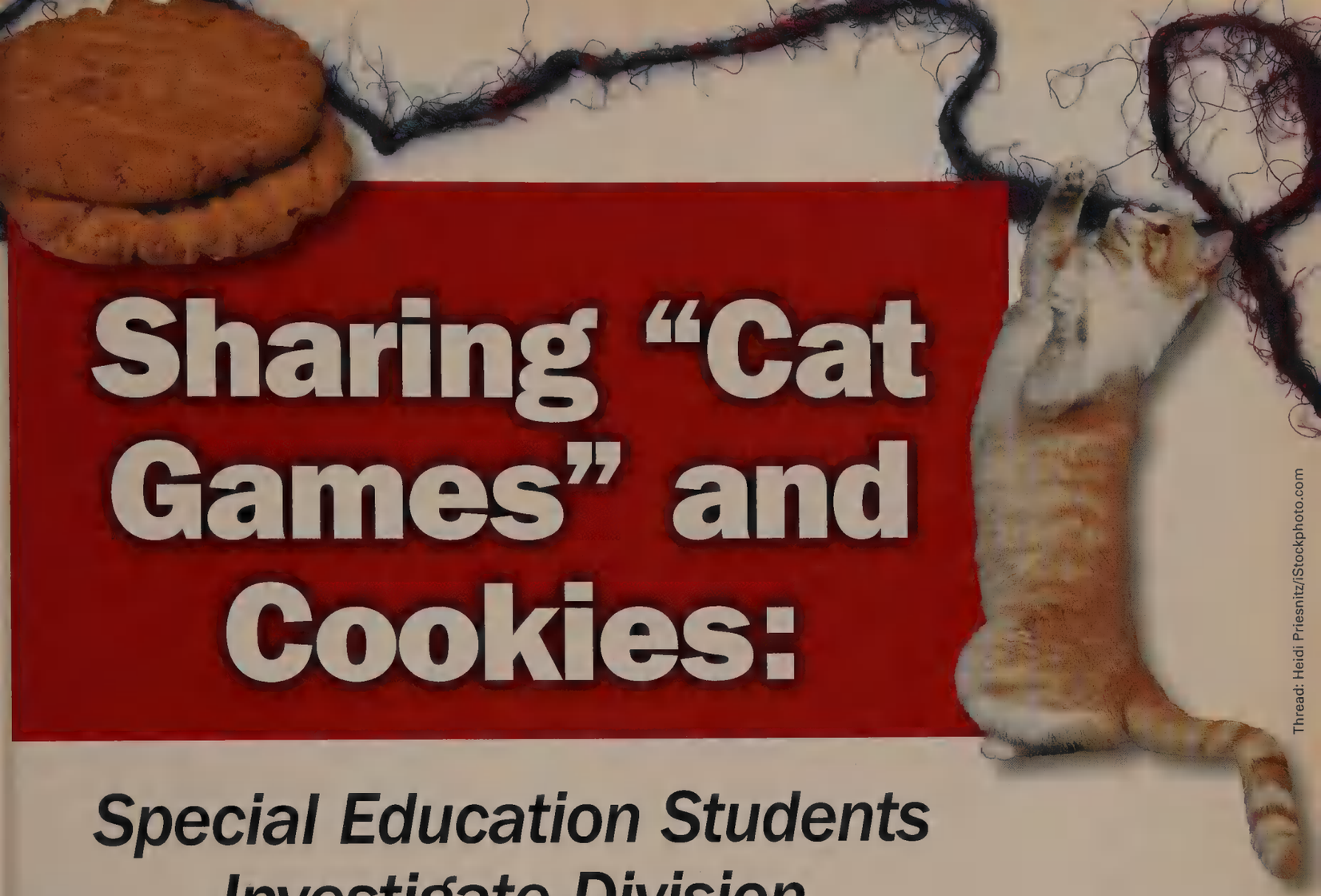
More info and course registration
Teaching Children Mathematics Menu of Problems
Member access to problem archive
Curriculum Focal Points-related resources
Weekly Problem
Draw a square on a piece of paper, and write a number in the center of the square. Use a calculator to find four corner numbers that will add up to the "target" number in the center. Can you find any other combinations of numbers that would equal the target number? Making a list of the different numbers that work may help you see patterns in the numbers. Can you find the target number using just addition or subtraction or both addition and subtraction? Recording the operation in front of the numbers in each corner may help. Try a larger target number.
Past Problems
Middle School
High School
Journals
Lessons and Activities
CFP Related Resources
Tips for Teachers



Sharing "Cat Games" and Cookies:

Special Education Students Investigate Division

Thread: Heidi Priesnitz/iStockphoto.com

Learning mathematics has traditionally been thought of as a sequential progression. Children learn to count to ten, then to twenty, and then to one hundred. They learn to add without regrouping and then with regrouping. We teach addition before multiplication and the two-times table before the six-times table. We usually teach division as a separate unit after multiplication. Organizing mathematics textbooks as a sequential progression of skills promotes this perspective on teaching mathematics. When we, a special education teacher of fourth-grade students and a mathematics teacher educator at a university, joined forces to teach a unit on division to a class of ten fourth graders, we purposefully adopted a very different perspective in planning the learning activities. We found it useful to think of students' knowledge as individual networks of concepts, ideas, and procedures that were linked together in distinctive ways. Many students in the class shared some knowledge, but each student also had unique interests, talents, knowledge, and—especially—an idiosyncratic organization of that knowledge.

Each student had experience with multiplication, but students' responses to tasks in individual interviews gave us more specific information about the particular components of each student's knowledge of multiplication and division. For example, Deuvorn knew many multiplication facts, but when he was asked to draw a picture to show three times six, he drew three animal faces and six human faces. Although he could quickly name correct products for many pairs of factors, this knowledge did not seem to be strongly connected to understanding that three times six means three groups of six. On the same problem, Andrea made an error in skip counting to arrive at twenty-one as the product of three and six. She drew a three-by-seven array to illustrate three times six and her result of twenty-one. Then, without prompting, she counted the columns in her

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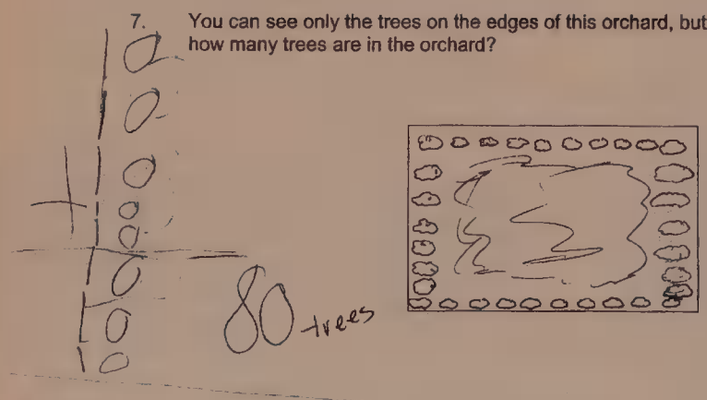
array, erased a column to create a three-by-six array, and changed her answer to eighteen.

When asked to state the number of squares in a four-by-eight rectangular array, six of the ten students counted each square of the array; one student multiplied four times eight; and one skip counted by fours to reach thirty-two. Two students did not count the squares accurately: Jason miscounted thirty-one; Antonio could not count past twenty-nine. A more challenging but related task had students determine the number of trees in a partially obscured array (see **fig. 1**). Six students either counted only the trees around the edges or else attempted to mark and count the obscured trees one by one. Antonio was one of only four students who used the eight-by-ten array structure to arrive at an answer of eighty trees. He counted the ten trees on the visible top row and the ten on the bottom row. He then counted down each side of the array and said, "There are eight rows of trees." Next, he generated a column of eight tens and added them. First, he added up the eight zeros; then he totaled the eight ones by counting them; and then he wrote his answer, "80 trees."

Rather than preparing individualized instructional activities that targeted each student's misconceptions or computational difficulties, we selected learning activities that would give students many opportunities to investigate, in a variety of contexts and problem situations, a few major ideas related to multiplication and division. We focused on division as sharing, division as the inverse of multiplication, and using rectangular arrays to represent factors and products. We selected activities that had several correct solutions or solution paths so that students could successfully develop their individual knowledge and skills. As students shared their individual perspectives and solutions with other members of the class, they had opportunities to learn from each other. Because many of the activities were open ended, eliciting a variety of possible correct responses, we were able to assess students' knowledge and skills during each day's lesson. In particular, we were better able to understand how students were making sense of the problem situations and how they used models, pictures, or written symbols to represent problems and solutions.

Figure 1

Antonio successfully found the number of trees in the partially obscured array.



Division as Sharing

Before the unit began, many of the students had difficulty using manipulatives to represent sharing situations. For a problem such as "Four people are going to share twenty-four cookies evenly. How many cookies will each person get?" many of the students laid out twenty-four counters to represent the cookies and four counters to represent the people. They had difficulty keeping track of which counters represented cookies and which represented people. When they finished distributing the counters, they were puzzled to have seven counters in each group instead of six. It seemed to us that these students were using the counters to represent the numbers in the problem but were having difficulty fully representing the problem's action of sharing to make equal-sized groups. This reminded us that a physical representation does not necessarily mean the same thing to a student that it might mean to adults. Representing a situation with a model sometimes presents unanticipated difficulties for students. When we suggested during the interview that students use pieces of paper to represent "plates" for each of the sharers, more of them successfully shared the twenty-four counters equally across the four "plates."

For one of the early investigational activities, we gave students opaque paper bags containing a number of objects. Working with a few students at a time, we asked each student to share the items in his or her bag evenly with the other members of the group. Each student succeeded in sharing the objects, stating the quantity in each share, and, when appropriate, stating the quantity in the remainder. Students dealt the objects one by one. Two weeks later when we repeated the activity using bags of larger numbers of objects, we observed that the students dealt the objects in groups of two, three,

or four. Some students were able to perform the division mentally and give the others their share in one turn. This indicated students' growing ability to partition a set of objects using composite units or to perform the operation of division mentally.

For the second day's lesson, we read *The Doorbell Rang* (Hutchins 1986) to the students and then showed them how to make an eight-section chart to represent each step in the sequence of events and the division problems (Wickett, Ohanian, and Burns 2002) that occurred during the story. **Figure 2** shows Annie's representation of each sharing event in the story and the division equations associated with each.

As a follow-up activity, we gave each student a box of colored candies and written instructions to share the candies at home with their families. Then they were to write a division equation to show how they shared the candies. We made overhead transparencies of the students' reports and discussed each student's solution with the class.

In a subsequent lesson, we asked students to write short stories or word problems about sharing situations. Many students set to work immediately, but others, like Jason, seemed to have difficulty getting started. When we suggested to Jason that he write a story about cats because he loves cats, he began to write and draw a story about cat games (see **fig. 3**). Deuvorn, who frequently wore a football jersey with the name and number of an NFL quarterback, wrote a story about football posters; and Amanda wrote a story about horses.

Figure 2

Part of Annie's chart of the division situations in *The Doorbell Rang*

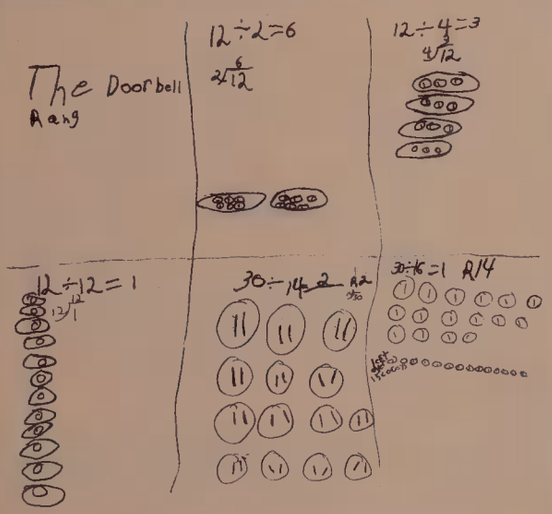
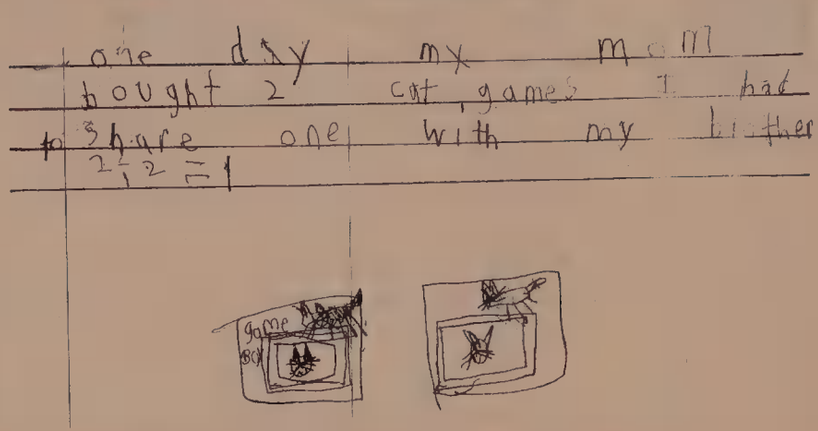


Figure 3

Jason's story about sharing two cat games

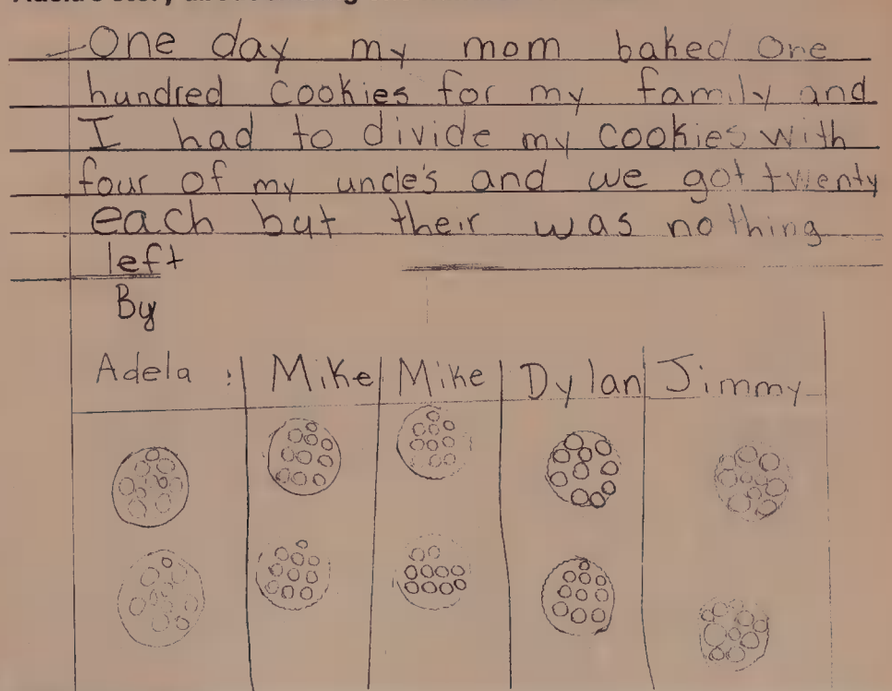


Many students' not only reflected their varied interests through the stories but also chose to work with interesting, challenging numbers. For example, Adela wrote, "One day my mom baked one hundred cookies for my family, and I had to divide my cookies with four of my uncles, and we got twenty each, but their [sic] was nothing left." **Figure 4** shows how Adela found the answer to her division story. Later she built on her success with one hundred by writing stories about two hundred or one hundred fifty objects.

Although many students selected numbers that challenged them, not everyone did so. For example,

Figure 4

Adela's story about sharing one hundred cookies


One day my mom baked one hundred cookies for my family and I had to divide my cookies with four of my uncle's and we got twenty each but their was nothing left
By
Adela
Mike
Mike
Dylan
Jimmy

Jason's story involved dividing two by two to equal one (see **fig. 3**). Therefore, we provided balance by sometimes asking students to write stories for given number sentences, such as $56 \div 8$ or $42 \div 5$, allowing them to write about situations that interest them while pushing them to use more challenging numbers.

When students went to the computer lab the next time, they typed their stories and created a list of problems for all students to solve. Adela's story prompted a class discussion of an important aspect of sharing: When one person is sharing with a number of people, how do we indicate how many shares will be created? We noticed that students took care to make this clear when they wrote subsequent stories. For example, Annie wrote, "I bought seventeen oranges and shared them with my classmates. There are seven people in my class. How many oranges will everyone get?"

Having students write stories and represent them with pictures as well as number sentences illustrated for us the connections that students were making between their story, a number sentence, and a pictorial representation. For example, Deuvorn often drew pictures that included numbers from his story but did not correspond exactly to the situation he described. When he wrote a story about distributing fourteen footballs among seven people, he drew two boxes with seven footballs in each box instead of showing seven groups of two footballs each.

The assortment of activities that students discussed and wrote stories about enabled them to represent division in a variety of ways, to write about mathematical situations that were interesting to them, and, many times, to work with numbers that were challenging to them.

Representing Factors and Products with Rectangular Arrays

We introduced the idea of using rectangular arrays to represent factors and products with the "Candy Box" activity from *About Teaching Mathematics* (Burns 2000). We had prepared bags containing either twelve tiles or twenty-four tiles, but all the students wanted to work with the numeral 24. Students worked in pairs to discover that twenty-four tiles could be arranged into rectangles with dimensions of three by eight tiles, four by six tiles, and two by twelve tiles. None of them made a one-by-twenty-four-tile

Figure 5

Antonio aligned the array card with his multiplication chart to find the product of 5 × 8.

Multiplication Grid

X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	27	30	33	36
4	0	4	8	12	16	20	24	28	32	36	40	44	48
5	0	5	10	15	20	25	30	35	40	45	50	55	60
6	0			5	X	8				54	60	66	72
7	0									63	70	77	84
8	0									72	80	88	96
9	0									81	90	99	108
10	0									90	100	110	120
11	0	11	22	33	44	55	66	77	88	99	110	121	132
12	0	12	24	36	48	60	72	84	96	108	120	132	144

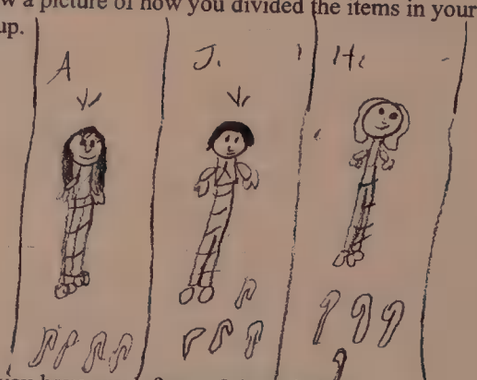
Figure 6

Hannah's representations of sharing 13 candy canes among 3 people

Item in bag Candy Canes

How many were in the bag? 13

Draw a picture of how you divided the items in your group.



Did you have any leftovers? (remainder) yes

How many? R1

Write a math problem showing how you solved this problem.

4 x 3 = 12

rectangle until we asked them if they could make a rectangle with a side longer than twelve tiles. During subsequent lessons, we had each student make a set of array cards as found in *Arrays and Shares* (Economopoulos, Tierney, and Russell 1995).

Representing factors and products as rectangular arrays gave students opportunities to recognize the relative size of multiplication products, to use the commutative property, and to determine the product of two factors by skip counting or repeated addition. Students were to cut out each array card, count the number of squares comprising the length and width of the array, and write the two sets of factors on the array side of the card (e.g., 8×4 and 4×8). They were then to find the total number of squares and write that product on the back of the card. We encouraged students to use skip counting or known multiplication facts to find the totals rather than to count by ones. Nevertheless, we saw many students counting by ones, and some students miscounted the number of squares, resulting in an incorrect product. We encouraged them to compare the labels on their cards with those of their partners, and we and the classroom aide spot checked as many cards as we could.

Antonio had much more difficulty cutting out and labeling his cards than the other students did, and he lagged far behind in completing the cards. To help him keep up with the class, we helped him with the cutting. Antonio was, however, determined to label all his cards himself. This was difficult for him because he could not consistently count beyond twenty-five. He soldiered on, working on his cards whenever he had a spare moment. In fact, Antonio invented a clever procedure that enabled him to accurately label the factors and products on each of his cards. Earlier in the year, each student had been given a multiplication chart with factors and products from 0×0 to 12×12. Antonio discovered that he could align the squares on one edge of his array card with the products on the chart in a one-to-one correspondence. For example, when he worked with the five-by-eight array, he went to the row of multiples of five, matched the side of the card with eight squares to the row with the multiples of five, and then found the product of forty above the eighth square of the array's top row (see **fig. 5**). He wrote "5×8" and "8×5" on the array side of the card, and then wrote "40" on the product side of the card. Neither of his teachers had thought of this accommodation for Antonio! It enabled him to complete his set of cards and to participate in all the games and other activities with the class.

Students Discover Square Numbers

The second time we did the paper bag activity, we asked students to draw a picture to show "how you divided the items in your group" and then to "write a math problem showing how you solved [the] problem" (see **fig. 6**) All the students' drawings correctly illustrated dividing the items, but only three of the nine students who were present that day wrote division equations. Five of them wrote multiplication equations, and Antonio wrote a repeated addition problem. Although the students could partition sets of objects and write stories and draw pictures about partitioning, not all of them were using division to represent those actions.

To help them make a stronger connection between a product's factors and the operation of division, we devised an activity for students that used their array cards. We invited the children to select any one of their array cards, write a multiplication equation that corresponded to the card, and then write the two division equations that also corresponded to the card. Because students were allowed to use any array card they wished, they were able to adapt the activity to their interests and their own knowledge. Deuvorn became restless after completing a few lines of the worksheet and said he wanted to do something else. We encouraged him to do a few more arrays. A moment later, he asked, "What do I do if I can only

Figure 7

Hannah's list of multiplication and division equations

Choose an array card and write two division problems that go with it.

Product		
21	21 ÷ 3 = 7	21 ÷ 7 = 3
54	54÷6=9	54÷9=6
144	144÷12=12	Perfect square
4	4÷2=2	Perfect square
9	9÷3=3	Perfect square
20	20÷2=10	20÷10=2
40	40÷10=4	40÷4=10
30	30÷10=3	30÷3=10
110	110÷10=11	110÷11=10
99	99÷11=9	99÷9=11
36	36÷9=4	36÷4=9
96	96÷8=12	96÷12=8
28	28÷7=4	28÷4=7
45	45÷9=5	45÷5=9
60	60÷5=12	60÷12=5
21	21÷3=7	21÷7=3
36	36÷6=6	Perfect square
27	27÷3=9	27÷9=3

write one division problem?" We questioned which card he was working with. "Sixteen," he said. We asked the class if anyone else had found a card for which they had been able to write only one division equation. Hannah said that she had found that 144, 4, and 9 had only one division equation. When we asked students what the cards 144, 4, 9, and 16 had in common, they answered that the numbers were squares. Deuvorn and Hannah decided to write "perfect square" in place of a second division equation (see **fig. 7**). Some students, however, preferred to write the equation twice (e.g., 100 ÷ 10 = 10 and 100 ÷ 10 = 10). After his discovery of square numbers, Deuvorn enthusiastically continued with the activity, writing three pages of division equations.

The individual interviews we conducted after four weeks of instruction provided evidence that the students' understanding and skill with multiplication and division had improved and that they were able to make more connections among various concepts and representations. For example, nearly all the students were able to use the array structure to find the number of squares in a rectangular array more quickly and easily than by counting individual squares. Each student was able to determine that sharing twenty-four cookies among four individuals would result in each person's receiving six cookies. Five of the ten students wrote a division or multiplication equation before modeling the problem with counters.

Reflecting on students' learning after four weeks of investigations, we recognized that many students had found unique methods for accomplishing the mathematical tasks. Antonio, for example, had created a way to determine the factors and product for each array card, and after this success, he engaged more enthusiastically in the learning activities and contributed more to class discussions. Because he was able to participate successfully with the rest of the class instead of requiring individualized tasks, Antonio liked school more. For the first time, he took home the work he had completed in class and proudly showed it to his family.

We also recognized that some students had set tasks for themselves that were more difficult than those we would have assigned. And they had succeeded at those tasks! For example, Adela had frequently written and solved division problems involving numbers larger than one hundred. We were especially pleased that the learning activities provided many opportunities for special education students with varying abilities to make sense of multiplication and division. We have described some of the ways in which the students provided successful adaptations and differentiated instruction for themselves.

Although we were working with a small class of students in a special education setting, we believe that learning experiences like these can be used successfully in other kinds of teaching situations. Many of the activities were originally designed for use in ordinary classrooms with twenty-five to thirty or more students. Activities such as the "Candy Box" activity and those with the array cards provide opportunities for students to work in small groups, foster support for one another, and monitor each other. Just as students with special needs possess unique and idiosyncratic interests, talents, experiences, and knowledge, so do students in regular classrooms. Because of the variety of solutions or solution paths for many problems and activities, students benefit from sharing their solutions during whole-class discussions at the conclusion of group time or independent work.

Teachers of larger classes can make time to work closely with small groups and observe their solution methods. Many of the learning activities we used can be set up at mathematics centers for independent or small-group work. The teacher can work with small groups on more challenging or unfamiliar activities while the rest of the class is working at centers or on other assignments. In a self-contained classroom, the teacher can work with small groups of students while the rest of the class is engaged in other kinds of learning activities, such as reading or social studies.

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Resources for a New School Year

Books

For Students

Math Matters en Español: Una Colección para Kate, *Barbara deRubertis, 2007. 32 pp., $5.95 paper. Ages 5–7. ISBN 1-57565-240-4. The Kane Press; (212) 268-1435; www.kanepress.com.*

The purpose of this Spanish version of *Math Matters: A Collection for Kate* is to integrate the concept of addition with literature using an interesting story and a realistic context in grades K–2 classrooms. In the story, Kate, the protagonist, is concerned about what collection to show to her class. Because she does not have many items of a single kind, she decides to wait and see the collections her classmates bring. As they show their collections, Kate counts the total number of items in each. In some cases, she has to add two numbers; in other cases, she needs to add three numbers. In one instance, Kate must use regrouping to find the sum of four numbers. At this stage, Kate is really worried and has to think of a way to get out of her predicament. She does so in a smart way: showing a collection of collections!

The book has two important pages at the end: One page graphically depicts the concept of sum and describes several strategies to perform addition (counting on, using doubles, using related facts, looking for a pattern, and the standard algorithm). The other page contains some pedagogical suggestions for teachers and parents about how to support and extend both the language and mathematics concepts beyond the story (reading together, making connections with mathematics, and applying mathematical concepts).

Based on the story, mathematics concepts, diversity of addition strategies, and the pedagogical suggestions, I recommend this inexpensive book to grades K–2 teachers and parents.—*José Contreras, University of Southern Mississippi, Hattiesburg, MS 39406.*

One Leaf Rides the Wind, *Celeste Davidson Mannis, illustrated by Susan Kathleen Hartune, 2005. 32 pp., $15.99 cloth. ISBN 978-0670035250. $6.99 paper. ISBN 978-0142401958. Penguin Group; (800) 526-0275; www.penguinputnam.com.*

Designed for children ages 4–8, *One Leaf Rides the Wind* is a counting book that introduces haiku. Each two-page spread has a drawing on the left-hand page with the number and poem on the right-hand page. Also featured at the bottom of the right-hand page is a description of the Japanese garden element depicted in the picture and poem.

A teacher could choose a variety of subject areas—including mathematics, language arts, and social studies—to develop lessons with this book. Older students may chose to write their own counting books, write haiku, or conduct further research about Japanese culture or gardens.

My third-grade students enjoyed looking at the colorful drawings as well as listening to the rhythm of the haikus. Because of the variety of ways to use this book, I would highly recommend it for first- through third-grade classrooms. Older students being introduced to haiku might also find the book helpful.—*Sarah Pullie, Latta Elementary School, Latta, SC 29565.*

Prices on software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

For Teachers

Encouraging Your Child's Math Talent: The Involved Parents' Guide, *Michael J. Bosse and Jennifer V. Rotigel, 2006. 200 pp., $14.95 paper. ISBN 1-59363-184-7. Prufrock Press; (800) 998-2208; www.prufrock.com.*

Written by parents and for parents of mathematically advanced children, this book outlines how to recognize advanced mathematical ability in children, navigate the school system in support of a child, and offer support at home. It also suggests a list of enrichment opportunities for children. Each of these topics has specific examples and ideas for parents to use with their children and their children's teachers.

A good book for both parents and teachers of mathematically advanced children to read and have on hand as a reference, one of this book's strengths is how it presents material in a manner any parent can understand. The authors give a balanced view of schools working with mathematically advanced children: what they do well and what they cannot do. Several times, the authors state their belief that every child has a right to learn new content and skills every day. However, they are realistic about what teachers and schools have time to do. The book points out ways children are turned off to mathematics, either by teachers who do not challenge them or parents who push too hard. The authors encourage parents to work with their school systems.—*Sue Runyon, K–12 Mathematics Consultant, Keystone Area Education Agency, Elkader, IA 52043.*

Knowledge under Construction: The Importance of Play in Developing Children's Spatial and Geometric Thinking, *Daniel Ness and Stephen J. Farenga, 2007. 282 pp., $32.95 paper. ISBN 978-0-7425-4789-6 (0-7425-4789-2). Rowman and Littlefield Publishing Group; (800) 462-6420; www.rowmanlittlefield.com.*

This is a fascinating book that explores how young children construct knowledge with regard to the spatial and geometric aspects of behavior during block play. The authors establish the value of informal play and how it relates to cognitive development and possible success in mathematics and science. First, the authors trace the origin and use of "space" terms in philosophical and psychological contexts, as well as phrasal histories relating to space and geometry. Second, they examine four theoretical frameworks that have shaped our current understandings of cognitive and intellectual development as it relates to humans' interactions with space in different contexts. Finally, the researchers give an overview of literature, uncovering "holes" where researchers have not yet examined young children's spatial and geometric thinking through naturalistic methods.

The researchers and an architectural professor examined videotapes of several children's block play within natural settings. The results show that young children have a natural inclination to incorporate mathematical and architectural principles in their block play. The authors discuss what teachers, parents, and researchers need to observe to foster and promote young children's cognitive strengths in the area of spatial and geometric knowledge.

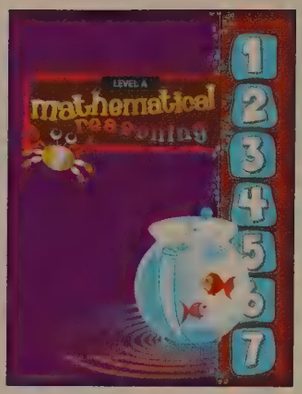
The book has a contribution to make in mathematics, early childhood education, and the field of cognitive sciences. The research is a pioneering study in an area where little data currently exists and serves as a catalyst for further investigations into the importance of play, cognitive development, and success in school.—*Darlinda Cassel, University of Central Oklahoma, Edmond, OK 73034.*

Et Cetera

Beginning Mathematical Reasoning, *Doug Brumbaugh and Linda Brumbaugh, 2005. 240 pp., $29.99 paper. ISBN 0-89455-882-X.* **Mathematical Reasoning Level A**, *Doug Brumbaugh and Linda Brumbaugh, 2006. 256 pp., $29.99 paper. ISBN 0-89455-884-6. Bright Minds; (800) 641-6555; www.brightminds.us.*

These two visually engaging and mathematically rich activity books are designed for preschoolers and kindergartners working with an adult facilitator. The authors align the books with NCTM Standards and age-appropriate skills in mind. Each book is divided into eight pastel-colored sections focusing on developmentally appropriate, successively difficult mathematics skills and concepts that spiral throughout the book. A table of contents in the form of a matrix allows facilitators to easily focus on a

specific skill or strand. On the top right corner of the prekindergarten book, the skill for each page is printed; in the kindergarten book, it is the NCTM strand.



Various learning styles are addressed by suggestions such as using manipulatives (concrete objects), including virtual manipulatives (electronic applets), and doing kinesthetic activities (a walking number line). The authors highly recommend that the books be completed by the students, page by page, taking into account different learning rates and styles.

The pages in both books are heavy stock with bright, engaging artwork and large, bold print. Children of this age group can easily understand the directions on each page. An added bonus is the emphasis on some important Big Ideas of mathematics. For example, the kindergarten book shows a cookie divided in half. The words *equal parts* are underlined to emphasize the idea that two equal parts are called *halves*. When any important mathematical word or concept—for instance, the word *between*—is first introduced it, too, is underlined. In the prekindergarten book, when an activity calls for the child to color parts of a whole, the name of the color they are to use is in colored print. When the child is to count and color blue shirts, the word *blue* is in blue type. Early reading skills are reinforced along with mathematics skills.

Because the audience for these products includes facilitators who may not be educators, one weakness of the books is their lack of a glossary. In the kindergarten Number and Operations sections, a dotted line shows children how to draw a line segment to match shapes, but nowhere does the book explain what a *line segment* is or how it differs from a *line*.

Teachers of English Language Learners (ELL) will find these books especially valuable in presenting vocabulary and concepts. Parents who home school students also will find the books extremely helpful when introducing mathematics skills and all NCTM concepts and strands. Parents wanting to enrich or reinforce what students are learning at school will find developmentally appropriate activities for their children to do in short time periods. For the money, both books are excellent buys and worthy supplements to any early childhood student's education.—*Maxine Pincott, Oliver Ellsworth School, Windsor, CT 06095.*

The Math Process Standards Series, *2007. $25 ea. paper (includes book and CD). Grades 3–5.* **Introduction to Problem Solving**, *Susan O'Connell. 208 pp. ISBN 978-0-325-00970-4.* **Introduction to Communication**, *Susan O'Connell. 208 pp. ISBN 978-0-325-01106-6.* **Introduction to Reasoning and Proof**, *Karren Schultz-Ferrell, Brenda Hammond, and Josepha Robles. 175 pp. ISBN 978-0-325-01033-5.* **Introduction to Representation**, *Bonnie Ennis and Kimberly Witeck. 160 pp. ISBN 978-0-325-01104-2.* **Introduction to Connections**, *Honi J. Bamberger and Christine Oberdorf. 176 pp. ISBN 978-032500999-5. Heinemann; (800) 225-5800; www.heinemann.com.*

This series isolates and provides strategies for instruction in each of the NCTM Process Standards. Each book is organized with an introductory chapter explaining the Process Standards followed by a thorough discussion of the targeted standard. The subsequent chapters offer practical instructional suggestions for teachers of grades 3–5. Examples of student work are consistently provided, and frameworks for assessment of each Process Standard are clearly outlined. In addition, each book comes with its own CD-ROM containing activities and instructional tools specific to the Process Standard. For example, the Problem Solving book has several instructional charts and over seventy sample problems. The CDs are easy to use, and each file can be modified to personalize activities. Each book presents specific suggestions for modifications.

One of the series' strengths is that the activities provided are easily differentiated to work with all depths of familiarity. Toward that end, sample problems on the CDs are particularly valuable, whether students have significant background knowledge of the skill or are seeing it for the first time. In addition, the authors anticipate likely hurdles for mastering each skill and offer helpful suggestions to guide students.

The books work well as stand-alone professional support to address individual Process Standards or as a series for improving practice in all Standards. Given the Standards' interconnectedness, some overlap between books is inevitable. Nevertheless, each does a good job of focusing on the particular Process Standard and providing useful information.—*Alison Eber, Brockett Elementary School, Tucker, GA 30084.* ▲

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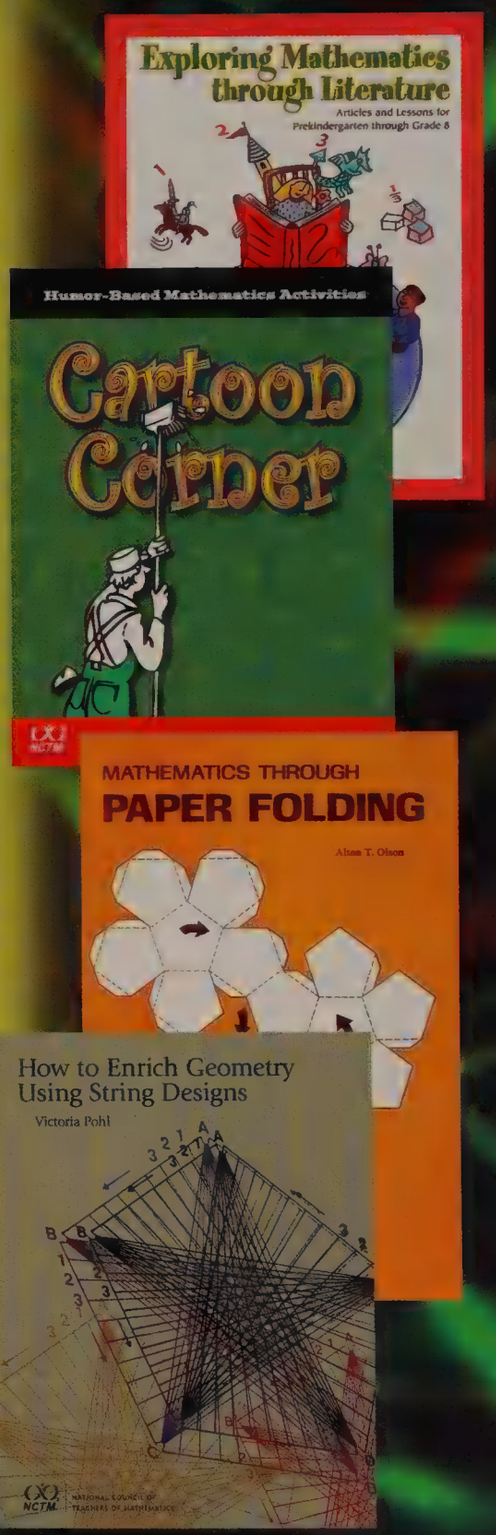

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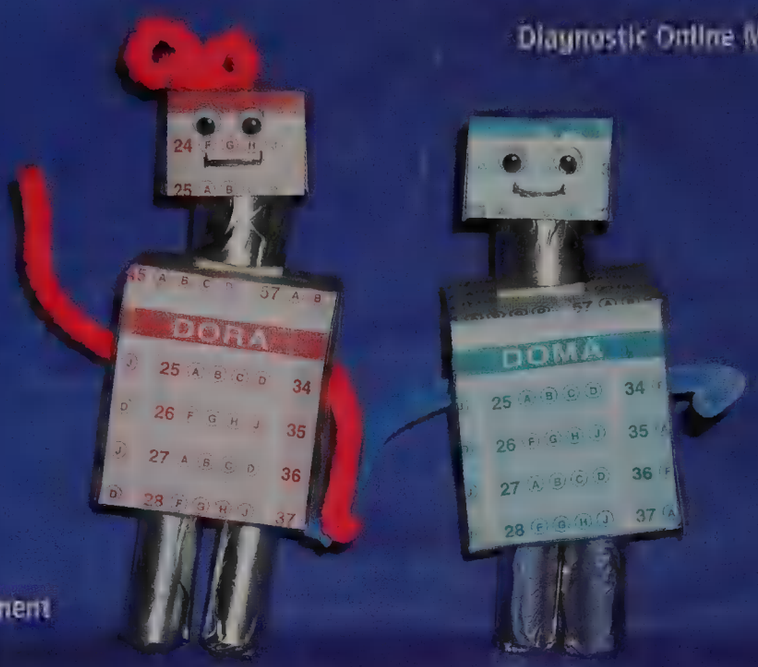

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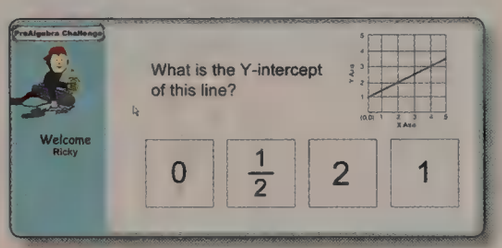

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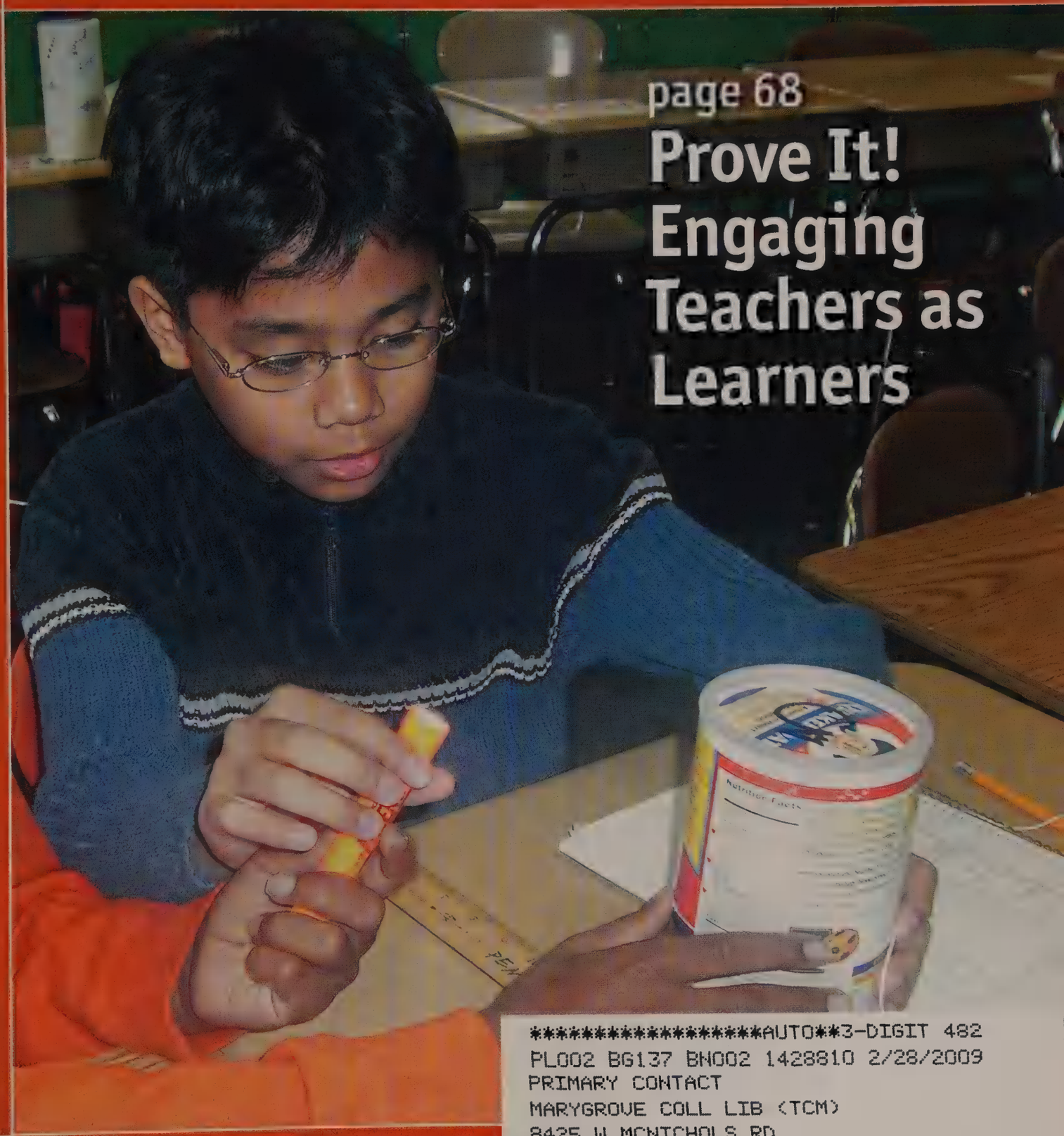

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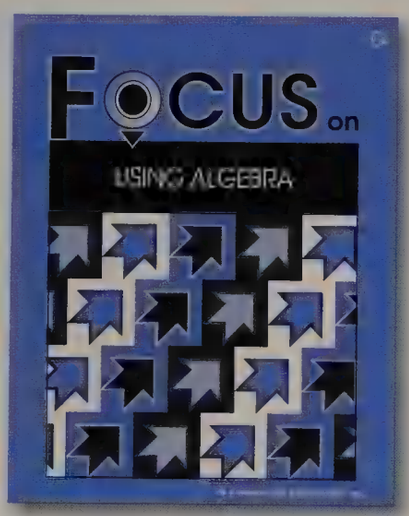

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Effective professional development engages, rather than bypasses, teacher thinking. Julie Sweetland and Meghann Fogarty's article, "Prove It! Engaging Teachers as Learners to Enhance Conceptual Understanding," describes strategies to heighten teachers' awareness of the difference between procedural knowledge and conceptual understanding. Photograph by Center for Inspired Teaching; all rights reserved

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HOT Topics

Call for Manuscripts

Teaching Children Mathematics is interested in publishing articles on "hot topics" that *TCM* readers have noted as subjects of high interest on reader surveys. By highlighting the issues and challenges that face mathematics educators today, the Editorial Panel hopes to provide teachers and teacher educators with resources to assist them in their efforts to meet the mathematical needs of all students. We invite authors to share their classroom experiences and their ideas. Manuscripts that include photographs and samples of student work or dialogue are especially encouraged.

The following list of topics and related questions are intended as a guide for authors. Manuscripts that address related issues are always welcome.

Differentiated instruction

- What are the challenges of differentiated instruction, and how do teachers deal with these challenges?
- How do we develop meaningful mathematical tasks that address students' diverse academic needs and ensure that all students have the opportunity for success in learning and understanding mathematics?
- What are the best ways to manage differentiated instruction in an inclusive mathematics classroom?

Assessment Strategies

- How are student interviews used to assess mathematics learning?
- How can we create authentic assessment tasks?
- What alternative assessment strategies are most effective?
- How can we help students understand and participate in assessment?

Curriculum focal points

- In what ways can NCTM's newly released *Curriculum Focal Points* inform the teaching and learning of mathematics?
- How can the focal points be addressed within the framework of state goals?
- How can the focal points be used in the mathematics classroom and shape student learning?

Intervention strategies

- What evidence supports the use of instructional intervention strategies to ensure that all students succeed in today's high-stakes testing environment?
- What is the most effective way to manage intervention while still moving students forward?
- What are effective classroom structures that support intervention strategies?
- How can teachers successfully manage intervention strategies while meeting the diverse needs of all students?

Teachers' knowledge of mathematics

- How can we close the gap between what teachers know and what they need to know to teach mathematics effectively in the twenty-first century?
- What role can a mathematics specialist serve in helping teachers develop content knowledge?
- How can evidence of student improvement based on teachers' increased content knowledge be collected?

Professional development strategies that have helped teachers and staff be more effective (e.g., professional learning communities)

- What are successful strategies for fostering communication among teachers at the same grade level? Across grade levels?
- How can vertical teaming activities be developed and implemented in a multigrade school?
- How do we get started using book studies, lesson studies, and team-planning initiatives within our professional learning communities?
- How does team scoring affect student work and teaching practices?

Elementary mathematics specialists

- What impact do elementary mathematics specialists have on a school's professional community?
- What are the most effective strategies and best practices used by specialists and coaches for collaboration, co-teaching, and mentoring teachers?
- What are the roles of mathematics specialists?

Using children's literature in mathematics

- How can children's literature aid in developing mathematics concepts (fractions, time, money, operations, number sense, etc.)?
- How can literature be used to develop problem-solving skills?
- How can language arts and mathematics content be joined to create an integrated curriculum?

If you have interesting ideas, research, or classroom-tested approaches concerning any of these topics, please share them by writing manuscripts for the journal. Submit a manuscript to *TCM* by accessing **tcm.msubmit.net**. For more information, refer to "Instructions for Submission of Manuscripts" at nctm.org/publications/write.htm. Questions can be directed to the Journal Editor at tcm@nctm.org.

Prove It!

Engaging Teachers as Learners to Enhance Conceptual Understanding

The faculty of an elementary school gathered for a staff development workshop and quickly completed the first mathematics problem assigned by the facilitator: "What is 1/5 × 1/4?" Within a few seconds, every teacher in the room arrived at the correct answer: 1/20. The follow-up question, at first glance, seemed to be equally simple: "How did you get your answer?"

"You multiply the numerators, and then you multiply the denominators," offered a participant.

The facilitator pushed harder. "That is the formula. You have described the steps you were taught to follow. I would like to challenge you to demonstrate why the formula works. In other words: Prove it!"

By Julie Sweetland and Meghann Fogarty

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The distinction between the facilitator's two requests—"Tell me the answer" and "Demonstrate the thinking behind the answer"—is an important one. To provide a correct answer, teachers needed only use a familiar procedure. By asking teachers to "prove it," the facilitator pushed them to dig a little deeper to the mathematical concept underlying the formula. This article describes strategies to heighten teachers' awareness of the difference between procedural knowledge and conceptual understanding. By pressing classroom instructors to discover and then articulate the reasoning behind familiar formulas, teacher educators can demonstrate the power of teaching for understanding. Many teachers need to develop their own mathematical proficiency, as they "may know the facts and procedures they teach but often have a relatively weak understanding of the conceptual basis for that understanding" (Kilpatrick, Swafford, and Findell 2001, p. 372). At the same time, by engaging teachers as learners, staff developers can deepen their own appreciation of how much teachers actually *do* know about mathematics—knowledge that teachers can come to articulate when and if they are led through carefully designed activities that allow for trial and error, collaborative problem solving, and self-reflection.

Demonstrate Your Thinking

Fraction multiplication is an example of a topic frequently taught through memorization of procedural steps. The traditional approach provides learners with an efficient route to the correct answer. Even teachers who do not cover fractions as part of their curriculum can usually recall and apply the formula quickly. Knowing the formula, however, does not necessarily lead to a conceptual understanding of what it means to multiply numbers that are less than one. In the scenario above, the teachers were first given a few minutes to "prove" that $1/5 \times 1/4 = 1/20$ using pencil and paper. Without further prompting, a few teachers started sketching groups of circles in fours or fives or generated other representations of fractions. Many did not move beyond the formula—at least until the facilitator reminded participants that a multiplication sign can also be read as "of." This hint was enough for nearly all the teachers to begin struggling individually with how to represent multiple sets of fractional numbers.

After each teacher had arrived at an answer or other "resting place," the facilitator invited participants to share with the group. The teachers demonstrated many different diagrams and thinking processes—an observation the facilitator highlighted to underscore the diverse ways of thinking among any group of learners. The facilitator then asked one teacher to come to the easel to develop a possible diagram. First, the teacher was asked to draw a square, divide it into fourths horizontally, and represent 1/4 by circling one row. Then, the teacher was to divide the paper into fifths by drawing vertical lines and coloring one-fifth of the already circled one-fourth. By shading one-fifth of one-fourth, the teacher colored one-twentieth of the grid (see **fig. 1**).

The visual representation yielded a few "aha" moments, but some participants needed another demonstration. The facilitator then physically involved the whole group in the mathematics problem, ensuring that even the teachers who had not arrived at the answer independently now became engaged in the process. Twenty teachers were chosen as "manipulatives." The remaining four or five teachers became "mathematicians." The task of using the manipulatives to model the explanation was given to the mathematicians, and the facilitator stepped back to allow the teachers to think, debate, and make a few false starts on their way to arriving at the correct solution. A first attempt consisted of lining up the manipulatives in a row of twenty, with

Figure 1

A collaboratively constructed diagram illustrates why one-fourth of one-fifth equals one-twentieth ($1/4 \times 1/5 = 1/20$).

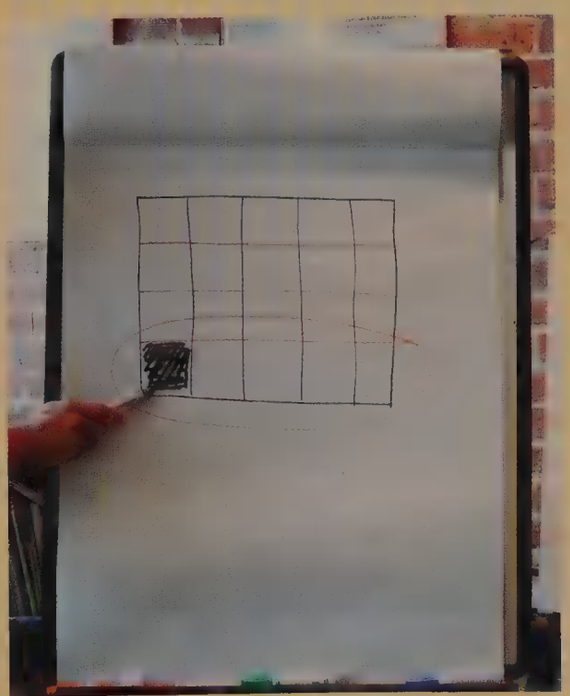

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Figure 2

Facilitator steps in a "Prove It" activity for staff development

1. Identify a formula or mathematical procedure that teachers can apply but are unlikely to be able to derive or articulate.
2. Demonstrate teachers' procedural fluency by providing a simple problem that requires them to apply the formula.
3. Demonstrate teachers' need for a more conscious conceptual understanding by asking for an explanation of why the formula works. Prove it!
4. Allow time for independent exploration and intellectual struggle. As teachers work, the facilitator listens, observes, and asks questions that guide teachers from one level of understanding to the next. Do not evaluate participants' attempts or provide answers.
5. Once teachers have reached individual solutions, allow for a brief discussion of their findings.
6. Facilitate the discovery of one or more alternative solutions using an entirely different mode of instruction. This step allows for a richer exploration of the concept as well as a greater appreciation for the importance of accessing multiple learning styles in mathematics instruction.
7. Allow time for reflection: Ask teachers to think aloud about what they learned and how they learned it; invite them to connect the learning experience to their own teaching practice; and highlight the social-emotional aspects of learning mathematics.
8. Reinforce and extend teacher learning by providing time for them to adapt the activity for their own classroom use or to begin to develop similar activities inspired by their experience.

nineteen standing and one sitting. "One-twentieth is the answer," agreed the facilitator, "but you have not shown me *why* it is the answer to 1/4 × 1/5."

The mathematicians then directed the manipulatives into four groups of five members. "Three groups need to sit down," announced one teacher, and the manipulatives complied. "This standing group is one-fourth of the whole. It has five people in it. One person is one-fifth of one-fourth, which is the same as 1/4 × 1/5." The mathematicians gently helped four group members to the floor, so that one manipulative was left standing. "You are one-twentieth of the whole."

After a collective sigh of relief and a brief smattering of applause, the facilitator guided a discussion allowing teachers to reflect on the implications of the experience for classroom practice. One teacher noted that the visual representation was confusing for her but that playing the role of mathematician to create a physical representation of the problem gave her some clarity. Her colleagues immediately connected her point to the needs of kinesthetic learners and also noted that providing multiple variations of an explanation is often beneficial for all learners. Another teacher spoke of feeling anxious when she realized she was unable to generate a satisfying explanation of the equation; she said she will remember her own discomfort the next time her fourth-graders struggle with a task. This discussion was as important as the mathematical demonstrations that preceded it. Effective professional development enables teachers to connect their own learning experiences to the context of teaching, and it does so in a way that builds educators' skills in analyzing instruction.

In its entirety, the "Prove It!" activity allowed the workshop participants to evaluate and enhance their own content knowledge. By asking teachers to extend themselves intellectually, emotionally, and physically, the activity pressed the teachers to reflect on their own conceptual understanding and, perhaps more important, that of their students. In the remainder of the workshop, teachers engaged in two additional activities to reinforce the theme and further illustrate the pedagogy associated with teaching for understanding. **Figure 2** summarizes a facilitator process for planning and implementing professional development activities that engage teachers in discovering and explaining the logic behind familiar procedures and formulas.

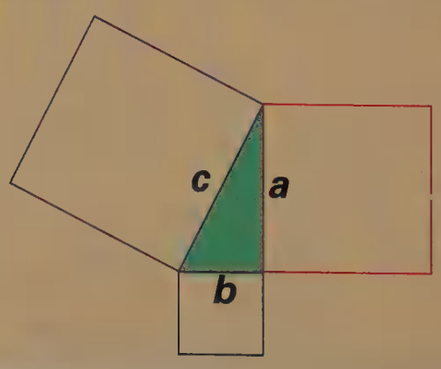
The Pythagorean Theorem

A second, similar activity involved the teachers in an exploration of the Pythagorean theorem. The goal was to allow teachers to discover the powerful difference between the ability to plug numbers into the equation $a^2 + b^2 = c^2$ and the ability to articulate that the equation means that for any right triangle, the area of the square made with the hypotenuse is equal to the sum of the two squares made by the two legs of the triangle (see **fig. 3**).

Using an activity adapted from teacher-educator Aleta Margolis (2007), the facilitator gave teachers several labeled cardboard squares of different dimensions—three inches by three inches, four by four, five by five … ten by ten—and so on. The for-

Figure 3

A diagram illustrates the Pythagorean theorem.



mula, along with its textbook definition, was posted on chart paper, and the teachers were directed to use the squares to demonstrate the Pythagorean theorem. That was all the instruction given. Although unsure of what to do at first, the teachers began to manipulate the squares in different ways. Some folded the squares in half, which produced right triangles but did not lead to a model of the theorem. Others began by placing the squares one on top of another but abandoned the idea when they realized that the method did not create triangles (see **fig. 4**).

Eventually, many began to move toward a solution by using the squares to create triangles in the negative space between the sides of the squares (see **fig. 5**). By experimenting to determine which squares made right triangles and which did not, participants discovered that the triangle in the negative space made with a three-by-three-, a four-by-four-, and a five-by-five-inch square was a right triangle; but the triangle in the negative space made with a five-by-five- and two three-by-three-inch squares was not a right triangle. From a learner's perspective, this activity helps build a visual and visceral understanding of an abstract concept.

The teachers talked as they worked, and as the activity came to a close, the facilitator noted how much mathematical vocabulary was used during the lesson: "I heard *right triangle*, *hypotenuse*, *similar*, *different*, *square*, *equation*, *equals, times, multiple*, and many more terms all used naturally in the course of peer-to-peer interaction." Participants agreed that this was the sort of atmosphere they would like to create in their own classrooms—an environment where mathematical terms come up as frequently as literary terms such as *book, poem*, or *sentence*. Providing a context where students can acquire and use mathematical terminology is a crucial step toward helping students build mathematical communication skills; such skill is an important element of students' overall mathematical proficiency (NCTM 2000).

Other reflections suggested the value of this sort of teacher support in overcoming "math phobia." One teacher noted that the hands-on experience sparked her creativity: "I'm a follow-the-book kind of teacher when it comes to mathematics. But as we were doing this, I was thinking of all the other standards and topics I could touch on through this activity—negative and positive space, congruent shapes, all kinds of stuff. I'm always coming up with spin-offs and extensions with other subjects but not usually with mathematics, so it was nice to see I have it in me!" Just as a student's "productive

Figure 4

A teacher makes a first attempt at using squares of different dimensions to illustrate the Pythagorean theorem.

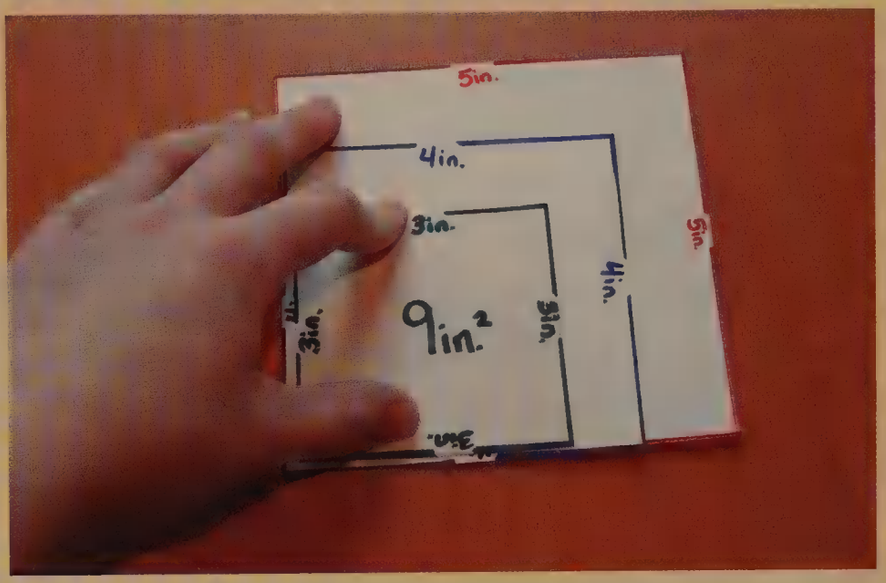


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Figure 5

Hands-on experience with manipulatives helps mathematics learners of all ages develop skills in explanation and justification. Students use squares of different dimensions to form edges of right triangles, demonstrating the Pythagorean theorem.

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disposition" toward mathematics is essential for mastering content and concepts, teachers also must be given the chance to develop the sense that they are capable of learning and teaching mathematics (Kilpatrick, Swafford, and Findell 2001, p. 384).

Finding π

The next step in the staff development workshop was to ask the teachers about pi. Volunteers quickly recalled that pi equals approximately 3.14, and one quipped that "pie aren't square; they're round," but no one could articulate that the significance of pi is that the ratio of a circle's circumference divided by its diameter holds true for all circles. For the next hands-on activity, adapted from Burns (2004), the group divided into pairs. Each pair of teachers was given a ball of yarn, a pair of scissors, a ruler, and a calculator. Additionally, the groups received an ordinary cylindrical object—a film canister, a box of salt, a soda can, and so on. Partners applied themselves to using the ruler to measure the diameter of their cylinder and then using the yarn to measure the circumference. (**Fig. 6** shows students working through this same problem.)

Each pair of teachers recorded their measurements on a large chart that everyone could see. Finally, using a calculator, the teachers divided the circumference of each object by its diameter and recorded their results on the chart. As the final column was filled with close approximations of pi, one teacher noted, "My kids would love this. It's like magic. You get the same answer every time." Along with gaining a better understanding of pi—the mathematical constant—the teachers gained a better conceptual understanding of the often-used and comfortable formula $C = \pi d$. The follow-up discussion highlighted the teachers' growing awareness of the importance of teaching for understanding: "These kinds of lessons would make a big difference for kids who are always mixing up which formula goes with what. I can see that doing this activity would help kids remember that πd is the formula for circumference, not area."

Figure 6

Sixth graders use a ruler and string to measure the diameter and circumference of cylindrical objects and demonstrate that pi is a constant that holds true for all circles.

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Summary and Conclusion

The form of these teacher learning activities is as important as the substance. When staff developers allow teachers to grapple with mathematical ideas, the resulting professional learning is also personal and often powerful. Such learning cannot be achieved through any shortcut. As a high school mathematics teacher put it, "I always found that when I used discovery methods in my class, I had to sum everything up for the students at the end to make sure they really learned it. Eventually, I figured, why not skip the exploration and just go straight to the explanation? I thought of myself as saving time. But this workshop has reminded me that the discovery phase is not something you can just skip. It is not an appetizer; it's the meal." Asking teachers to develop physical representations of familiar formulas treated teachers as learners, engaging them intellectually. Too often, in-service teacher professional development provides practitioners with "the answers" and bypasses teachers' thinking in the process. In contrast, the problem-based, collaborative nature of the activities described here puts teachers in action: thinking, debating, and rethinking. Learning for understanding is as critical for teachers as it is for students, for only then can the benefits of the professional development opportunity extend beyond the specific activity or technique modeled in the in-service session: "Perhaps the most important feature of learning with understanding is that such learning is generative. When students acquire knowledge with understanding, they can apply that knowledge to learn new topics and solve new and unfamiliar problems" (Carpenter and Lehrer 1999).

The temporarily destabilizing nature of the "Prove It!" process has an important emotional

impact on teachers as well. After an activity, reflecting on their experience is important for the teachers. In addition to cementing or enhancing teachers' understanding of the mathematical concept at hand, debriefing allows the teachers to reflect on the process of being a learner. Some teachers experience confusion, frustration, and even a brief intellectual paralysis when initially tackling novel mathematical problems. For adults accustomed to the luxury of always knowing what will be expected next in the classroom, challenging learning activities can be a valuable reminder of the feelings that children experience on a daily basis as they continually face new and unfamiliar intellectual terrain. Skilled facilitation leads teachers to reflect on the social-emotional aspects of mathematical learning and connect their own learning experiences to the experiences of their students. By reentering the role of learner, teachers can reflect on past experience in order to shape their students' future experiences.

Making such a link to classroom practice and student learning is the hallmark of effective professional development. The "Prove It" process underscores the critical difference between procedural compliance and true understanding for teachers—but the ultimate goal of the process is to lead teachers to teach for understanding in their own classrooms. Traditional, dominant mathematics instruction methods ask children to perform rote memorization or apply plug-it-in formulas. Presenting mathematics as a series of rules to be followed is like teaching history as a series of dates to be memorized—such approaches cut off learners' thinking, sacrifice understanding for efficiency, and often stifle learners' intrinsic interest in the subject as well. The alternative, teaching for understanding, requires more time and a bit more tolerance for the messiness of intellectual struggle but results in a fuller, more substantive appreciation of content.

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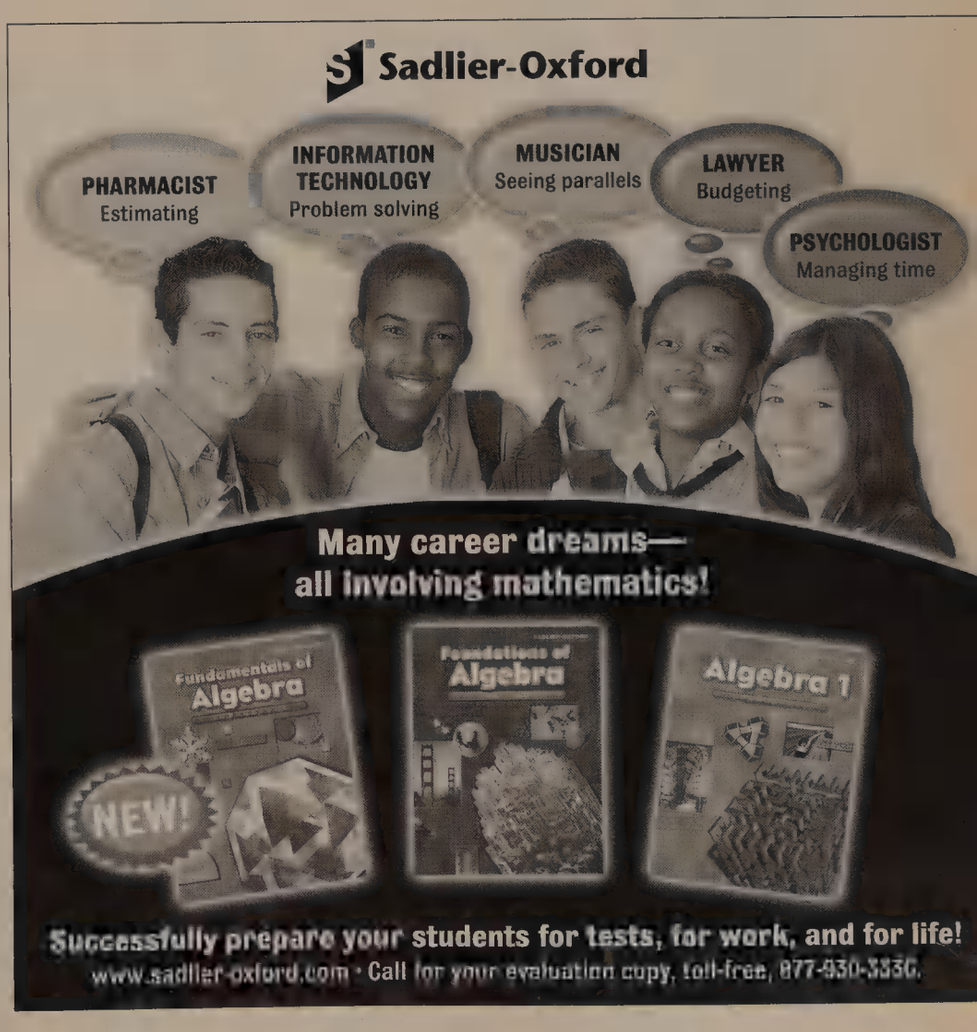
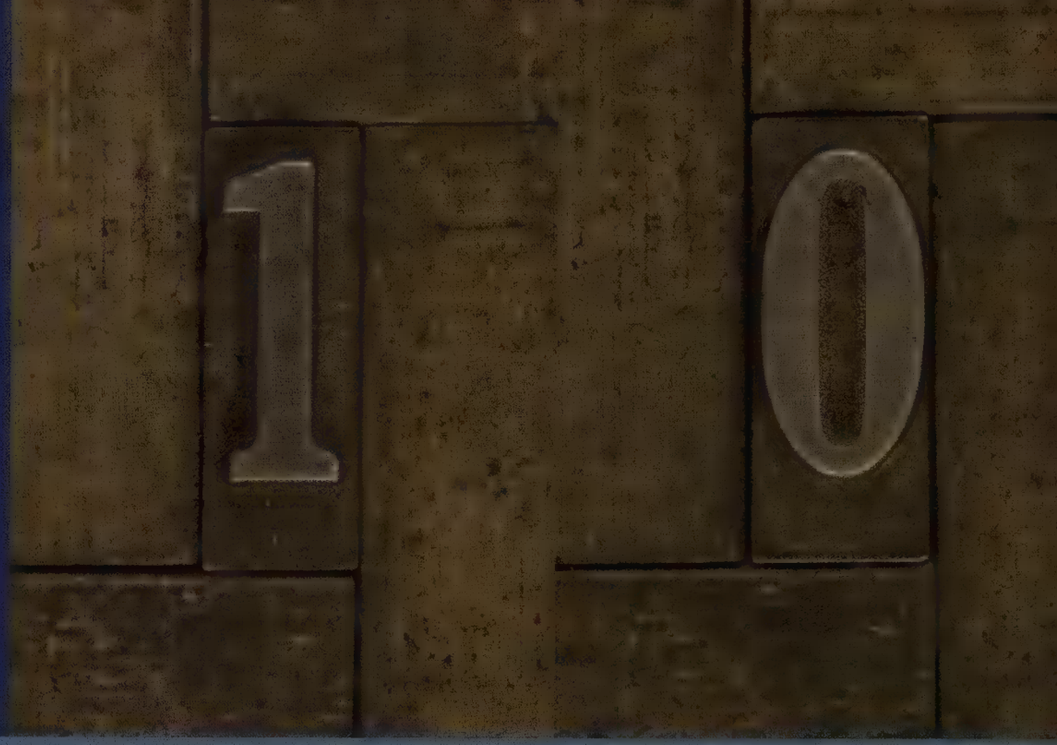


"We All Have Something That Has to Do with Tens":



Counting School Days, Decomposing Number, and Determining Place Value

By Anne M. Goodrow and Kasia Kidd

It is the seventieth day of school. As in many other second grades, the children in this class keep track of how many days they have been in school. In the following vignette, the children share number sentences that equal seventy. The second graders and Mrs. K., their teacher, are gathered on the rug in front of the whiteboard. Each child holds a small blue mathematics journal, in which they record their work. Keaton begins sharing expressions that equal seventy. As he dictates, Mrs. K. writes "139 – 39 – 30" on the chalkboard.

Keaton: One hundred thirty-nine, then if you minus thirty-nine, you'll be back at one hundred. Um, and then minus thirty. One hundred, then you go ninety, eighty, seventy.
Mrs. K.: Oh, beautiful. You can even do that mentally.
Keaton: I did that.
Mrs. K.: You did that. You did it mentally, and you didn't even use the number grid.

Winston shares next. He reads aloud how he has made seventy, and Mrs. K. writes his expression on the chalkboard: "1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 20 + 10 – 10 – 1 – 1 – 1 – 1 – 1."

Mrs. K.: Do you want to explain this one to us?
Winston: Well, my grandpa told me a trick. You add the one with the nine, the eight and the two, the three and the seven. [*He draws a line connecting the 1 and the 9 to make a 10.*]
Mrs. K. [*to the class*]*:* Do you see how he made that line there?
Mrs. K. [*to Winston*]*:* You see how you showed us how the one and the nine go together and make a ten? Could you do it for the other numbers you are combining so we can see?
[*Winston connects the 2 and 8, 3 and 7, and 4 and 6.*]
Winston: It equals ten plus ten from here [*pointing to 1 and 9, 2 and 8. He then counts aloud.*] "One, two, three, four, five" [*as he writes*] "*10 + 10 + 10 + 10 + 10*" [*under the long expression*].
Winston: Then plus a five. So that equals five tens [plus 5]—equals fifty-five. And the fifty-five plus twenty equals seventy-five. Seventy-five plus ten equals eighty-five. Minus ten equals seventy-five. Minus one, minus

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one, minus one, minus one, minus one. Minus five ones. Minus ten and minus five ones equals seventy.
Dr. A: What do people think about what Winston did?
Alex: He added one plus two, plus three, plus four, but he knew he could make them all ten by adding the 1 to the 9, the 2 to the 8, the 3 to the 7, the 4 to the 6. Then [he added] the 5 is by itself. Then you still have the 10. And that's ten, twenty, thirty, forty, fifty, fifty-five ... plus twenty equals seventy-five, plus ten equals eighty-five, minus ten equals seventy-five, minus one equals seventy-four, minus one equals seventy-three, minus one equals seventy-two, minus one equals seventy-one, minus one equals seventy.
Dr. A: Neat. So it really is helpful to know how to make ten, isn't it?
Mrs. K.: That making ten idea. Look at this. That kind of reminds me of Keaton's because he told me that 139 minus 39 would give me what?
Children: One hundred.
Mrs. K.: Now I've got one hundred minus thirty equals seventy. Does anybody see a ten in there? Gianna?
Gianna: Thirty and seventy and a hundred.
Mrs. K.: And one hundred, right? So, that thirty and seventy makes ...?
Gianna: One hundred.
Mrs. K.: So, three tens and seven tens make ...?
Gianna: One hundred.
Mrs. K.: And that's the same as? Ten?
Gianna: Tens.
Mrs. K.: Ten tens. Exactly right.

A few minutes later, Maggie takes her turn sharing: "One hundred minus ten, minus twenty."

"Minus twenty," Mrs. K. repeats as she writes on the chalkboard.

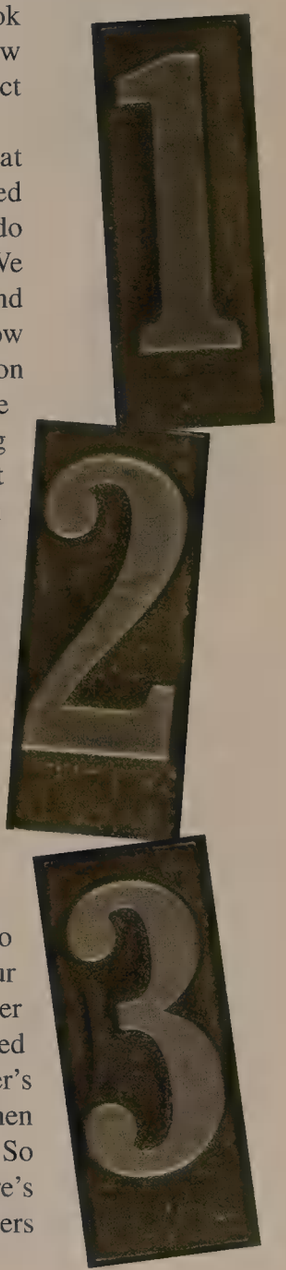
Mrs. K.: You want to tell us about that one?
Maggie: I knew one hundred minus thirty equals seventy, but instead of using the thirty, I [changed] it to one hundred minus ten. I knew twenty plus ten equals thirty. So I did the minus ten, and then I did the minus twenty.
Mrs. K.: Nice going. You know what Keaton said as soon as you went up there to talk about it? What did you say?
Keaton: I said, "It's just like mine."
Mrs. K.: He said, "It's just like mine."
Maggie: All of us who shared have something that's the same—Keaton, Winston, Jailene, me—we all have something that has to do with tens.
Mrs. K.: Exactly. You do. You all have something that has to do with tens. That's great.

Counting School Days

Counting the days of school is a popular routine in many elementary mathematics classes. It is often used only as a counting and grouping activity, and on the hundredth day of school, the children bring collections of one hundred objects to school. The intention is to help them develop an understanding of place value. Mrs. K.'s classroom is no exception in this regard, but it *is* an exception when her students break apart the day's number to create their own equivalent numerical expressions.

What do the children learn about place value by decomposing numbers in this activity? What does their teacher learn about the children's mathematical thinking when they share? How does Mrs. K. respond to their sharing? How does she help her students move forward in their mathematical thinking? This article looks at the events that took place on day 70 of the school year to illustrate how decomposing numbers can help children construct their understanding of place value.

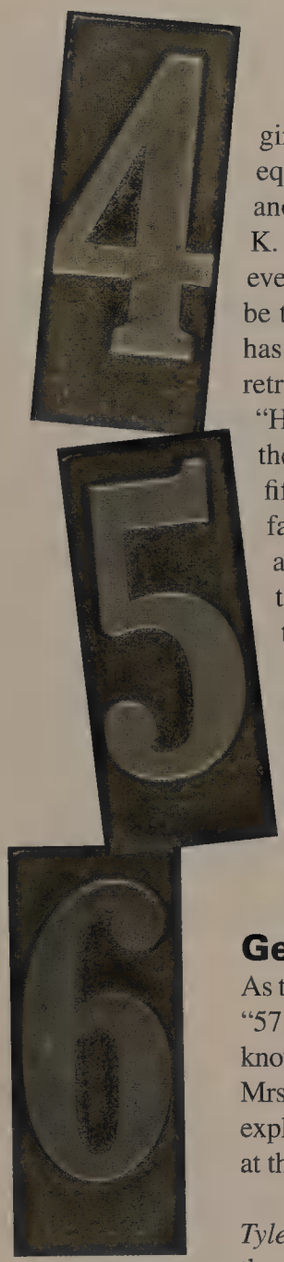
Thus far in their sharing, we learned that Keaton can count back by tens from one hundred with ease, although we do not know if he can do so from a number that does not end in zero. We saw that Winston is very interested in numbers and how they combine. He and other children know pairs of numbers that equal ten, and when Winston shares and Mrs. K. responds, the students see that making tens can be helpful when combining a string of single digits. Mrs. K. also pointed out to the children that just as three ones plus seven ones equal ten ones, three tens plus seven tens equal ten tens. She helps the children extend what they know about making ten by adding ones to making one hundred by combining ten tens.



Tyler shares

As the class continues, place value and ways to make ten remain the focus of the discussion.

Tyler: Fifty-seven plus thirteen.
Mrs. K.: This is one I'd like you guys to talk to each other [about]—you're going to talk to your partner. Now listen carefully. I look at the number seventy, and I look at that, and I think "Oh, I need to have seven tens there." Now look at what Tyler's got. He's got a fifty, so that's five tens, and then he's got a thirteen, so I see another ten there. So that's five, six tens. So I'm just wondering, where's that other ten? Why don't you talk to your partners about that?



A buzz of children's voices fills the room. One girl says to her partner, "Fifty plus a ten would equal sixty. And then seven plus a three will equal another ten. That would equal seventy." Here Mrs. K. has used a "turn and talk" technique to involve everyone in the mathematics; she wants everyone to be thinking about ten. She makes explicit that fifty has five tens and thirteen has one ten, although, in retrospect, she might have first asked the children, "How many tens does fifty have?" or "What does the five in fifty mean?" before telling them that fifty has five tens. Nonetheless, highlighting the fact that seventy has seven tens, fifty has five tens, and thirteen has one ten and asking the children to find the remaining ten communicates to them the importance of tens in our number system.

Mrs. K.: All right. Who would like to, who would like to explain that to us? Liza, what's your thought?
Liza: Fifty plus the ten equals sixty, and the seven plus the three is another ten. And that equals seventy.

Getting "unstuck"

As the discussion continues, Tyler, the boy who wrote "57 + 13," encounters difficulty explaining how he knows that fifty-seven plus thirteen equals seventy. Mrs. K. gently supports and guides him through his explanation, helping Tyler to clarify his thinking and at the same time reinforcing mathematical concepts.

Tyler: You know, like the five, the fifty-seven and the thirteen, so I added the thirteen, and then I added up seven, so then I went [*he stops talking and is quiet*].
Mrs. K.: So you had this [*she writes "57 + 13"*], and you wanted to check and see if it really was seventy?
Tyler: Yeah.
Mrs. K.: So how did you start?
Tyler: I started with the fifty-seven.
Mrs. K.: OK.
Tyler: So then I went up—fif—no, I went up thirteen again, and then …
Mrs. K.: You went up thirteen. What do you mean by that?
Tyler: I mean that when I was at fifty-seven, I went up thirteen.
Mrs. K.: Were you using a number grid [hundreds chart]?
Tyler: Yeah.
Mrs. K.: Go on.
Tyler: So then I went up thirteen again, and I thought to myself, that didn't equal seventy. So I went back thirteen.
Mrs. K.: I'm a little confused. So you're saying fifty-seven plus thirteen is not equal to seventy, or it is equal to seventy?
Tyler: It is.
Mrs. K.: It is. And how do you know for sure?
Tyler: Because thir—no.
Mrs. K.: You want to start with your fifty-seven?
Tyler: Yeah.
Mrs. K.: Okay. So I'm at fifty-seven, and what would you like to add to it first?
Tyler: I added a, a ten.
Mrs. K.: All right. So fifty-seven plus ten got you where?
Tyler: Got to … [*pause*]… Got to sixty-seven.
Mrs. K.: Did this help? [*She has drawn "57" and "67" in boxes, vertically, as they are on the hundreds grid and has written "57 + 10 = 67."*]
Tyler: Yeah.
Mrs. K.: Now he's at sixty-seven. So now what are you going to do?
Tyler: So then I plus. [*Tyler is silent.*]
Mrs. K.: Where did this ten come from?
Tyler: It came from the fifty-sss—from the thirteen. [*Mrs. K. points to the 1 in 13.*]
Mrs. K.: So this one stands for your ten, right? So, what have you got left?
Tyler: Three.
Mrs. K.: Three. And sixty-seven plus three more makes …?
Tyler: Equals seventy.
Mrs. K.: Nice job.

As we examine Mrs. K.'s words and actions, we note that first she asked Tyler for clarification: "You went up thirteen. What do you mean by that?" She then asked about the tool he may have used. Moving "up" thirteen suggests using a number line or number grid (hundreds chart). When Tyler was confused, Mrs. K. suggested that he start at fifty-seven. Tyler chose to add ten but could not go any further until Mrs. K. drew a visual—two boxes arranged vertically and labeled "57" and "67." After Tyler said, "Sixty-seven," she wrote, "57 + 10 = 67." In the discussion about where the ten came from, Mrs. K. reinforced place value. The ten is part of the thirteen, and now Tyler knows he has three left to add to sixty-seven. Mrs. K's scaffolding has helped Tyler, and very likely other children, to think about place value (the meaning of the one in thirteen) as well as how to add one ten to a number.

Adding tens and adding to make ten

The class continues as children share their ways to make seventy and Mrs. K. asks for one more child to explain where the seven tens are in 57 + 13.

Mrs. K.: Is there one more person who can tell me where the seven tens are? Okay, let's see. Gabriella.

Gabriella: Um … um … I took away the three and the seven for a while. And then …

Mrs. K.: So you ended up with a fifty and a ten.

Gabriella: Umhmm, and I knew that a fifty and a ten would equal sixty.

Mrs. K.: Umhmm.

Gabriella: Then plus three plus seven would equal, would equal ten. So fifty plus ten will equal sixty plus a ten will equal, will equal seventy.

Mrs. K: Nice going. So you broke that fifty-seven and that thirteen into fifty, ten, three, and seven. Nice job.

Dr. A: So she knew how to add tens. She knew that seven and three made ten.

Mrs. K: That's handy.

Child: We have it on our desks from you. That's why me and Jailene went to get it.

Mrs. K: Oh, because it's …

Child: Ways to make ten.

Gabriella: Equals ten.

Mrs. K.: And that's pretty important …. It came in useful when …

Gabriella: Winston.

Mrs. K.: Winston. When Keaten needed seventy; one hundred minus thirty.

Child: One hundred thirty-nine minus thirty-nine.

Mrs. K.: He had that one hundred minus thirty right in there.

Dr. A: And when Winston had added all those numbers.

Mrs. K.: And when Winston added all those numbers too. Nice job.

Gabriella, like Liza, sees that groups of ones, in this case seven and three, can be added to make ten. However, these children treat this sum of ten as another group of ten. As Liza put it, "Fifty plus the ten equals sixty, and the seven plus the three is another ten." Their discussion suggests that they see that ten can be ten ones or one group of ten, an important developmental step in understanding place value.

Decomposing the Number of the Day

Through this classroom routine (entitled "Today's Number" in *Mathematical Thinking at Grade 2* [Economopoulos et al. 1998, p. 40] and "The Name Collection Box" in *Everyday Mathematics* [Bell et al. 2002]), students have many opportunities to look at how numbers are composed, and, in doing so, they develop their understanding of the structure

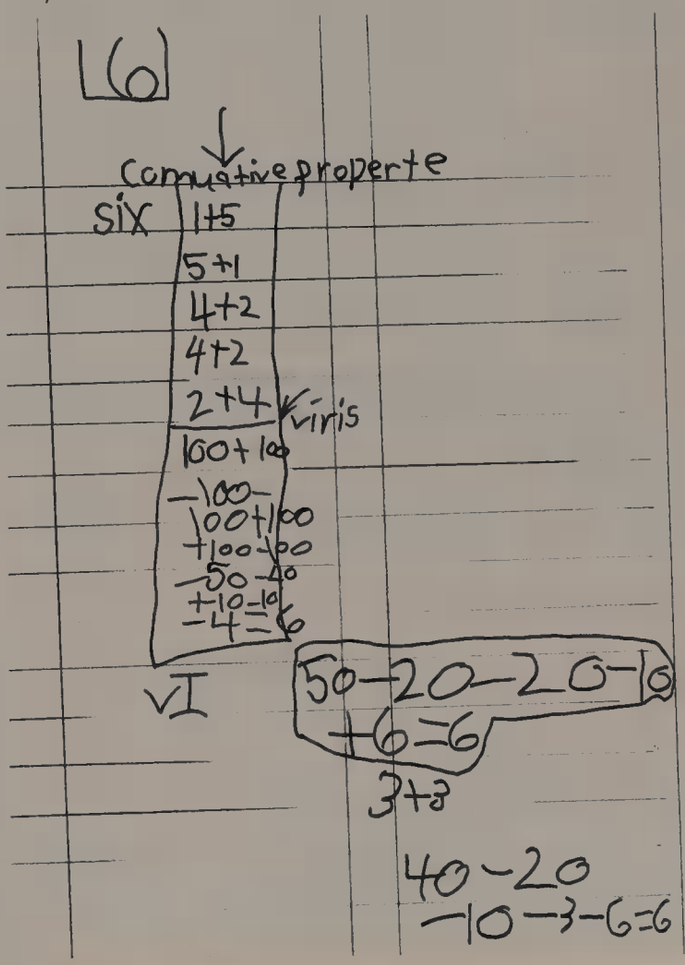
Figure 1

Examples of children's expressions for 18 are from early in the school year.

(a) Mrs. K. used the mathematical terms *expanded notation* and *commutative property*. She inserted parentheses to show how one student added fives (Kidd 2005).

10 + 8 = 18 expanded notation [the teacher added the label]
18 + 0 = 18 0 + 18 = 18 cmoohivpopde (commutative property)
19 – 1 = 18
13 + 5 = 18
42 – 24 = 18
(2 + 3) + (2 + 3) + (2 + 3) + 3 = 18 [the teacher added the parentheses]
17 + 1 = 18
100 + 100 –100 + 100 – 80 – 20 – 3 – 7 – 3 – 7– 3 – 7– 3 – 7 – 3 – 7– 3 – 7– 3 – 7 – 3 –7 – 3 –7 – 3 – 7 + 8 – 8 + 8 – 8 + 4 + 4

(b) The *commutative property* makes sense, and children remember the term when they have used it in their work.



of our number system as well as explore arithmetic operations.

As the children break apart numbers, they work with part-whole relationships. Part-whole relationships are often considered in the context of number concepts and conservation of number, as, for instance, the different combinations that make a certain quantity (e.g., 5 can be 2 and 3, or 1 and 4, etc.). Our view of part-whole relationships includes place value. Place value is more than identifying the ones, tens, and hundreds columns. Understanding groups of ten grows out of children's own logic as they work with numbers in meaningful activities such as decomposing numbers to make expressions equivalent to the number of the day. Over time and with many numerical experiences, children come to understand the relationship between numbers and groups of tens and ones (Burns 1993; Kamii 1985, 1989; Ross 1989), and the idea that a digit can simultaneously represent a number of tens or a number of ones (e.g., the 5 in 52 can represent five tens or fifty ones).

Figure 2

The constraint of using three addends was given on day 94. This student first decomposed the 4 and then the 90.

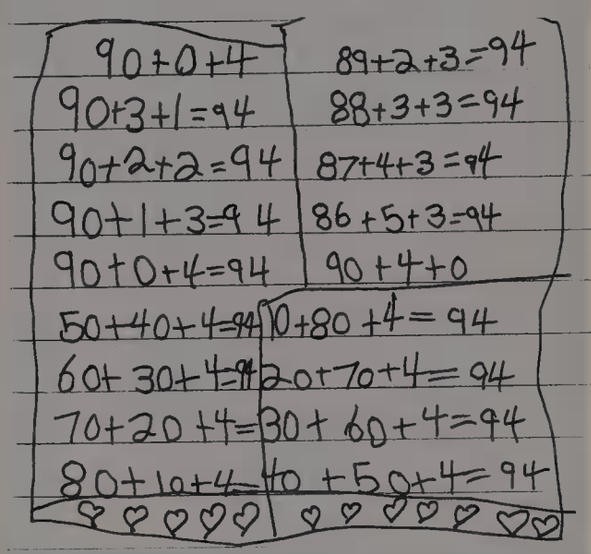


Figure 3

Here are five examples from day 108, when students were asked to write expressions using subtraction in this format: ___ – ___ = 108. Some children decomposed larger numbers, and others wrote story problems.

108

8,000 – 7,892

300 – 192

708 – 600

Heather has 109 dollars. She loses 4 quarters. How much money does she have now?

Liza knows 110 Russian words. She forgets 2. How many Russian words does she know?

How the activity evolves

A typical mathematics class, like day 70, begins with children working individually in their mathematics notebooks to create mathematical expressions equivalent to a particular number. As the children work, we circulate, ask questions, and offer help if needed. Our goal during this time is to encourage the children's own mathematical thinking. Questions can be general, such as, "How did you think about that?" and "How does your pattern work?" or more specific, such as, "What will happen if you add another ten?" and "Is there something you know about [another number] that you can use to help you?"

Next, the class comes together, and the children share and discuss their numerical expressions. Discussion is a critical component of the activity; children's ideas are made public, and mathematics is made explicit, as we saw in the vignette. We focus the children's attention on mathematical ideas that otherwise may remain embedded within the children's work. Although we have topics in mind, such as place value, topics may also come from the children themselves. Discussion topics have included the inverse relationship of addition and subtraction, negative numbers, and, of course, place value.

Early in the year, few constraints are imposed on the task. Very long expressions, such as the last one in **figure 1a**, capture children's interest; however, children have to learn to keep track of their computation. They learn about chunking numbers to subtract and about making zero; in one memorable class, the children come to the conclusion that if you subtract the same amount as you add (or vice versa), the result is zero. Opportunities arise for using mathematical language in context. Children remember the commutative property, and it makes sense to them, when it is connected to their own work (see **fig. 1b**). As the year continues, Mrs. K. provides mathematical constraints and challenges, and the children's work increases in sophistication as their knowledge of our number system grows (see **figs. 2** and **3**).

We believe that children's continued experiences decomposing number as they keep track of their school days encourages development of understanding place value as well as practice using the number system. Such experiences may very well provide a much-needed foundation for children's work with algorithms for addition and subtraction; these algorithms can be viewed as particular ways of decomposing to combine or separate numbers based on place value. Such experiences provide additional rewards, too. They support children in the quest to learn mathematics meaningfully and well, in taking risks, and in learning from one another.

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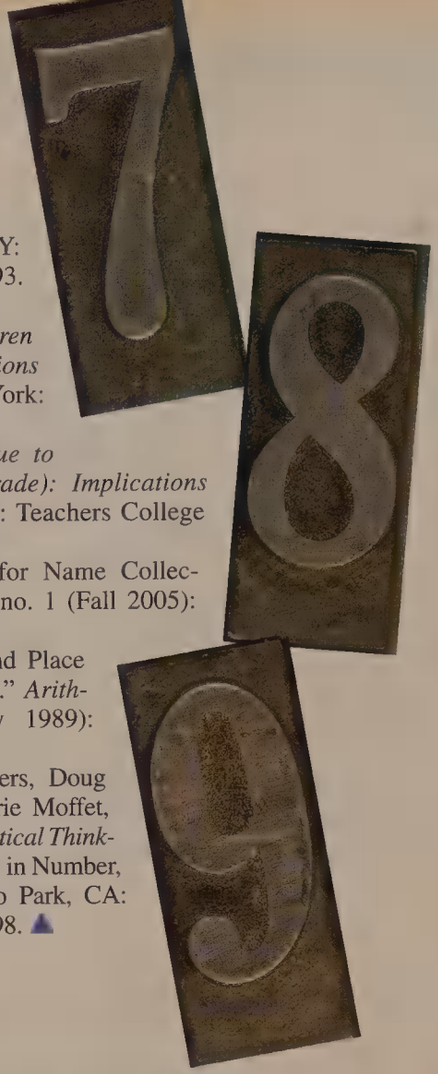

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Jodelle S. W. Magner

Reflections and Kaleidoscopes: Not Always What You Expect

The "Investigations" department features children's hands-on and minds-on explorations in mathematics and presents open-ended investigations to enhance mathematics instruction. These tasks invoke problem solving and reasoning, require communication skills, and connect various mathematical concepts and principles. The ideas presented here have been tested in classroom settings.

A mathematical investigation is—

- multidimensional in content;
- open ended, with several acceptable solutions;
- an exploration requiring a full class period or longer to complete;
- centered on a theme or event; and
- often embedded in a focus or driving question.

In addition, a mathematical investigation involves processes that include—

- researching outside sources;
- collecting data;
- collaborating with peers; and
- using multiple strategies to reach conclusions.

Although this department presents a scripted sequence and set of directions for a mathematical exploration for the purpose of communicating what happened in this particular classroom, *Principles and Standards for School Mathematics* (NCTM 2000) encourages teachers and students to explore multiple approaches and representations when engaging in mathematical activities. In the Reflections and Kaleidoscopes: Not Always What You Expect investigation, students actively engage in investigating the geometric concepts of reflections, as well as making conjectures about new situations. At points, students may experience dissonance between what they expect will happen and what actually occurs. The investigation features the history and mathematics of kaleidoscopes. Students first use miras and then two hinged mirrors to investigate reflections. The investigation culminates with the students creating their own kaleidoscopes and discussing the mathematical reasoning behind the number of images that they see. This investigation addresses NCTM's Geometry Standard (NCTM 2000).

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Edited by **Sue McMillen**, *mcmillse@buffalostate.edu, who teaches mathematics and mathematics education courses at Buffalo State College; and* **LouAnn H. Lovin**, *lovinla@jmu.edu, who teaches mathematics methods and content courses to prospective elementary teachers and classroom teachers at James Madison University in Harrisonburg, Virginia.* *"Investigations" highlights classroom-tested multilesson units that develop conceptual understanding of mathematics topics.* *This material can be reproduced by classroom teachers for use with their own students without requesting permission from the National Council of Teachers of Mathematics.* *Readers are encouraged to submit manuscripts appropriate for this department by accessing* **tcm.msubmit.net**. *Manuscripts must include no more than twelve double-spaced typed pages and two reproducible pages of activities.*

The Investigation

Learning goals, rationale, and pedagogical context

This investigation highlights the fact that "Geometry [and all areas of mathematics for that matter] is more than definitions; it is about describing

relationships and reasoning" (NCTM 2000, p. 41). Throughout the study of mathematics, students need to develop and apply geometric reasoning, visualization, and spatial reasoning skills.

During this investigation, students are encouraged to develop their reasoning and justification abilities by posing predictions, making observations, and explaining their observations. Students explore the concepts of reflection and symmetry. These lessons were taught to two classes of sixth graders at Waterfront Elementary public school. Each class had English Language Learners (ELLs) and students with individual educational plans.

Objectives of the investigation

Students will—

- understand and represent reflections by using sketches;
- describe and identify reflections in the plane; and
- explore the concept of symmetry and its connection to reflections.

Materials

For all the lessons, students should be in groups of four consisting of two pairs. It is best that students do not work alone.

Lesson 1

Each student will need—

- markers;
- miras (single mirrors can be used);
- **activity sheet 1**; and
- **activity sheet 2**.

Lessons 2–3

Each student will need—

- 2 copies of the curlicue rectangles (see **fig. 1**). Students cut out the images and fold them along the dotted line;
- hinged mirrors;
- **activity sheet 2**;
- **activity sheet 3**; and
- **activity sheet 4**.

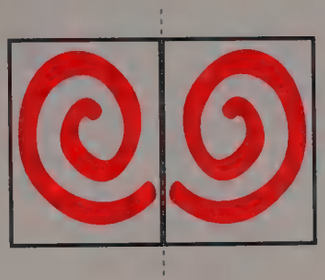
Lessons 3–4: Kaleidoscope-Making Supplies

Each student will need—

- 1 cardboard toilet paper roll (not the mega roll size);
- 1 disposable 60 mm plastic petri dish;

Figure 1

Curlicue rectangle



- 1 adhesive Mylar sheet with backing (available in poster-size sheets from art supply stores), cut into a 4 3/8-inch by 5 1/4-inch rectangle;
- pencils and markers; and
- scissors.

Each group of four students will need—

- 2 sheets of patty paper; and
- 1 card-stock square the same size as the patty paper.

Other supplies to have available include—

- duct tape, one roll for each group, your choice of color; and
- plastic beads of various shapes and colors. Beads should be at least 5 mm in diameter. For a class of 20, use approximately one 160 gram box of 5 mm beads, one bag of 18 mm 210-count starflake beads, and one bag of 875 tri-shaped beads.

Previous knowledge

The students in this investigation had no formal instruction on transformations. They had some experience justifying their mathematical thinking but limited experience in making and justifying conjectures.

Lesson 1: Reflections in One Mirror or Mira

Begin the investigation by asking students what they see when they look in a mirror. Typical responses might include such statements as "Yourself" or "Whatever is in front of it." The word *reflection* will likely be mentioned in the discussion. Most students know that a reflection is an opposite image,

but describing the phenomenon can prove challenging. Have students focus on precise language in their descriptions.

The mirror game is one way to help students gain understanding of reflections. Students play the mirror game in pairs. Two students stand facing each other. One partner is the "mirror" and has to reflect the other's actions. After a few minutes, students switch roles so that each student has a turn to be the mirror. When they finish playing the game, have the entire class develop a definition of *reflection* together. Students can describe how they created a reflection when they were acting as the mirror. For example, if the first student lifted her *left* hand, then the "reflection" had to lift his *right* hand.

Distribute **activity sheet 1** to the students. Explain that just as a mirror shows reflections, so does a mira. Ask students to draw the image they think they will see in the mira and then describe it in words. Allow students who have difficulty visualizing the reflection to manipulate the two curlicue rectangles that they created from **fig. 1**. This offers them the opportunity to physically create a reflection with the curlicue rectangles and avoid relying solely on mental visualization or drawing skills. After students have had time to complete this on their own, have them share what they drew and why. A common error at this point is for students to show a translation and not a reflection (see **fig. 2a**). The use of the mira will help students visualize the correct image.

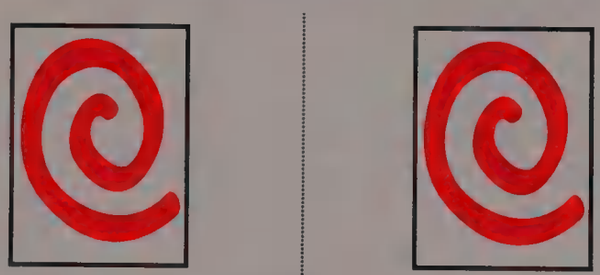
Next, distribute the miras and ask students to use them to see the reflection and record what they see. As a class, have students discuss their predictions and compare them to the image they see in the mira. This is a good time to highlight the fact that the reflection is the same size and shape as the original, yet the orientation has changed.

To reinforce students' ability to predict a reflection, ask them to write the first initial of their name, predict what a vertical line of reflection will look like, and then use the miras to see if their prediction is correct. If time allows, challenge students to find capital letters whose vertical reflection looks exactly the same as the original (see **fig. 2b**). This is an excellent time for the teacher to connect this concept to an object's lines of symmetry. If a letter or shape has a vertical line of symmetry, then its reflection across a vertical line will look exactly as the original looks. For example, the letter **A** has a vertical line of symmetry; if one looks at a reflection of half the letter, one will see the other half. The mira is not placed on the letter's actual line of symmetry but is used to verify that a vertical line of symmetry exists.

Before the next class, have students color at least some of the patterned triangle on **activity sheet 2**. Colors will allow students to describe what they see when they discuss the reflections in the hinged mirrors. Using different colors in the strips near the right angle is particularly useful.

Figure 2

(a) An incorrect prediction of a reflection



(b) A capital letter whose vertical reflection looks exactly the same as the original

Lesson 2–3: Reflections in Two Mirrors

Start this day of the investigation by asking students to share what they know about kaleidoscopes and what is inside them. Some students may know that kaleidoscopes are based on mathematics. Others may think of them as somewhat mysterious toys with a collection of pictures inside.

Introduce students to the people behind the invention of the kaleidoscope. Scottish scientist Sir David Brewster (see **fig. 3a**) is credited with inventing the first kaleidoscope in 1816. It had only two mirrors. In 1852 August Ferdinand Möbius (see **fig. 3b**) created a different type of kaleidoscope by using three mirrors that met at a common, fixed point.

Photograph by Gay Dillin; all rights reserved

Hand out **activity sheet 3**. Students will again use the curlicue, but this time they will use a hinged mirror. Ask the students to predict and justify what they might see now. In our investigation, Liznette commented that she drew two images of the curlicue as her prediction, one for the mirror on each side of the hinge. Other students in the class nodded their heads in agreement.

Next, distribute the hinged mirrors and have the students place the mirrors on the dotted lines on **activity sheet 3**, forming a right angle around the curlicue. Most students are very surprised to see three images rather than two. Discuss why there are three images. Guide students to observe that they are seeing the reflection of a reflection. The reflections can be created by using a set of four curlicues with the image also traced on the back. The reflections in each mirror are the ones the students have predicted.

Next, increase the angle between the mirrors on the activity sheet. Return the mirror to the original position, reflect the curlicue, and then reflect it again across the expanded mirror. You will see the other image. So, when you set the hinged mirror on the corner of the rectangle, you will see a total of four curlicues: the original curlicue and three reflections.

Now ask students to place **activity sheet 2** in front of them and predict what they will see (and why) if they place the mirrors on each leg of the triangle, with the hinge at the right angle. The most common response is, "I will see three triangles: the original and one in each mirror."

Now have the students look in the mirrors. Lining up the mirrors and the legs of the triangle is an important detail for getting the best images. Our classroom filled with exclamations: "Cool!" "Wow! That is so neat! I see a circle!"

"A circle?" I responded, "But you started with a triangle." A student quickly pointed out that the triangle arc is reflected three times, and so we see a circle (see **fig. 4**). Things are not always as expected!

Have the students investigate informally with the mirrors on their triangles. After some initial exploration, ask them to place their triangles in front of them with the right angle (the hinge) on the left side of the paper and a mirror on each leg of the triangle. Ask them to investigate what happens when they change the angle between the mirrors by keeping the left mirror still and moving the right mirror. One student in our investigation shouted, "I see more [images] as the mirrors get closer [together]." More formally stated, as the angle decreases between the mirrors, the number of images increases.

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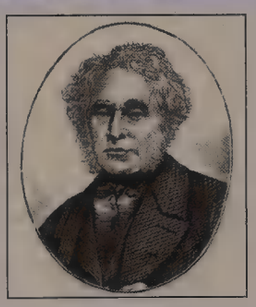
Figure 3



(a) Sir David Brewster (en.wikipedia.org/wiki/David_Brewster)

(b) August Ferdinand Möbius (en.wikipedia.org/wiki/August_Ferdinand_M%C3%B6bius)

Figure 4

When the triangle arc is reflected three times, we see a circle.

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Distribute **activity sheet 4**, a guide to exploring the connection between angle measurement and the number of images. Have students move the mirror to the 60 on the arc, creating a 60° angle between the two sides of their mirrors. They can look into the mirrors and see something wonderful. Direct students to look at the shorter, dark line segment toward the center and its reflections, which form a shape similar to a star (see **fig. 5**). Have them count the number of points they see on that star. Next, ask them to look at the closed figure near the center of the mirror and count its number of sides. Have them record their observations in the table on the activity sheet. They should then change the angle, moving only their right hand, and repeat the process until they have completed the table. If students are having difficulty with seeing the sides of the figure, have them place a pencil between the mirrors and slide the pencil left to right, counting as the pencil travels around the image.

Figure 5

The picture of a star highlights the star points.

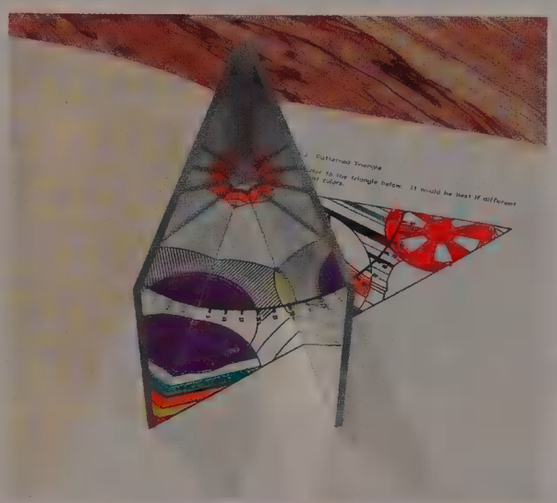

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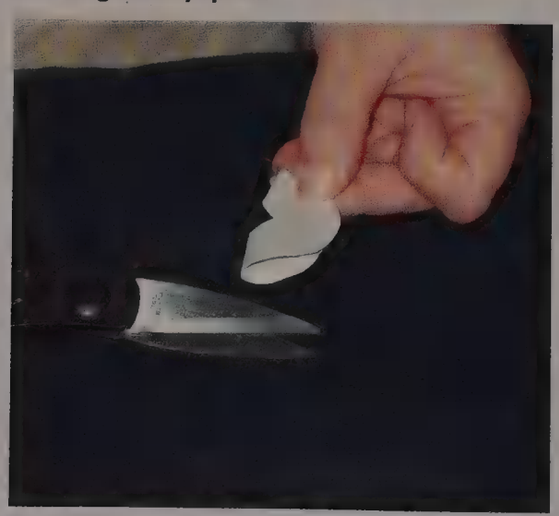
Figure 6

Cutting the eyepiece



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After students have completed **activity sheet 4**, ask them to make conjectures about the angle between the mirrors, the number of star points made by the dark line segment, and the closed figure's number of sides. Students may notice that each of the angles they used were factors of 180. Guide students to note the connection between the number of sides, the number of star points, and the factors of 180 (angle measure). In particular, when the angle measure n is a factor of 180 and is less than 90 degrees, the number of star points on the star made from the dark line segment is $180/n$, and the number of sides in the closed figure is $360/n$ (two times the number of star points). Also, the number of reflections in the mirrors is $360/n - 1$, because one side of the closed figure is the original segment.

Lessons 3–4: Making a Kaleidoscope

This investigation culminates with students making their own kaleidoscopes and then discussing the mathematical reasoning behind the images they see. Each student may make each component of the kaleidoscope on his or her own, but I have found it best to assign each student a number (1 through 4) and set up tasks so that each student creates a specific piece of the kaleidoscope for everyone in their group. When all the pieces are complete, the whole class can assemble the kaleidoscopes at the same time. Each group of four should have two sheets of patty paper, one sheet of card stock, four petri dishes (the smaller part and the larger part), four cardboard tubes, four sheets of Mylar, and a roll of duct tape. To avoid unnecessary fingerprints, you may want to distribute the Mylar after the eyepieces have been attached.

Make the following student assignments:

- Person 1 traces the small part of the petri dish four times onto patty paper and cuts out the circles.
- Person 2 traces the end of a cardboard tube onto card stock four times and cuts out the circles.
- Duct tape is a bit difficult to cut, so students will need to work together to keep it taut while cutting. Person 3 holds the duct tape.
- Person 4 cuts the tape. Working together, person 3 and person 4 cut eight tape strips so that

the strips are long enough (about eight inches) to go around the cardboard tubes one-and-a-half times.

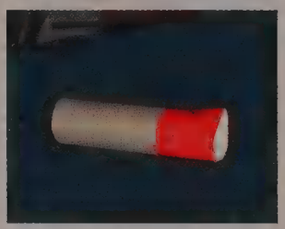
Once all the pieces are made, each person should have one circle of patty paper, one circle of card stock, two pieces of duct tape, one cardboard tube, and one petri dish. Their first task is to place the patty paper circle in the larger side of the petri dish and set it aside. The next step is to make the eyepiece by trimming a small hole out of the card stock circle. Do this by pinching the circle in the middle and trimming a small triangle out of the middle (see **fig. 6**). To check the size, place the eyepiece on top of the cardboard tube. It does not need to extend over the tube too much, but it should not fall through.

Next, take a piece of duct tape and wrap it around one end of the cardboard tube (see **fig. 7a**). Cut the tape in four equidistant points on the circumference and peel it back to make an **X** (see **fig. 7b**). Place the eyepiece on the **X** and fold the tape over the eyepiece, being careful not to cover the hole (see **fig. 7c**).

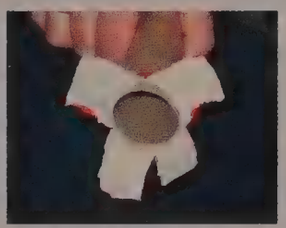
Show students how to fold the Mylar in half, shiny side to shiny side, matching up the 4 3/8-inch sides while keeping fingerprints to a minimum. Try to make the creases sharp, because the creases hold the prism together. Unfold the Mylar and fold each side into the middle seam. Finally, shape the Mylar into a triangular prism. If the prism does not stay together, place a small piece of Scotch tape on the side seam. Next, place the prism into the cardboard roll and attach the second piece of duct tape to the end of the tube opposite the eyepiece. Cut the tape, creating another **X** shape.

Figure 7

Attaching the eyepiece



(a) Wrap the duct tape around one end of the cardboard tube.



(b) Cut the tape on four equidistant points and peel the sections back as far as the tube to make an **X**.

(c) Center the eyepiece on the **X** and fold the tape over the eyepiece, being careful not to cover the hole.

Photographs by Jodelle S. W. Magner; all rights reserved

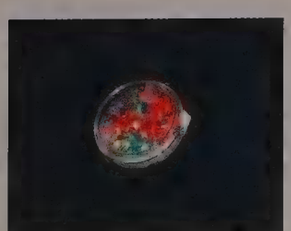
While some students are taping and cutting the rolls, others can select beads for the petri dishes. Caution students to choose enough beads to be seen but to leave enough room in the petri dish so that the beads can move when the lid is on (see **fig. 8a**). They will put the beads in the smaller part of the dish, place the lid on it, and then place the dish on the desk, large side down.

Have each student now complete a kaleidoscope by pressing the **X** on the tube's open end onto the top of a petri dish, small side up, and wrapping the strips of tape around the dish (see **figs. 8b, c,** and **d**). They may use the triangle they colored on **activity sheet 2** as a wrapper for their kaleidoscopes (see **fig. 8e**).

Encourage students to look into their kaleidoscopes at the point in the center where the angles share a common vertex. Engage the students in a discussion about why they see six images. If the students are having difficulty explaining, you may

Figure 8

Completing the investigation

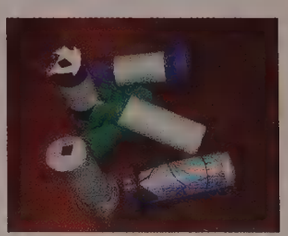


(a) Choose enough beads to be seen but leave enough room in the petri dish so that the beads can move when the lid is on.

(b) Press the open side of the tube onto the top (small side) of the petri dish.

(c) Wrap tape strips around the dish.

(d) A kaleidoscope!



(e) Use the colored triangle as a wrapper.

Photographs by Jodelle S. W. Magner; all rights reserved

want to call their attention to **activity sheets 2** and **3**, or arrange the hinged mirrors on **activity sheet 2** with a 60-degree angle.

Beyond the Lesson

Although we created a three-mirrored kaleidoscope, this investigation focuses on reflections with one or two mirrors. If you have advanced students, you may want to investigate similarly with three mirrors as David Gay does in *Geometry by Discovery* (1998).

If computers and Internet access are available, the following Web site allows students to create a virtual kaleidoscope: www.zefrank.com/byokal/kal2.html. Students can drag various shapes into the "dish" and see the image inside the kaleidoscope. They can explore what happens when they use different-shaped pieces and when they move shapes closer to the center of the circle. Older students and teachers may enjoy the following Web site, which allows for more creativity: www.zefrank.com/dtoy_vs_byokal/index.html.

Concluding Remarks

This investigation models the exploration approach, which can be fostered throughout the school year. Students enjoy being surprised by the number of images in the hinged mirrors. Invite another adult or two to assist in the class on the day you make the kaleidoscopes. Having students select their own beads is very time consuming. If you are at all pressed for time, you may want to hand out baggies of beads or put the beads in the petri dishes ahead of time.

Students' investigations with both the mirrors and the mirror game strengthen their conceptual knowledge of reflections and symmetry. Using the curlicue, the hinged mirrors, and the kaleidoscopes can help students connect concepts with physical objects.

References

Gay, David. *Geometry by Discovery*. New York: John Wiley & Sons, 1998.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000. ▲

Activity Sheet 1. Reflections and Kaleidoscopes Name _____

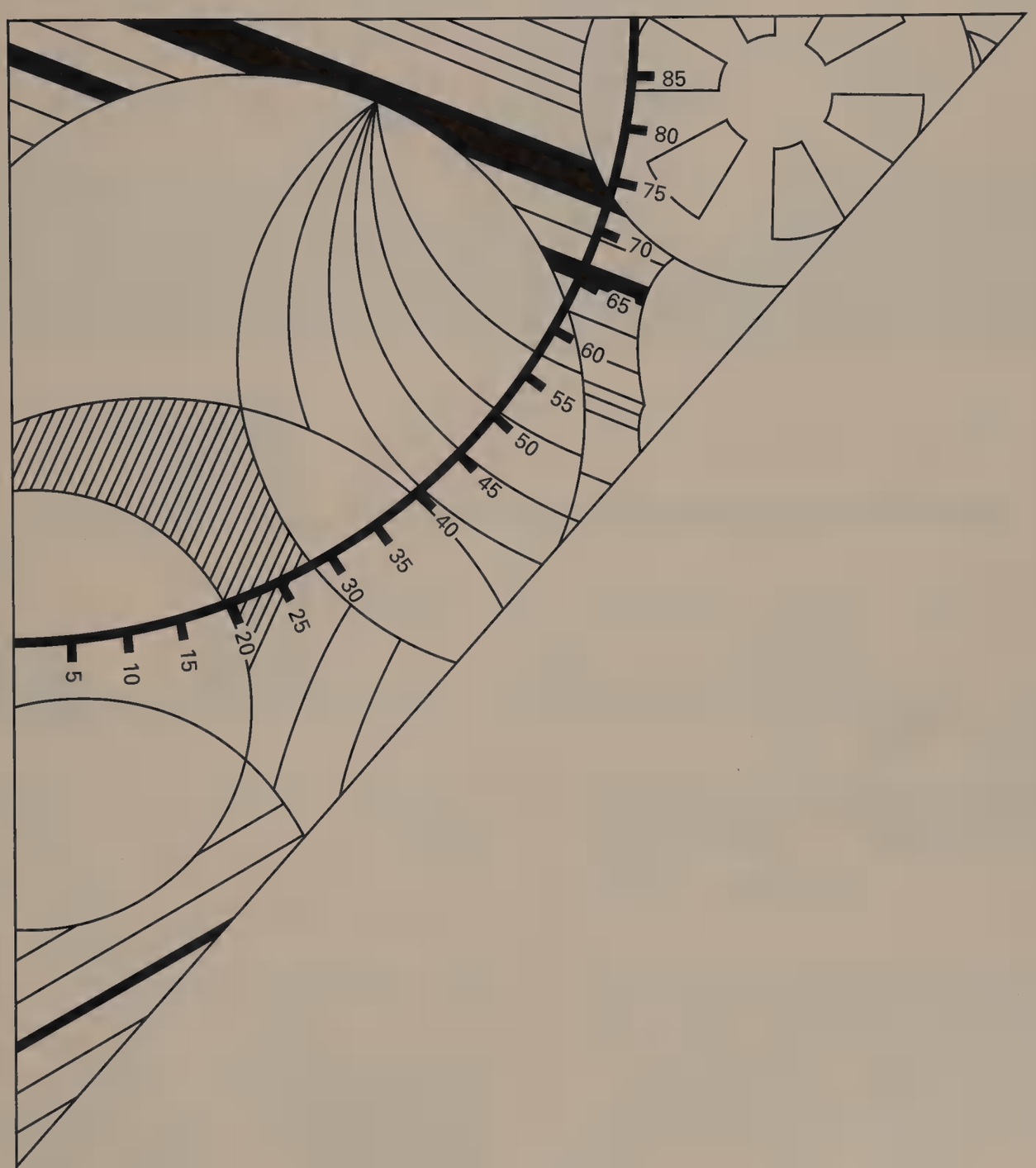
REFLECTIONS IN A MIRA OR ONE MIRROR

Predict what you will see in the mira when you place your curlicue rectangle next to it. Draw your prediction and use words to describe your drawing.	After you have made a prediction, use the mira and draw what you see. Use words to describe your drawing.
Mira	Mira

Activity Sheet 2. Reflections and Kaleidoscopes Name _____

PATTERNED TRIANGLE

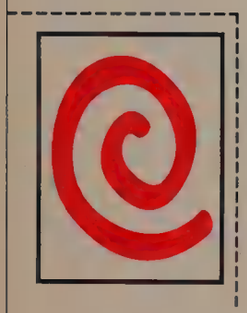
Add some color to the triangle below. Coloring different sections in different colors works best (Gay 1998, p. 191).



From the September 2008 issue of TEACHING CHILDREN Mathematics

Activity Sheet 3. Reflections and Kaleidoscopes Name _____

REFLECTIONS IN TWO MIRRORS

Predict what you will see in the hinged mirrors when you place your curlicue rectangle next to it. Draw your prediction and use words to describe your drawing.	After you have made a prediction, use the hinged mirrors and draw what you see. Use words to describe your drawing.
	

_____ _____

_____ _____

_____ _____

From the September 2008 issue of TEACHING CHILDREN Mathematics

Activity Sheet 4. Reflections and Kaleidoscopes Name _____

ANGLE OBSERVATIONS

Angle in Degrees	No. of Star Points on Star	No. of Sides on Closed Figure	No. of Reflections in Mirrors
60			
45			
36			
30			
20			
18			

From the September 2008 issue of TEACHING CHILDREN Mathematics

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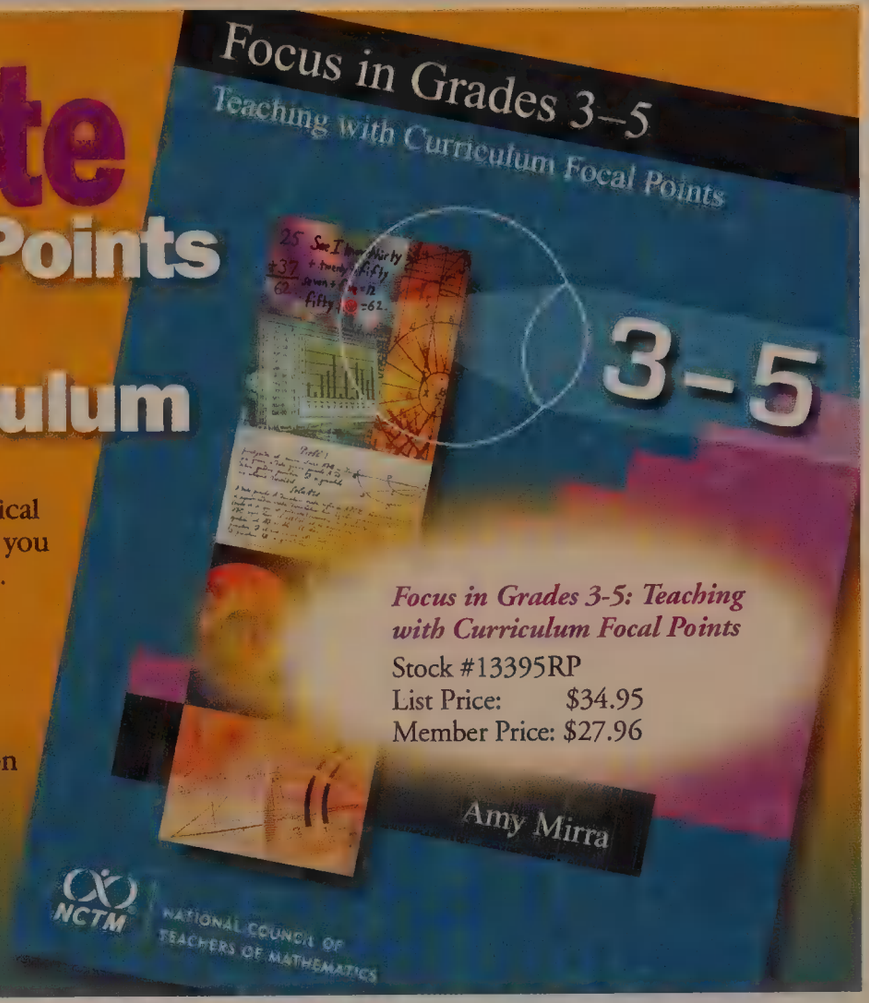

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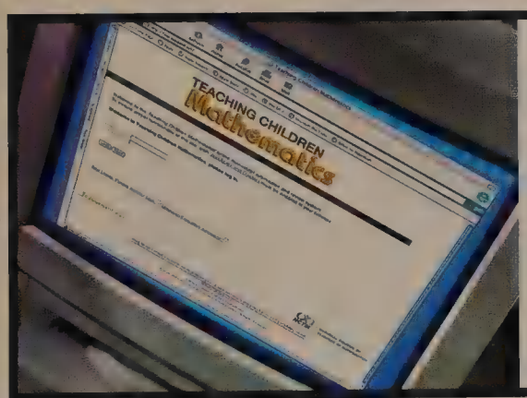

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Good news! NCTM has simplified the manuscript submission process This new streamlined process allows you to submit your material online and then, by accessing your personal Web page on the submission site, monitor it as the manuscript progresses through the steps of the review process. The process saves both time and mounds of paper while allowing you, the author, to get your accepted manuscript published much faster.

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1. Begin by accessing the *TCM* electronic submission Web site at **tcm.msubmit.net**.
2. Log in to the site. (*Note:* The password is *not* the same password as used in the Members Only section of the nctm.org site.) If you do not have a login name and password for the manuscript submission site, click on New Users: Please register here. You need to register as a *TCM* author only once. Your registration information will be stored in the system, saving you even more time with subsequent submissions.
3. Under **Author Tasks**, click on Author Instructions. After you have followed the step-by-step instructions and your manuscript is complete, you are ready for the last step.
4. Click on Submit Manuscripts.

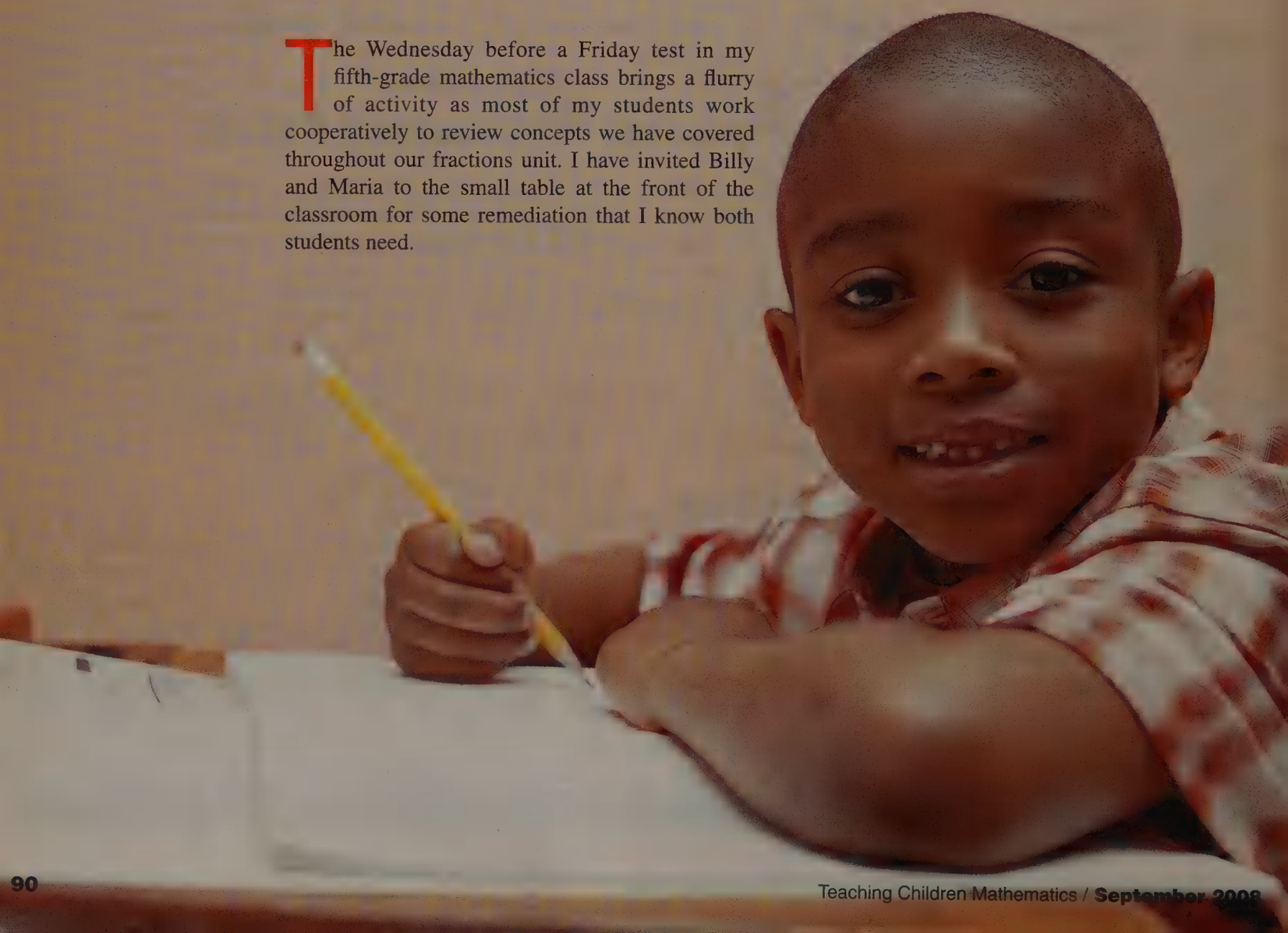
You will upload your manuscript file to this site, and the process will begin. A red arrow will alert you to any information required or action you need take. An acknowledgment e-mail will be sent to you, explaining how to log back into the system and check the status of your article at any time. If you have questions, clicking on HELP in the top bar will direct you to various question marks—noted as (?)—spread throughout the site.

We look forward to receiving your online submissions. Thank you for writing.

Students Control Their Own Learning:

A Metacognitive Approach

The Wednesday before a Friday test in my fifth-grade mathematics class brings a flurry of activity as most of my students work cooperatively to review concepts we have covered throughout our fractions unit. I have invited Billy and Maria to the small table at the front of the classroom for some remediation that I know both students need.



"Let's talk again about finding equivalent fractions," I hear myself saying. "We have a big test coming up."

"Ugh, I hate equivalent fractions," they groan, almost in unison.

Realizing that fractions are here to stay, Maria decides to explain her thinking. "Well," she says, "you multiply the first fraction by fifteen and get bigger numbers for both the numerator and denominator. Like when you multiply one-half by fifteen, you get fifteen-thirtieths."

"But, when you multiply any number by fifteen, the product is always fifteen times larger than the original number," I reply. "For instance, one times fifteen equals fifteen. Isn't that true when I multiply a fraction by fifteen?"

The students look at each other and then back at me. I know that as they are pondering my question, they are simultaneously wondering if I am trying to trick them. I wait, silently hoping they will see where I am going with my question. As the seconds tick by, it is clear to me that these students still are not fully grasping the identity property. Finally, Billy realizes what I am saying.

"Oh, I know. You mean that we have to multiply one-half by fifteen-fifteenths, not just by fifteen, to get the [equivalent] fraction because fifteen-fifteenths is the same as one whole. You know, how like 'Secret Agent One' can disguise himself as any number over itself!"

Off to a slow start but increasingly gaining speed, we continue the review that might never have happened without preassessment student self-reflections.

How Do I Study?

I admit that not too long ago, I was guilty of saying such things as, "Study for your fractions test on Friday," and then not revisiting the idea of studying again. After some enlightening discussions with my students about what they do to prepare for assessments, I started to wonder how effective this directive was. I soon realized that most of them have no idea where and how to begin reviewing, let alone what it means to "study." If they even remembered to take home their binders, most of them admitted they would just flip through the papers inside, glancing over work that I had returned to them. How could they study without any idea of what they know and do not know? I tried different strategies, such as giving students a list of concepts, with little success. If they did anything in response, they would look over the list and mentally check off each concept, believing they already knew everything about each one.

After additional discussion, I discovered that once students take an assessment, rarely do they take the time to sift through the concepts to determine which ones they have or have not mastered. Once a test is over, most of them do not think about it again.

I teach fifth grade in a high-risk, low-income school. The student body is composed of 47 percent limited-English speakers, 49 percent on free or reduced-cost lunch status, and 16 percent special education students. Parent involvement is generally minimal. Nevertheless, I believe it is not only possible but also critical for *all* students to learn how to take responsibility for their learning. Over the past few years, I have learned several points that benefit my students and make me a better teacher:

- Many students in the elementary grades do not know how to "study" for mathematics assessments when teachers ask them to do so.
- When students have a solid rapport with their teacher, they are honest about what they do and do not already know.
- Students must internalize information about their own mathematical understanding to make the information effective (i.e., they need to use a metacognitive approach).

To help facilitate accountability, I have developed a preassessment student self-evaluation diagnostic instrument, as well as a postassessment student self-evaluation instrument. After just a few opportunities to see these self-evaluations modeled, students can take control of their own learning before, during, and after the assessments.

What Do You Know?

The goal of preassessment student self-evaluations (hereafter called preassessments) is to help students recognize what essential knowledge they have mastered and which concepts they still struggle with. Armed with this knowledge about their understanding,

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Ellen T. Wieser, Ellen.Wieser@fcps.edu, teaches fifth grade at Westlawn Elementary in Falls Church, Virginia. She continues to encourage her students to self-evaluate their skills both before and after assessments and enjoys watching her students take responsibility for their own learning and achievement.

students can then determine what they need to study and which strategies they can use to master material that is still eluding them.

Preassessment student self-evaluations

Preassessments give students study concepts in advance and require them to explain their thinking for each one. For instance, usually four or five days before an assessment (i.e., a test or quiz), students receive a preassessment with six target concepts or sample problems listed down the left column. Students individually review each concept and then determine whether they know the content. Those who know the content solidify their understanding by completing the "Explain" section of the preassessment, using words, pictures, numbers, or symbols.

However, students who realize they cannot explain a concept in detail must follow a different route. They determine which strategy to use to learn the material. The choices include consulting the teacher, reviewing notes, looking in an interactive notebook, and so on. Once students have sought specialized help and feel they have mastered a concept, they complete the "Explain" section and give the preassessment to me for review. This allows me (and, potentially, parents) to catch any misconceptions and offer extra assistance before the assessments are given.

Billy, from the opening scene of this article, is a good example of how preassessments work. When Billy took the preassessment for our second fractions quiz (see **fig. 1**), he recognized that he did not fully grasp two of the concepts (ordering fractions with unlike denominators and ordering fractions and decimals). He chose the strategy "Ask a Friend or Family Member for Help." After receiving the help, he filled in the "Explain" section of the preassessment with what he had learned. In both cases, he ended up with correct answers; however, he still clearly lacked understanding, namely of the identity property. Being aware of Billy's misconception *before* the assessment prompted me to sit down with him and help him understand the two concepts, as evidenced by how the scene played out.

Figure 1

Billy's preassessment for a second fractions quiz

Review each concept below. If you feel that you understand the concept, explain it. If you feel that you don't quite understand the concept fully, determine which study method would best help you.

CONCEPT or SAMPLE QUESTION	DO YOU UNDERSTAND THIS CONCEPT?	
• Put the following fractions in order from least to greatest: $\frac{1}{2}, \frac{5}{6}, \frac{3}{10}$	YES ---------->	Explain: $\frac{1}{2} \times 15 = \frac{15}{30}$ $\frac{3}{10}$ $\frac{1}{2}$ $\frac{5}{6}$ $\frac{5}{6} \times 5 = \frac{25}{30}$ $\frac{3}{10} \times 3 = \frac{9}{30}$
	NO, but I am going to do this to help my understanding	review past assignments/notes; ask a friend/family member; draw picture; other:_____; make flashcards
• Which decimal is equivalent to $\frac{1}{4}$?	YES ---------->	Explain: $\frac{1}{4} = 0.25$
	NO, but I am going to do this to help my understanding	review past assignments/notes; ask a friend/family member; draw picture; other:_____; make flashcards
• Order the following numbers from **greatest** to least: $\frac{1}{5}, 0.9, \frac{4}{5}$	YES ---------->	Explain: $\frac{1}{5} \times 2 = \frac{2}{10}$ $\frac{9}{10}$ $\frac{8}{10}$ $\frac{2}{10}$ $\frac{4}{5} \times 2 = \frac{8}{10}$ 0.9 $\frac{4}{5}$ $\frac{1}{5}$ $0.9\ \frac{9}{10}$
	NO, but I am going to do this to help my understanding	review past assignments/notes; ask a friend/family member; draw picture; other:_____; make flashcards

Postassessment student self-evaluations

Postassessment student self-evaluations (hereafter, postassessments) have also proved to be powerful tools for students, parents, and me. After completing each assessment but before turning it in to be graded, students rate each question based on how they feel they did on it. Options range from "It Is Very Easy" to "I Have No Idea What the Question Is Asking of Me." Postassessments allow me to determine what specific concepts still need remediation and what areas have been sufficiently mastered.

For postassessments to be effective, building rapport, trust, and mutual respect with all my students is imperative. I am, after all, asking them to admit their weaknesses to themselves and to me. Before each assessment, I remind them that postassessments are just between us. This reminder encourages the students to feel comfortable enough to admit when they still might not understand a concept.

Postassessments also eliminate the problem of "lucky guessing." If a student marks the choice, "I Have No Idea What the Question Is Asking of Me" for a specific question but still manages to guess the correct answer, I will not mistakenly assume that the student understands the content. This was exactly the case with Gary's postassessment for our decimals test (see **fig. 2**). He received a perfect 15/15

on the test yet marked on his postassessment that he had "No Idea What [Question 14] Is Asking of Me." Simply looking at his score might lead me to believe that he has mastered all of the concepts and is ready to move on. However, because Gary was honest and explained that he did not understand question 14 at all, I was able to review and reteach the concept.

Beyond catching Gary's misunderstanding of how to divide decimals (question 14), further analysis of my other students' postassessments led me to fascinating and useful results about my teaching. I soon noticed a remarkable trend once I read Becky's postassessment (see **fig. 3**). She seemed very confident with questions 1–11 and 15, marking that she felt that each question was "Very Easy." Because she got each of those questions correct on her test, Becky, her parents, and I were all assured that she had mastered the material. However, she clearly marked question 14 as one that she did not feel she had mastered; she marked that she needed "A LOT More Help." Several other students also singled out question 14 as being extremely difficult, so I concluded that my approach to teaching decimal division was not effective. Here is a classic case where teaching and learning were not connecting; I was failing to make a concept clear to my students. It was evident that I needed to approach decimal division from a different angle and try it again. As a result, I spoke with several other mathematics teachers, including our school's Title 1 mathematics specialist, to ask how they tackled decimal division. I was able to apply some of their strategies to reteach the concept.

In some postassessments, I include a final question: "Which numbers, if any, did you correct while checking over your work?" to emphasize to students the importance and value of reviewing their work before turning it in to be graded, as demonstrated with Erica (see **fig. 4**). Seeing how much a grade is affected by taking the time to check their work can be empowering for students.

"Erica, super job on getting a perfect score on your test! Tell me about question 14 since you wrote that you caught a mistake and corrected it."

"Well, at first I got 45.5 for my answer, so I circled A."

"Uh-huh."

"But, then I was going back through, and I tried to reverse the division by using multiplication."

"Keep going."

"Well, when I multiplied 45.5 by 4, I got 182.0. But the number under the house, uh, the dividend, for the division problem was 18.2."

Figure 2

Gary's postassessment

Decimals Unit Test Reflection

Put a check next to each number that describes how you feel about it. Remember, this is between you and me; be honest.

Question Number	It is Very Easy	I Feel Pretty Good About This	I Could Use Some More Help	I Need A LOT More Help	I Have No Idea What the Question is Asking of Me
1.	✓				
2.	✓				
3.	✓				
4.	✓				
5.	✓				
6.	✓				
7.	✓				
8.	✓				
9.	✓				
10.	✓				
11.	✓				
12.	✓				
13.		✓			
14.					✓
15.	✓				

What part of the test did you think was easiest?
numbers 1-12, 15.

What part of the test was most challenging for you?
number 14

You should have checked your work before turning it in. Please tell me which numbers, if any, you were able to correct because you checked your work.
number 13.

Figure 3

Becky's postassessment

Decimals Unit Test Reflection

Put a check next to each number that describes how you feel about it. Remember, this is between you and me; be honest.

Question Number	It is Very Easy	I Feel Pretty Good About This	I Could Use Some More Help	I Need A LOT More Help	I Have No Idea What the Question is Asking of Me
1.	✓				
2.	✓				
3.	✓				
4.	✓				
5.	✓				
6.	✓				
7.	✓				
8.	✓				
9.	✓				
10.	✓				
11.	✓				
12.		✓			
13.		✓			
14.				✓	
15.	✓				

What part of the test did you think was easiest?
Numbers 1, 2, 3, 4 and 5

What part of the test was most challenging for you?
Number 14

You should have checked your work before turning it in. Please tell me which numbers, if any, you were able to correct because you checked your work.
12

Figure 4

Erica's postassessment

Decimals Unit Test Reflection

Put a check next to each number that describes how you feel about it. Remember, this is between you and me; be honest.

Question Number	It is Very Easy	I Feel Pretty Good About This	I Could Use Some More Help	I Need A LOT More Help	I Have No Idea What the Question is Asking of Me
1.					
2.					
3.					
4.					
5.					
6.					
7.					
8.					
9.					
10.					
11.					
12.					
13.					
14.					
15.					

What part of the test did you think was easiest?

All the questions ex. 14.

What part of the test was most challenging for you?

I think question 14 because dividing with decimals is pritty hard

You should have checked your work before turning it in. Please tell me which numbers, if any, you were able to correct because you checked your work.

I think #14 because I used multiplication after I divided

"So, what did you do when your product and dividend didn't match?"

"I tried multiplying 4.55 by 4, and then I got 18.2, so I knew that had to be the right answer."

"So, Erica, because you took the time to check back over your work, you found a mistake and corrected it, which is wonderful! You would have missed that question if you had just turned in your paper. Instead, you got 100 percent on the test! That makes a big difference in your grade."

This type of brief conversation with my students can be the proof they need to change their attitudes (and general disdain) for reviewing their work before handing it in. I usually check in with students who need such conversations on the day that I return the tests after grading, while the rest of the students are analyzing their results independently or in small groups.

Conclusion

Pre- and postassessments are tools that let students take responsibility for their own learning. Students internalize what they know and do not know and begin to learn how to approach learning metacognitively. Parents can use both tools to get an understanding of their child's understandings. Finally, these tools allow me to collect data on each individual student, as well as the entire class, for each different assessment. I use the data to become a better teacher.

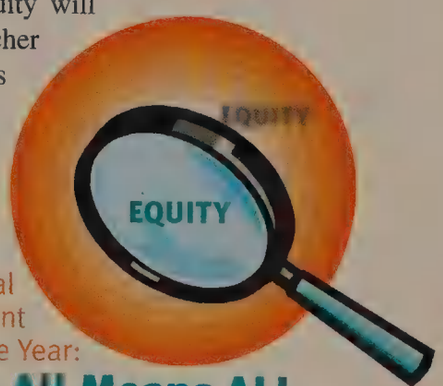
While I use these pre- and postassessment self-assessments mainly in fifth-grade mathematics and science classes, they could easily be adapted for any other content area. I believe they have a place in every classroom. For example, an elementary music teacher could customize the assessment to ask about mastering songs on the recorder. Just as easily, a high school history teacher could make the reflections relevant to her focus concepts. I worked with kindergarten teachers in my school to create a simple form of these reflections for their students (see **Appendix**). Their format consisted more of checklists for the five-year-olds, but the idea of students being responsible for their learning—before, during, and after assessments—was consistent. Giving all students opportunities and tools to reflect on their strengths and weaknesses is one of the most influential life lessons that teachers can offer.

Editor's Note: Pseudonyms have been used throughout to conceal students' identities. ▲

Equity: All Means ALL

As students prepare to thrive in a global society, equity in mathematics education becomes increasingly more important. All preschool through college students must have access to a variety of instructional tools, such as manipulatives and technology, and a coherent, rich, challenging curriculum taught by qualified, effective mathematics teachers.

NCTM's year-long emphasis on equity will help teachers, school leaders, and teacher educators examine equity and its salience in the mathematics teaching and learning process and hold high expectations for all students. For more information and additional resources on the NCTM Focus of the Year, visit www.nctm.org/focus.



NCTM 2008–09 Professional Development Focus of the Year:
Equity: All Means ALL

Appendix

Primary grades student self-assessment

Name _____

	I have got it!	I need a little more time.
I know the letter ___.		
My words match the picture.		
I can read this back to you.		
Additional concepts to be filled in by teacher		

From the September 2008 issue of Teaching Children Mathematics

Math quiz reflection

Name _____ Date _____ Homeroom ___

Describe each quiz question, using a 1, 2, or 3. Remember, this is between you and me; be honest.

1 = Easy 2 = Medium 3 = Hard

Question 1 _____
Question 2 _____
Question 3 _____
Question 4 _____
Question 5 _____

What did you do to make sure you got the correct answers?

_____
_____
_____

What problems did you find the most challenging? Why?

_____
_____
_____

What can you do to help yourself on the problems you struggled with?

_____
_____
_____

From the September 2008 issue of Teaching Children Mathematics

Decimal quiz #1 reflection (Use after assessments have been graded and returned)

Name _____ Date _____ Homeroom ___

I want you to take some time to really think about each quiz question. How well do you know the answer? Please be honest with yourself and with me!! Thank you.

Did you get the question right?	If you got it right, how comfortable do you feel with the problem type?	If you got the problem wrong, why?
1. ☐ Yes ☐ No	☐ It's so easy. ☐ It's O.K. ☐ I guessed, because it's hard.	☐ I made a simple mistake. ☐ I don't know how to do it.
2. ☐ Yes ☐ No	☐ It's so easy. ☐ It's O.K. ☐ I guessed, because it's hard.	☐ I made a simple mistake. ☐ I don't know how to do it.
3. ☐ Yes ☐ No	☐ It's so easy. ☐ It's O.K. ☐ I guessed, because it's hard.	☐ I made a simple mistake. ☐ I don't know how to do it.
4. ☐ Yes ☐ No	☐ It's so easy. ☐ It's O.K. ☐ I guessed, because it's hard.	☐ I made a simple mistake. ☐ I don't know how to do it.
5. ☐ Yes ☐ No	☐ It's so easy. ☐ It's O.K. ☐ I guessed, because it's hard.	☐ I made a simple mistake. ☐ I don't know how to do it.
6. ☐ Yes ☐ No	☐ It's so easy. ☐ It's O.K. ☐ I guessed, because it's hard.	☐ I made a simple mistake. ☐ I don't know how to do it.
7. ☐ Yes ☐ No	☐ It's so easy. ☐ It's O.K. ☐ I guessed, because it's hard.	☐ I made a simple mistake. ☐ I don't know how to do it.
8. ☐ Yes ☐ No	☐ It's so easy. ☐ It's O.K. ☐ I guessed, because it's hard.	☐ I made a simple mistake. ☐ I don't know how to do it.
9. ☐ Yes ☐ No	☐ It's so easy. ☐ It's O.K. ☐ I guessed, because it's hard.	☐ I made a simple mistake. ☐ I don't know how to do it.

What part of the test was the hardest for you? _____

_____

What would you like to spend more time practicing? _____

From the September 2008 issue of Teaching Children Mathematics

Student self-assessment

Name_____ Date _____

Use the following scale to let me know what you thought of the questions on our quiz/test. Remember, this is just between you and me, so please be honest!

1. This was very easy for me.
2. This was pretty easy for me.
3. This made me think a while.
4. This made me think A LOT!
5. I have no idea how to solve this problem.

Question #1	1	2	3	4	5
Question #2	1	2	3	4	5
Question #3	1	2	3	4	5
Question #4	1	2	3	4	5
Question #5	1	2	3	4	5
Question #6	1	2	3	4	5
Question #7	1	2	3	4	5

Which question(s) were hardest for you to answer?

_____

Why do you think those were the most difficult?

_____
_____

What can you do to prepare yourself for those types of questions?

_____
_____

From the September 2008 issue of Teaching Children Mathematics

Lois Brandts, Tom Hoopingarner, Christine Gallego, Dora Saldamando, Grace Tapia-Beltrán, and Karolyn Williams

Math Everywhere

"Math by the Month" activities are designed to engage students to think as mathematicians do. Students may work on the activities individually or in small groups, or the whole class may use them as problems of the week. No solutions are suggested, so students look to themselves for mathematical justification and develop confidence to validate their work.

The following activities are designed to help your students become familiar with their immediate surroundings and their location in the world through mathematical investigations involving number sense and operations, collecting and representing data, geometry, measurement, logic, communication, representation, and technology. ▲

WEEKLY ACTIVITIES

MATH EVERYWHERE K–2

SEPTEMBER 2008

8 **Where do you stand?** *Ordinal* numbers describe position: first, second, third, and so forth. The next time you stand in a line of people, count how many people are in front of you. How many are behind you? What is your position in the line?

15 **Mapping our classroom ...** After reading the book *Mapping Penny's World* by Loreen Leedy, imagine that you are a spider on the ceiling. On a blank piece of 8 1/2" × 11" paper, draw a map of your classroom from the spider's viewpoint. Represent each student's desk with a square. Place an X in the square that represents your desk. Draw the path you would follow to go from the door to your desk. Use a different color to draw a different route you could take. Exchange maps with a friend. How are your maps similar? How are they different? **Challenge:** Draw maps of the playground and school campus. Show how you would go from your classroom to the playground, to the library, and to the office.

22 **Fruit Bowl Turnover ...** Bring one of these fruits to make a salad to share with your classmates: a banana, a kiwi, a pineapple, or a bunch of grapes. Bananas and kiwi were originally grown in China. Now kiwi is grown in California. Brazil is the leading producer of bananas. Pineapples originated in Paraguay and southern Brazil, but most of the pineapples we eat are from Hawaii. Ancient records show that grapes grew along the Nile River in Egypt. Many types of grapes that we enjoy grow in the western United States, especially California. Your teacher will list the fruits that everyone brought to class and then ask different people to draw each fruit and write its name under the drawing. Your teacher will help you tape the drawings on a large world map to show where your fruit grows. Which fruit traveled the farthest? How do you know? Ask an adult to help you cut the fruit into bite-size pieces, mix all of it together, and share it with your classmates. Think about the taste and texture of fruit: sweet, crunchy, soft, tart. Sort the fruit by these attributes and then enjoy eating the fruit salad.

29 **Where would you go?** List four places you would like to visit. Locate each place on a map. Which is farthest from where you live? Which is closest? Put your list in order from closest to farthest. Compare your list to those of your classmates. Are some places more popular than others? Make a graph showing the four most popular places your classmates would like to visit.

Lois Brandts teaches second grade at El Camino School in the Goleta Union School District of California. Tom Hoopingarner is the mathematics curriculum and instructional specialist for Sunnyside Unified School District in Tucson, Arizona. The other team members teach at Pueblo Gardens Elementary School in Tucson Unified School District: Christine Gallego teaches first grade, Dora Saldamando teaches fourth grade, Grace Tapia-Beltrán teaches fifth grade, and Karolyn Williams teaches a third-fourth multiage class.

Edited by **Dana Islas**, *who teaches kindergarten at Pueblo Gardens Elementary School in Tucson, Arizona. Readers are encouraged to submit problems to be considered for future "Math by the Month" columns to the department editor at* **dana.islas@tusd1.org**. *Receipt of the problems will not be acknowledged; however, those selected for publication will be credited to the author. Readers are also invited to submit creative solutions and adapted problems to* **tcm@nctm.org** *for potential publication in "Reader's Exchange." Please include "Reader's Exchange" in the subject line. Because of space limitations, responses beyond the 250-word limit are subject to abridgement. Submissions are also edited for style and content.*

WEEKLY ACTIVITIES

MATH EVERYWHERE 3–4

SEPTEMBER 2008

8 **Color me stately.** Ask your teacher for a black-and-white copy of an enlarged portion of a map that includes your state (or province). Study the map and then make a prediction about the fewest number of colors that can be used to color your map if no borders share the same color. (You may use a color twice only if it does not share a border with an area that is the same color.) Test your prediction by coloring your map using the fewest number of colors. When you finish, share your map. Why is it important for mapmakers to use different colors for areas that share borders?

15 **How far is your state capital from a nearby one?** Ask your teacher for a U.S. map that shows all state capitals, state borders, and a scale in inches to represent miles or in centimeters to represent kilometers. With a partner and a piece of string, measure distances between cities on your map. (Place the end of the string on a capital city; your partner rolls out the string and cuts it at the point it reaches another capital. Use the map scale to find the distance in miles or kilometers between the two.) What are the longest and shortest distances you measured between capitals?

22 **Mapping their resources ...** The Anazasi built pit-house dwellings with roofs of strong northern Arizona forest beams. They cut mountaintop trees and carried them on their shoulders, walking about 60 miles back to their homes. If they traveled 13 miles per day, how many days would it take them to reach the mountaintops? Do you think the trip back to their village would take more, less, or the same amount of time as the trip to the mountains? Why? Share your ideas with a partner.

29 **Where do books come from?** Books are published all over the world. The publishing information is usually printed within a book's first few pages. Sometimes more than one city is listed. (For this activity, use the first city listed.) Look at a sample of the books in your classroom library and list all the places where they were published. Use your list to create a graph showing where the books were published. In 2004, the School Library Journal stated that the average cost of books for children and young adults was $19.31. Using this average cost, how much money did the publishers in each city on your graph generate in book sales for your classroom? Order the cities on your graph from the city generating the most money in book sales to the city generating the least. How much money in book sales was generated altogether for your classroom?

WEEKLY ACTIVITIES

MATH EVERYWHERE 5–6

SEPTEMBER 2008

8 **Post clocks** around your classroom to represent some of the Western Hemisphere's standard time zones from east to west: Newfoundland (NST), Atlantic (AST), Eastern (EST), Central (CST), Mountain (MST), Pacific (PST), Alaskan (AKST), and Hawaiian (HST). When you awaken, what time is it in Makaha, Hawaii? In Toronto, Canada? In Boulder, Colorado? Predict what fifth and sixth graders are doing in each time zone as you eat lunch. What are you doing when it is lunchtime in Los Angeles, California? What might fifth and sixth graders be doing on the island of Newfoundland when you go to bed?

15 **Where does junk mail come from?** Ask your parents for the junk mail that your household received on one day. Count how many pieces arrived and list the cities and states shown in the return addresses. Bring your list to school and compare it to your classmates' lists. Does more junk mail come from one particular place? After you determine where most of the junk mail originates, locate the city and state and find their populations. What businesses are in that city? What is the bulk rate for junk mail? How much would it cost to mail all the junk mail that your class counted? If every class in your school collected the same amount, how much would it cost to mail it all?

22 **Monarch butterflies** are remarkable insects. Each year, they migrate about 3100 miles from Canada and the northern United States to Zitacuaro in Michocan, Mexico. Their flight speed averages about 12 mph, although they can fly as fast as 30 mph with a good tail wind. In one day, they fly about 80 miles. Calculate how many days it takes for a Monarch butterfly to make its journey. An elite marathon runner averages 12 mph. If such an athlete runs 6 hours per day, how many days would it take the runner to cover the same distance as the butterfly? If you drove the same distance in a car, averaging 50 mph for 7 hours per day, how many days would it take you to make the journey?

29 **Where can dreams take you?** John Gage, 74, a New Jersey sailor, dreamed for 30 years of circumnavigating the globe in his sailboat. After extensive preparation, he set off on December 18, 2003. He planned to sail from Sandy Hook, New Jersey, to the British Virgin Islands, the Panama Canal, the Galapagos Islands, the Marquesas Islands, the Society Islands, Fiji, New Zealand, Australia, South Africa, the Caribbean, and back to Sandy Hook. On a world map or globe, calculate how many miles he planned to sail. If one mile is approximately 1.15 nautical miles, how far did he sail in nautical miles? His boat's average cruising speed was 6 knots per hour. A knot equals 1 nautical mile per hour. How many hours of sailing time would it take to sail his course? Be sure to show how you know. (Gage completed his voyage in May 2007.)

Measuring the "Grand" in the Grand Hotel

Problem

As we travel from place to place, we see many buildings with interesting architecture that we can relate to various geometric shapes. One such building is the Grand Hotel on Mackinac Island in Michigan. With its 660-foot porch, the hotel, sitting high on a hill, looks like one long rectangle. The 66-foot porch height seems small compared to its tremendous length. When you compare the porch length to its height, you notice that the length is exactly 10 times the height. Explore the ratio of length to height for a variety of rectangles and record their values. Sketch a representation of each different rectangle that you find. Find a rectangle in your classroom that is similar to the Grand Hotel's proportions (www.grandhotel.com/).

Photo courtesy of www.grandhotel.com. All rights reserved.

Variations

- Our travels could also take us to the Lincoln Memorial. The front of this memorial is 188 feet long. Its height is 99 feet. Find a rectangle in the classroom that has the same ratio (www.visitingdc.com/memorial/lincoln-memorial-picture.htm).
- Research the dimensions of a local building or another famous landmark and then find its similar shape within the classroom. Report your findings to the class.

The goal of the "Problem Solvers" department is to foster improved communication among teachers by posing one problem each month for teachers of grades K–6 to try with their students. Every teacher can become an author: Pose the problem to your students, reflect on your students' work, analyze the classroom dialogue, and submit your resulting insights to this department. Through contributions to the journal, every teacher can help us all better understand children's capabilities and thinking about mathematics. Remember than even students' misconceptions provide valuable information.

Edited by **Joseph Georgeson**, *jgeorgeson@usmk12.org, middle school mathematics department chair and teacher of eighth graders at the University School of Milwaukee in Wisconsin;* **Brian Schad**, *schad@aaps.k12.mi.us, a fifth-grade teacher at Lawton Elementary School in Ann Arbor, Michigan; and* **Sarah Slater**, *sslater@gmail.chelsea.k12.mi.us, a second-grade teacher at North Creek Elementary School in Chelsea, Michigan. Each month, this section of the "Problem Solvers" department features a new problem for students. Readers are encouraged to submit problems to the editors to be considered for future "Problem Solvers" columns. Receipt of problems will not be acknowledged; however, problems selected for publication will be credited to the author.*

Classroom Set-up

This month, "Problem Solvers" provides two different approaches for introducing the problem. Choose the one that is most appropriate for the grade level that you teach. Explain that this problem is one step toward developing the ability to think proportionally. This type of reasoning is fundamental to understanding many advanced mathematical concepts, from slopes and lines to the calculus where ratios are constantly changing. The next time you or your students wonder about a structure's height or use the scale on a map, you can be sure that ratios and proportional reasoning will help you understand what you are doing. (See "Where's the Math?" for more detail.)

Before the students begin their search for rectangles within the classroom, have them develop a system for making observations and recording data. This will enable the students to analyze and make conclusions about the rectangles that

have been measured as well as make predictions about the types of rectangles that need to be measured. For all grade levels, monitor the students as they prepare their strategies and measure the rectangles they identify. Listen to the students as they discuss their solution strategies. Interject ideas only if a student is having difficulty. The measurement and problem-solving strands are intertwined throughout this problem. As students measure the various rectangles, they will use the problem-solving strategies of guess and check, eliminating possibilities, and looking for patterns.

Remember that the process for finding the answer in problem solving is sometimes more important than the answer. Give students a later opportunity to review their problem solving. As they do, keep their focus on the different strategies they used rather than on their answers.

For the lower grades

Begin by explaining to students that they are going to "travel" to the Great Lakes to complete a mathematics problem. Show them a map and discuss the fact that there are five Great Lakes. The Great Lakes contain fresh water and are so large that you cannot see across them.

On a map of Michigan, show students where Lake Huron (one of the Great Lakes) and Mackinac Island are located. To get to the island, people take a ferry from the towns of Mackinaw City or Saint Ignace. Cars are not allowed on the island; people ride horses and bikes to get around. Ask if anyone has ever been to Mackinaw Island. Ask what the students think it would be like to visit a place where no cars are allowed.

Show a picture of the large hotel on Mackinac Island called the Grand Hotel (www.grandhotel.com). Discuss with the class that the hotel's porch

Where's the Math?

On the surface, this month's problem seems simple, but recognizing similar shapes and mathematically proving their similarity involve understanding and applying higher-level mathematics concepts of ratios, proportionality, and graphing. To compare one side of a rectangle to another, students use additive thinking. Ratios are comparisons. Solving this problem requires proportional thinking and producing an answer expressed as a ratio.

Some famous ratios are pi and the golden ratio. Pi is the ratio of the circumference of a circle to its diameter, about 3.14. Every circle is proportional to every other circle and therefore similar. If a circle's circumference is compared to the diameter—even the earth's circumference and diameter—the ratio is always this constant ratio (allowing for the inaccuracy of measurement). The ratio can be shown graphically by plotting points on a scatter plot where the x-coordinate is a circle's diameter and the y-coordinate is its circumference. Regardless of how many circles you plot, the scatter plot will show them to lie on a line—even circles whose diameters vary widely. This concept is later interpreted as a linear relation where the slope is always constant. The slope shows the ratio pi, and it is always the same.

The golden ratio represents what some people think is an ideal proportion, that is, a proportional relationship most pleasing to the eye. Take a length and split it into two pieces, a and b. If the whole length $(a + b)$ compared to one of the pieces $(a + b)/a$ is the same ratio as one piece to the other piece (a/b), it shows the golden proportion. Some architecture and many living things exhibit this proportion. For example, the ratio of a person's height to the height of her bellybutton from the floor is close to the golden ratio. Rectangles with this ratio are also called *golden*. The ratio of the long side to the short side in such a rectangle should be about 1.618. Rectangles whose length and width are found in consecutive numbers of the Fibonacci sequence are close to being golden. For more information on the subject of golden ratios, consider the following Web sites: www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html and pass.maths.org.uk/issue3/fibonacci/index.html.

Ratios and proportional thinking form many of the underpinnings of algebra. Slope and linear relations, where slope is constant, are examples. We can also use ratios for indirect measurement, too. For example, the ratio of a shadow to the actual height of an object should remain constant at any particular time of day. Scale models are another use of ratios. We want maps to reflect the same proportions that exist in the world that they represent. Models should be similar, or proportional, to the real object. Whenever size is changed so that the result is similar to the original, the ratios are equal.

Photo courtesy of www.grandhotel.com. All rights reserved.

length is ten times its height. Talk about what it means for something to be ten times longer than something else. Introduce the number pattern of one, ten, one hundred, one thousand. Lead a discussion to ensure that students understand the idea of multiplying numbers by ten. Explore what happens when a number is multiplied by ten. Use the Grand Hotel's porch dimensions (660 feet long by 66 feet high) as one of the explorations.

Ask students to look around the room to find a rectangle that is long and narrow like the Grand Hotel's porch. Measure the rectangle's width (the shorter side) and multiply it by ten. Is the product equal to the rectangle's length (the longer side)? Round off all measurements to the nearest inch or centimeter. As a class exploration, measure and compare the dimensions of several more rectangles. Then have the students work in small groups to analyze other rectangles.

For the upper grades

This month's activity begins with the topic of ratios. Display a rectangle that has four equal side lengths (a square). After measuring one side, divide the length by the width. The students will discover that the ratio of the sides is equal to one. Have the students measure other rectangles with congruent sides (i.e., a 2 × 2 square, a 5 × 5 square, etc.) to determine whether this is always the case. Conclude this exploration by discussing the fact that any time a rectangle has four congruent sides (a square), the ratio of the sides is equal to one.

Now "stretch" two of the rectangle's sides so that the length is twice that of the width. What happens to the ratio? Repeat but "stretch" the length to three times the width. Continue the activity with a rectangle that has a ratio involving a decimal (for instance, a rectangle fifty centimeters long by thirty-five centimeters wide). Explain to the students that if you compute 50 ÷ 35, you get a ratio of approximately 1.43. Discuss how this rectangle compares to the ones already mentioned.

Consider the many different shapes and sizes of the buildings in the world around us. Explain to the students that today's activity will focus on rectangular buildings. One such building is the Grand Hotel on Mackinac Island, Michigan. Mackinac Island lies in the northern portion of Lake Huron, between Michigan's Lower and Upper Peninsulas. Show a picture of the hotel to the students (www.grandhotel.com). The Grand Hotel's porch is 660 feet by 66 feet, exactly ten times longer than it is high. Instruct the students to find rectangles within the classroom that are close to or the same shape as the Grand Hotel (i.e., if you divide its length by its width, you get a quotient of ten). For this activity, students need a measuring tool and a calculator.

Share Your Students' Work

Please try this problem in your classroom. We are interested in how your students responded to the problem, what problem-solving strategies they used, and how they explained or justified their reasoning. Include information about how you posed the problem, samples of student work, and photographs. E-mail your thoughts, reflections, scanned student work, and photographs to **Joseph Georgeson** at jgeorgeson@usmk12.org by **November 1, 2008**, or send them to his attention at University School of Milwaukee, 2100 West Fairy Chasm Road, Milwaukee, WI 53217. Include your name, grade level, and school name with your submission.

Additional Resources

Did you know that NCTM has published a collection of some past "Problem Solvers" columns? Visit **nctm.org/catalog** for more on this and other NCTM resources, including professional development offerings, other publications, and online resources.

Sakshaug, Lynae E., Melfried Olson, and Judith Olson. *Children Are Mathematical Problem Solvers*. Reston, VA: National Council of Teachers of Mathematics, 2002. ▲

(Solutions to a previous problem begin on the next page.)

Solutions to the Stained Glass Window Designs Problem

Problem

Mrs. Estrada, the principal at Sierra Park Elementary School, wants to have a stained glass window added to the school's main entrance. She has decided to hold a contest for students to create a window design. Each design must fit the following specifications:

- It must exactly fill one of the window design templates (**fig. 1** or **fig. 2**).
- It must use at least four different geometric shapes (pattern block shapes are recommended).
- It must use at least four different colors (encourage students to vary these).
- The shapes must not overlap.

After making the design, students must answer the following questions and complete the tasks:

- What fraction of the whole window does each *shape* fill?
- Demonstrate mathematically that the fractions of all shapes total to 1 (a whole).
- What fraction of the whole window does each *color* fill?
- Demonstrate mathematically that the fractions for the colors total to 1 (a whole).

Figure 1

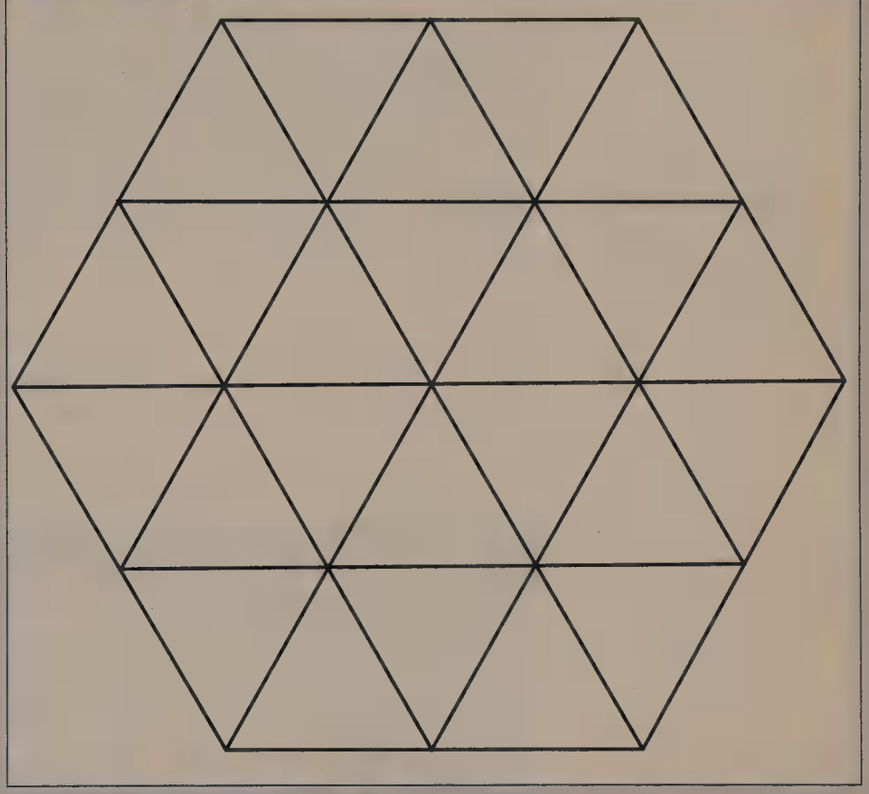
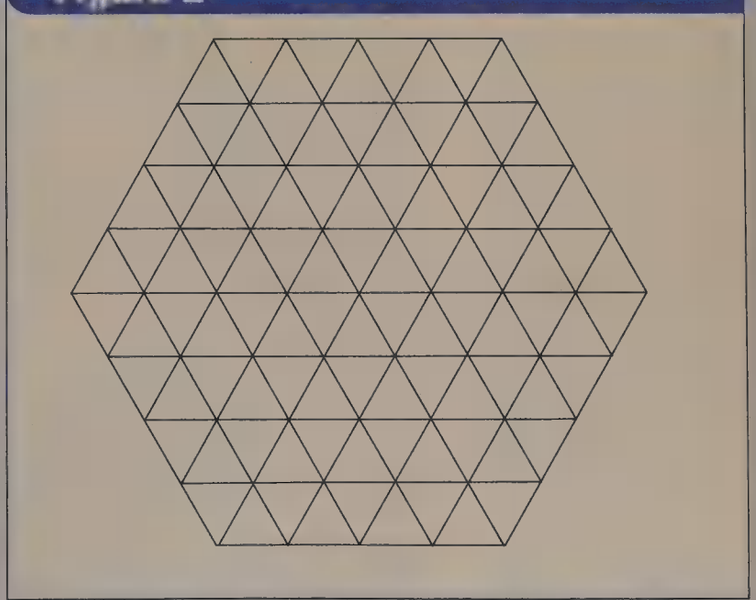


Figure 2



Note for classroom use: Figure 1 is actual size. To make figure 2 actual size, enlarge approximately 320 percent.

Variations

Two different hexagon templates (**figs. 1** and **2**) are available for the window design template. You may modify the assignment by choosing the template that is most appropriate for your students' understanding of fractions. The larger template requires students to work with fractions that may have a denominator as large as 96, while the smaller template requires students to work with fractions with a denominator of only 24. For each template, each triangle is equivalent to one small pattern block triangle.

This problem can be modified for younger students by omitting the fraction part of the problem and instead having students figure out the number of each geometric shape they used in their design. They can create a table that lists each shape by name and gives the number of each shape used in the design. The students can then repeat the process for the colors they used.

On the surface, the problem in the September 2007 "Problem Solvers" section required students to recognize and name basic geometric shapes. Beyond that, the Stained Glass Window Designs problem challenged student understanding of fractions as part of a whole and encouraged work with equivalent fractions.

The second-grade students from the classes of Marie Wright, Hannah Couch, and Anastasia Kalona at the River School in Washington, D.C., worked on the Stained Glass Window Designs problem as an informal exploration of the concept of fractions. The problem was introduced in their mathematics time block, during which the students are referred to as number detectives and assigned special agent names. In this context, the teachers posed the problem as follows: "Today in Agent School, we are going to make an unbreakable, weather-proof window for our Agent Headquarters." The teachers began by showing the agents the shapes they could use in creating the window and asking them to identify each by name. Some children had not heard the term *rhombus* before, so they wrote the word and practiced saying and tracing it.

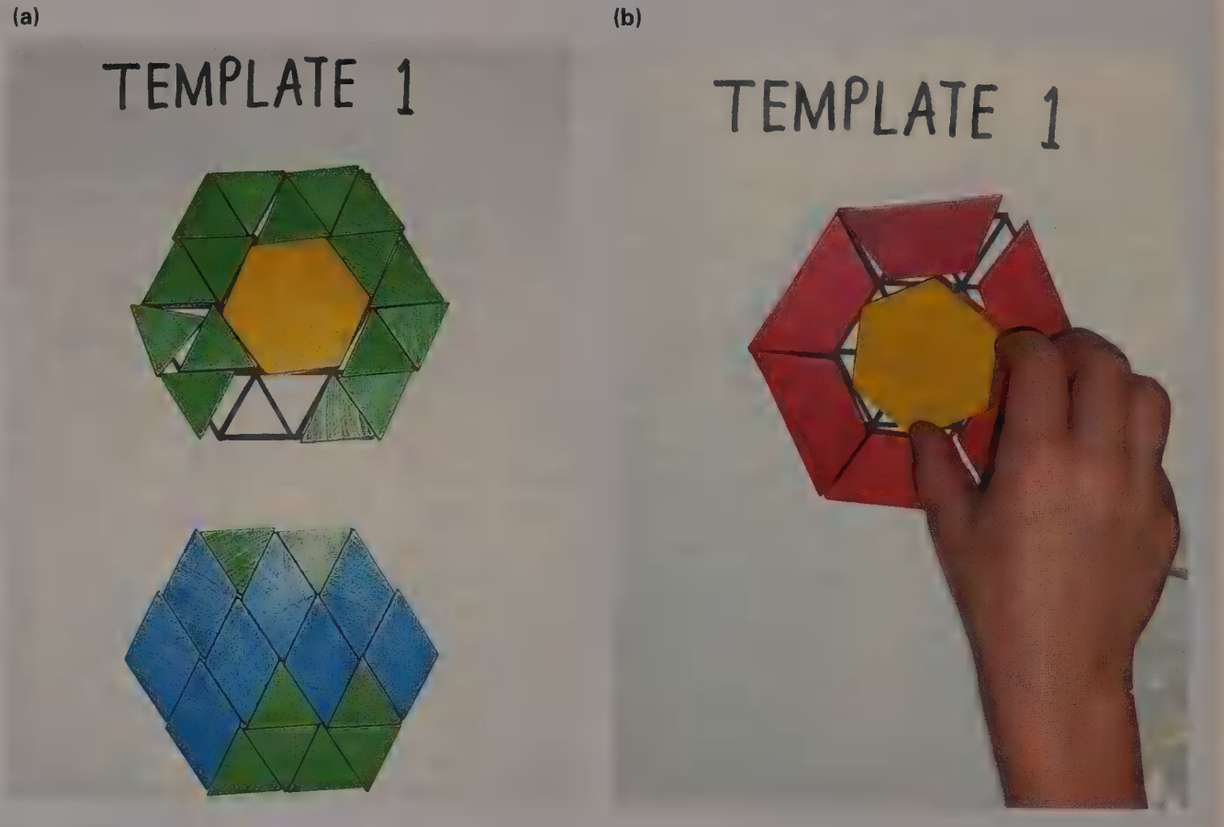
As the students made their window designs (**see figs. 3a** and **b**), many noticed mathematical relationships between the shapes. Following are some of the comments they made:

> I used six trapezoids, and I know it takes three triangles to fill one trapezoid. I have six spaces left. I know six divided by three is equal to two. So, I need two more trapezoids. One trapezoid covers three triangles.

> I put the trapezoid around the hexagon. I used hexagons, trapezoids, and triangles. It took three rhombuses to cover one small hexagon. I knew two triangles covered the rhombus. So, I doubled it up. I knew that three triangles covered one trapezoid. So, two trapezoids covered the small hexagon. I tried to think of it like a fraction like this. Three rhombuses equal one hexagon. Two triangles equal one rhombus. Three triangles equal one trapezoid.

Figure 3

Number detectives from the River School in Washington, D.C., made imaginary unbreakable windows for agent headquarters.



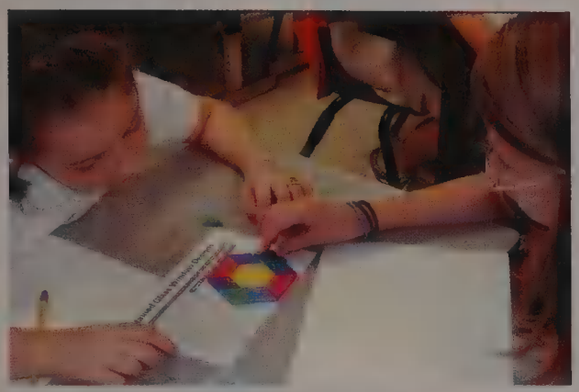
Photographs by Marie Wright; all rights reserved

Edited by **Mark Ellis**, *mellis@fullerton.edu, an assistant professor in the College of Education at California State University–Fullerton, where he prepares teachers of middle school mathematics and works with local classroom teachers in creating learning environments that support sensemaking in mathematics; and* **Cathery Yeh**, *catyeh@aol.com, who teaches third grade at Arnold Elementary School in Cypress, California. Each month this section of the "Problem Solvers" department discusses the classroom results of using problems presented in previous issues of* Teaching Children Mathematics.

Figure 4

Fourth graders from Tara Soudelier's classes surprised their teacher with their discoveries about equivalent fractions.

(a)



(b)

Photographs by Tara Soudelier; all rights reserved

The teachers summed up the lesson by showing a pictorial representation of a whole hexagon divided into several shaded pieces. The teachers showed different fractional parts and asked the students to express the values. When asked, "Why does twenty-four over twenty-four equal one?" one student confidently responded, "It is equal to one because it is a whole!"

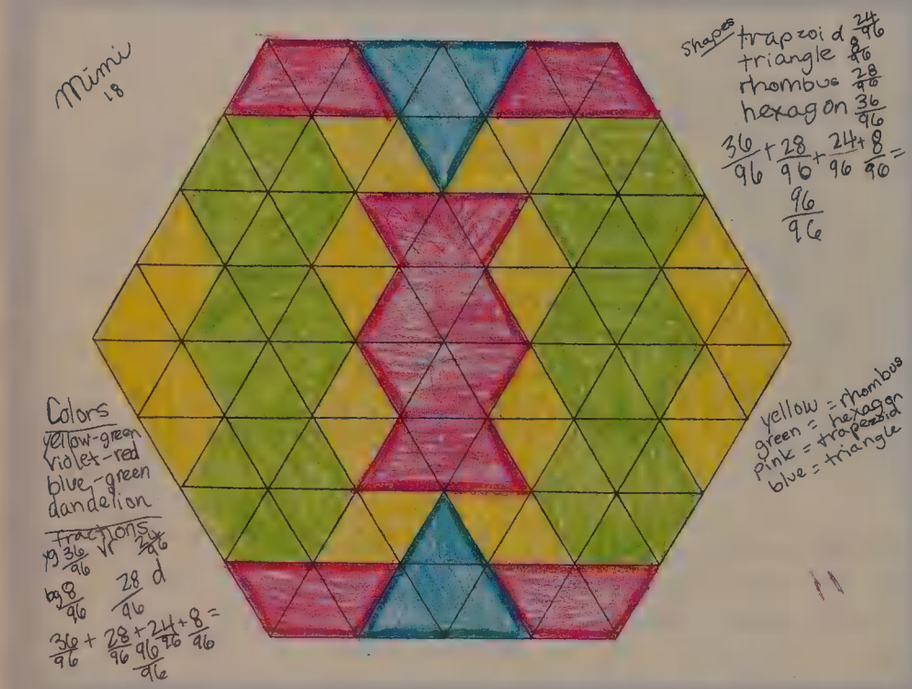
Tara Soudelier is a National Board Certified Teacher with five classes of fourth-grade mathematics students at Mulberry Elementary School in Houma, Louisiana. In each class, she gave students copies of the Stained Glass Window Designs template and the pattern blocks to create their own design, and then asked them to respond to the questions about what fraction of the whole each shape and color represent. During their discussions, Soudelier was surprised by students' discoveries about equivalent fractions. What follows is a portion of Soudlier's description of her implementation of the Stained Glass Window Designs problem with her fourth graders (see **figs. 4a** and **b**):

> After creating their designs with the pattern blocks, a few students took out red, blue, green, and yellow crayons and colored their windows. Then they counted all the yellow triangles to get a fraction for the part covered by hexagons, all the blue triangles to get a fraction for the part covered by rhombuses, and so on. Students had no trouble adding their fractions and were quick to realize that if their total was 24/24, they must have calculated correctly. Students who did not get 24/24 went back and recounted triangles on each polygon. As soon as students began to name each fraction of the window that each color filled, they quickly realized that their answers would be the same as the fraction each shape filled. They also realized that their total for the different colors filling the window would be the same as their total for the different shapes that filled the window.
>
> Also, while observing one group in particular, I asked what fraction of their window the hexagon filled. One student in the group replied, "One-fourth." This surprised me because he had made the connection that 6/24 is equal to 1/4.
>
> My students had previously learned that when adding fractions, the denominator stays the same. From this activity, they learned why the denominator stays the same. They also learned that to be a fraction of a whole, all parts have to be of equal size. Some began making connections to equivalent fractions as they manipulated the pattern blocks to discover that one trapezoid equals one-half hexagon and three triangles equal one trapezoid. Therefore, 1/2 = 3/6. Students were very excited about this activity and said they had fun in class that day. They also expressed that they enjoyed working with the pattern blocks and were excited to learn more about geometry and fractions.

Mary Kay Varley, a teacher at Fort Worth Country Day School in Texas, saw the Stained Glass Window Designs problem as a perfect wrap-up to her fourth graders' geometry unit and as an opportunity to gain insight into their understanding of fractions. Varley extended the problem by having students write a letter trying to persuade

Figure 5

A fourth-grade student in Mary Kay Varley's class colored this design.



the principal, Mrs. Estrada, to choose their design. They discussed some of the points they might want to include in their letters to make a good argument for their designs. **Figure 5** shows one student's design for the large window.

The responses to the Stained Glass Window Designs problem indicate the problem's usefulness for students across several grade levels. Students in the primary grades can use this activity to practice identifying basic geometric shapes and explore the concept of fraction in a hands-on manner. Upper-grade students can use this activity to deepen their understanding of equivalent fractions and develop insights into the relationship between fractional parts of differing sizes and a common whole. Through their work on this problem, students actively engage in problem solving, communication, and representation as they create their own artistic stained glass design and describe some of the mathematical properties of it with fractions. The work that students generate allows their teachers to assess their knowledge of geometric terms and their understanding of important fraction concepts.

Special thanks to those teachers and students who contributed to this article:

Marie Wright, Hannah Couch, Anastasia Kalona, and the students at the River School in Washington, D.C.

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Mary Kay Varley and her fourth-grade class at Fort Worth Country Day School in Texas

Kristina S. Kalb and the fourth graders at Barley Sheaf School in Flemington, New Jersey

The third graders at Arlington Elementary in Indianapolis, Indiana

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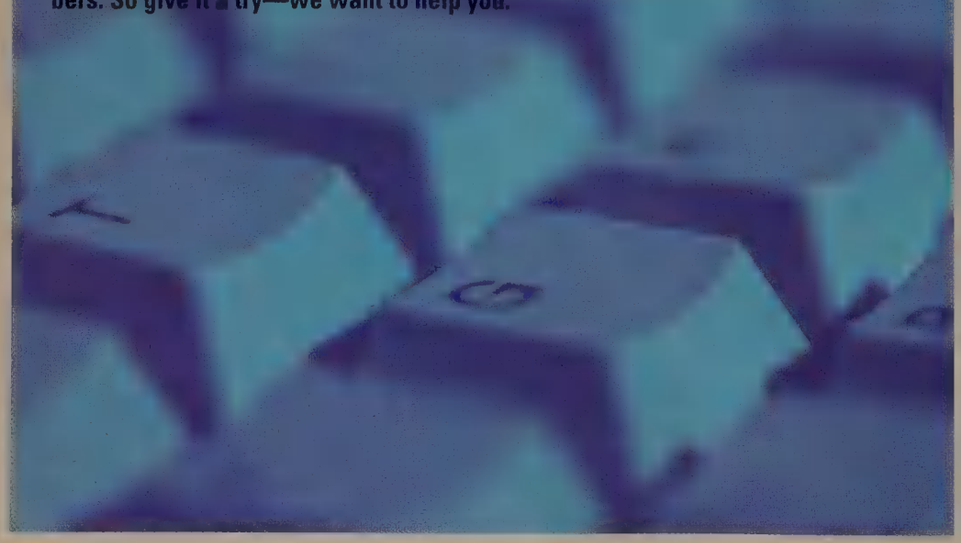

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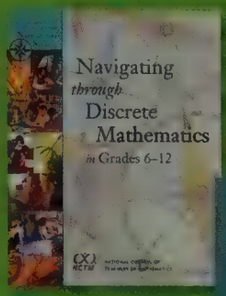

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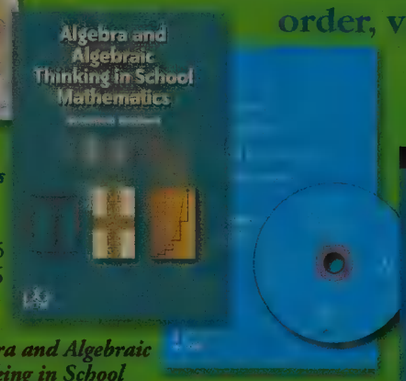

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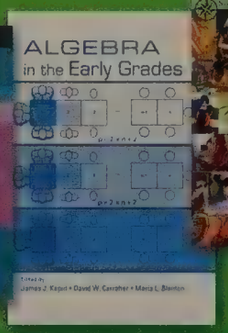

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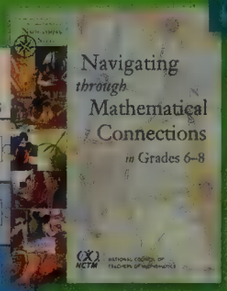

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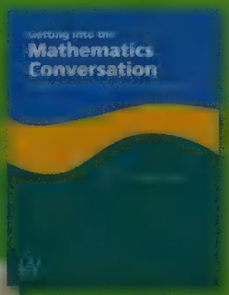

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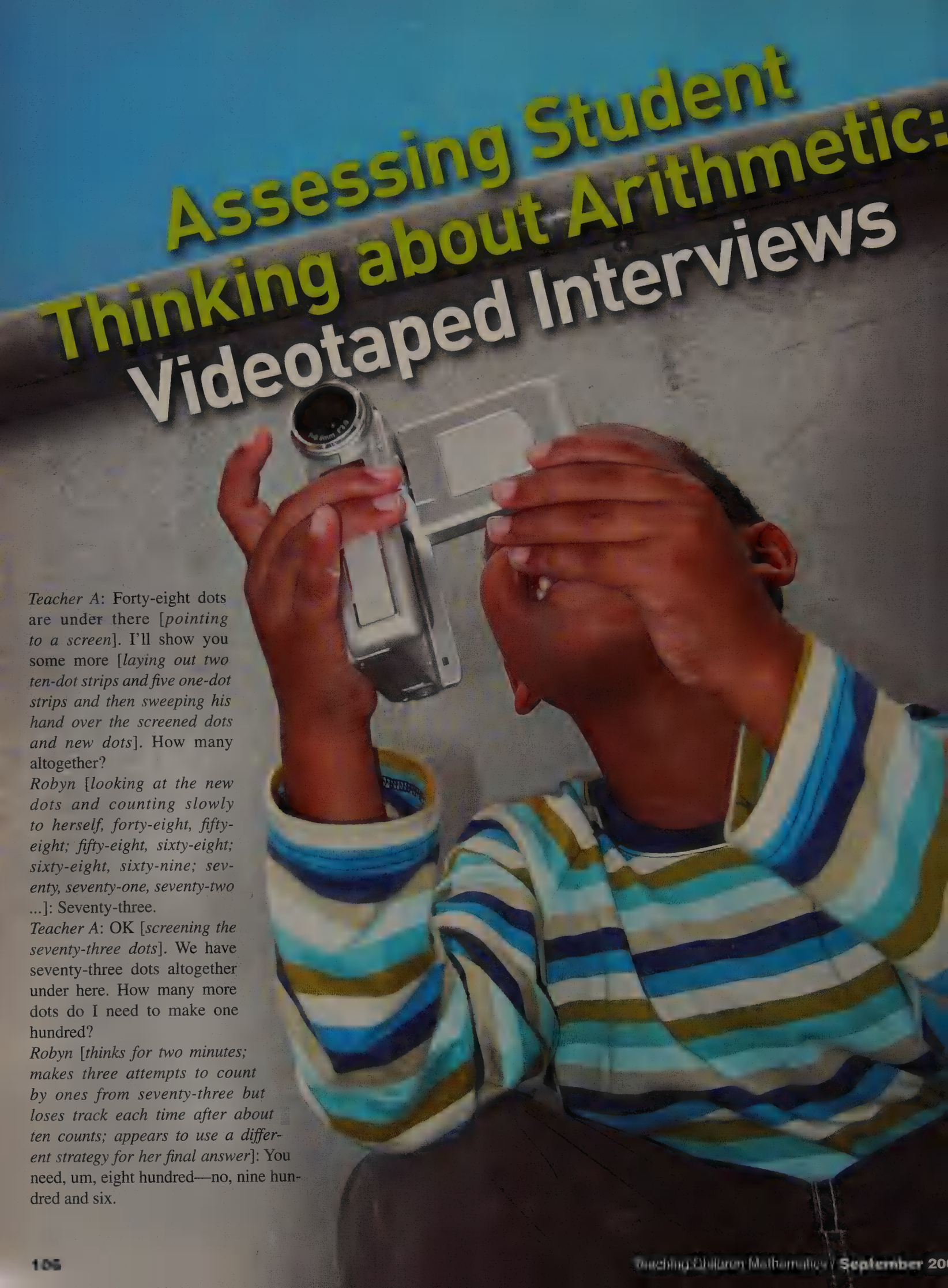
Assessing Student Thinking about Arithmetic: Videotaped Interviews



Teacher A: Forty-eight dots are under there [*pointing to a screen*]. I'll show you some more [*laying out two ten-dot strips and five one-dot strips and then sweeping his hand over the screened dots and new dots*]. How many altogether?

Robyn [*looking at the new dots and counting slowly to herself, forty-eight, fifty-eight; fifty-eight, sixty-eight; sixty-eight, sixty-nine; seventy, seventy-one, seventy-two ...*]: Seventy-three.

Teacher A: OK [*screening the seventy-three dots*]. We have seventy-three dots altogether under here. How many more dots do I need to make one hundred?

Robyn [*thinks for two minutes; makes three attempts to count by ones from seventy-three but loses track each time after about ten counts; appears to use a different strategy for her final answer*]: You need, um, eight hundred—no, nine hundred and six.

The excerpt on page 106 is from a videotaped assessment interview conducted by a fourth-grade teacher as part of a research project with children eight to ten years old who are considered low-attaining in arithmetic. Teachers and researchers watched this video excerpt several times to arrive at an understanding of how Robyn was thinking about two-digit addition (see **fig. 1**). The distinctive approach of videotaped interview-based assessment (VIBA) was central to the project. The teaching and research team appreciated the opportunity to analyze how each student was thinking about mathematics, and, in turn, to notice their own deepening knowledge of students' learning: "This has been the most powerful way I have found to get an insight into what the child is thinking" (teacher B, Numeracy Intervention Research Project 2006). This article describes the videotaped interview-based assessment, or VIBA, approach and explains the benefits of VIBA as an intensive assessment tool and a major source for teachers' professional learning.

Figure 1

Assessment of Robyn's two-digit addition thinking

Robyn—

- coordinates units of ten and one with base-ten materials;
- uses jumping by ten off the decuple: 48, 58, 68; and
- without base-ten materials, reverts to thinking in ones; she cannot think of ten as a unit.

Robyn's next learning needs to be constructing ten as a unit in *mental* arithmetic.

Figure 2

Pointers for videotaping interviews

- Ensure that you have parental or guardian permission to record the student.
- An additional person to operate the camera is unnecessary.
- Seat the student beside the teacher on the same side of the table.
- Position the camera on a tripod facing them, close (~2m) and high.
- Adjust the frame to capture the student's face and hands, the teacher beside the student, and the desk in front.
- Take care with lighting: Light the faces and avoid shining light into the camera.
- Take care with sound quality: Ensure that the student's voice is audible and minimize noise from air-conditioning, other people, etc.
- Label videotapes immediately with interview date, student's name, and school name.

What Is VIBA?

Videotaped interview-based assessment consists of a dynamic one-to-one interview of a student. The teacher poses mathematical tasks, observes the student's responses, and selects follow-up tasks based on the observations. The teacher may also ask students to explain their strategies. A wide range of mathematical tasks are suitable for an interview-based assessment (see the Further Readings section for extensive examples of tasks). Educators who use interview assessments have recommended their potential for decades (Cross and Hynes 1994; Labinowicz 1987; Weaver 1955). Long and Ben-Hur (1991) recommend interview assessments to enable teachers to "see problems through the eyes of the students, to respond to each student's particular needs, and to focus on stages of learning rather than answers."

The VIBA approach is interview-based assessment with two main refinements. First, the interview is recorded on videotape. A camera is positioned on a tripod to capture the student, the teacher, and the desk (see **fig. 2**). The assessment of the student's learning is then based on analysis of the recorded interview. Video analysis is particularly effective in collaboration with colleagues and a knowledgeable instructional leader. Second, VIBA specifies that the basic goal of the assessment is to determine the edge, or limit, of the student's knowledge and strategies. Determining this edge is achieved through attentive observation and questioning during the interview and skilled video analysis afterward. Below, we describe when VIBA is best used and then discuss three key aspects of VIBA: using video, determining the edge of a student's knowledge and strategies, and cultivating respectful attention in the interview.

When to Use VIBA

The VIBA approach was originally developed as a research tool for use in the study of early number learning (e.g., Steffe, Cobb, and von Glasersfeld 1988; Wright 1994). Because conducting the interview and reviewing the video are time consuming, VIBA is not used routinely. However, for the last fifteen years, the approach has been widely used in several circumstances:

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David L. Ellemor-Collins, David.Collins@scu.edu.au, is a researcher, and Robert J. Wright, Bob.Wright@scu.edu.au, is a professor in mathematics education, both at Southern Cross University in New South Wales, Australia. They are interested in intervention in early number learning and in developing teachers' knowledge of children's learning of mathematics.

- By teachers in Australia, the U.S., and elsewhere (Bobis et al. 2005) as suitable for an intensive, detailed, initial academic-year assessment, such as in *Count Me In Too* (New South Wales Department of Education and Training 2003)
- For diagnostic assessment in individual intervention, as used in Mathematics Recovery (Wright, Martland, and Stafford 2006)
- For extended professional development programs

Our current project, the Numeracy Intervention Research Project (NIRP), uses VIBA toward the goal of developing materials for intervention in the number learning of low-attaining third and fourth graders (eight- to ten-year-olds). During a three-year period, we worked with eight or nine teachers per year to pilot and refine assessment interview tasks, a learning framework, and instructional procedures. During the year, every one of these teachers participated in two days of training to conduct assessment interviews, assess the number knowledge of twelve low-attaining students using VIBA, and discuss analyses of the videotaped assessment interviews. They went on to teach eight of the students intensively for a term, supported by further professional development. The VIBAs were a diagnostic tool for the students in intervention, a source of professional learning for the teachers, and a source of data for our research.

Why Videotape Assessments?

Videotaping students is now used for a range of purposes in mathematics education. For example, researchers for the Learner's Perspective Study (Clarke, Keitel, and Shimizu 2006) videotaped students to investigate classroom culture and practices; Jacobs and others (2006) used videotapes as a catalyst for teachers to discuss their own teaching; and the IMAP project researchers (Philipp et al. 2007) used videotaped interviews so prospective teachers could begin reflecting on children's mathematical thinking. With VIBA, teachers videotape their own interviews to enable professional learning and detailed student assessment.

The process of conducting a videotaped assessment is fundamentally different from administering a written test or running record assessment that does not involve videotaping. Appreciating the value of video may be difficult without experiencing it. In our research and professional development projects, each new group of educators who has the opportunity to analyze the videotaped assessments is struck by video's potential to help them understand individual students' thinking. We describe below some reasons for videotaping assessment interviews.

Focus on the student

Videotaping an assessment session frees a teacher from the requirement to write a record of the student's responses while conducting the interview. Instead, the teacher can focus on observing and interacting with the student and can tailor the assessment tasks accordingly.

Review the interview

When a student's thinking is idiosyncratic or unexpected, a teacher might have difficulty determining in real time how the student is thinking about a task. "Often your perception of what happened or what you were doing is different from what actually happened" (teacher C). In assessing Robyn, teacher A reviewed several key interview moments and showed the videotape to colleagues to confidently determine that Robyn was thinking in tens with the ten strips.

See subtle clues

Full audiovisual records of students at work offer our best chance to interpret how they are thinking. Clues to a student's mathematical thinking may be subtle. Spoken words or written marks often sketch an insufficient picture, especially with younger students. During Robyn's second assessment task, small movements of her fingers, eyes, and mouth were critical in revealing her ones-based thinking: We could see that she was unable to turn ten ones into one ten to keep track of her count. Further, the videotape preserves a rich portrait of Robyn's mathematics, such as her readiness to engage with the problems and articulate her thinking, her energetic body language, the length of time she needed to process, and so on. Such idiosyncrasies cannot be captured by traditional pen-and-paper assessments. Teachers find it valuable to take the time to notice such features of students' thinking and to reflect on how to approach them in teaching.

Discuss with others

The videotape portrait is an invaluable assessment record for communication with colleagues, parents, and other professionals. "I was able to use the footage as a learning tool for staff at a professional development session and have them [identify] where the students' weaknesses were. I also shared it with

a parent to show where Luke's weaknesses were and how we were addressing them" (teacher D). Digital recordings can be readily edited, annotated, and reproduced, so teachers can show critical moments or particular tasks. Interviews that are not videotaped lack an authentic record, which weakens the interview's usefulness in communicating with others.

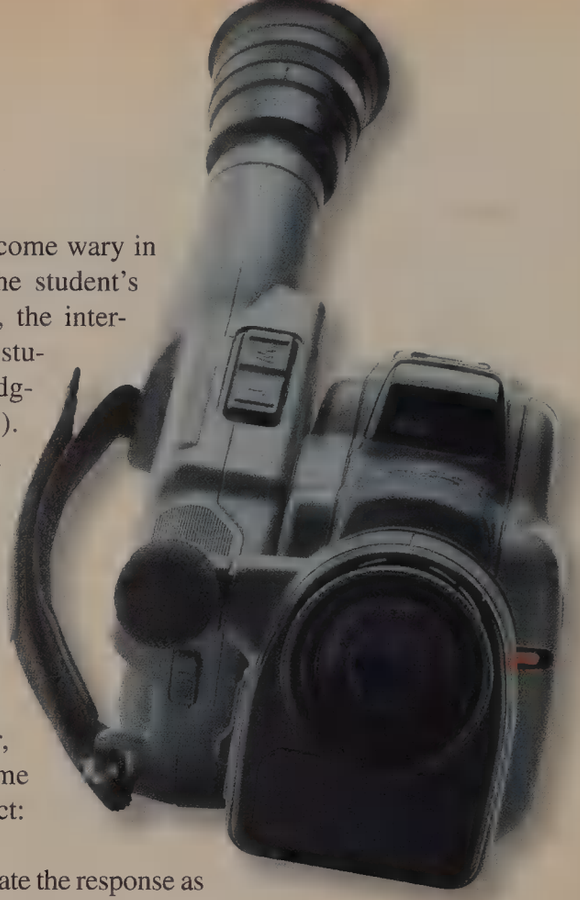
Determine the edge

In VIBA, the teacher's primary interview goal is to find the "cutting edge" (Wright et al. 2006, p. 28) of the student's knowledge in each topic. That is to say, a student's most sophisticated thinking in a particular topic and next potential learning step are highly valuable to a teacher. A great advantage of the dynamic interview approach is the teacher's opportunity to deliberately select tasks to assess where the student's knowledge reaches its limit. We are assessing not only the difficulty of the tasks but also the sophistication of the student's strategy in solving the tasks. Robyn's interview excerpt shows that she can add a number using tens when the tens are represented in materials but she cannot do so when the tens are not represented. This marks the upper limit—the cutting edge—of her sophistication in addition using tens.

Determining a student's most sophisticated strategy can be a challenge. Students frequently use less sophisticated strategies than those they are capable of. For example, a student might solve eight plus five by counting from one, although she is capable of counting on. Trying to elicit the most advanced strategy, interviewers might ask, "Can you do it another way?" Or they might pose supplementary tasks, as Robyn's teacher did with the second task in the excerpt. Using the VIBA approach, teachers have the opportunity to ask students directly about their strategies: "How did you do that?" However, it can be difficult for students—and for adults!—to explain how they thought about a task. Instead, as Robyn's teacher models, we rely on discerning as much as possible from close observation in the interview and from analysis of the video. "Through practice, review, and reflection, the interviewer becomes very attuned to the nature and range of children's responses to the tasks and the techniques involved in posing additional questions when necessary" (Wright, Martland, and Stafford 2006, p. 32).

Develop respectful attention to student thinking

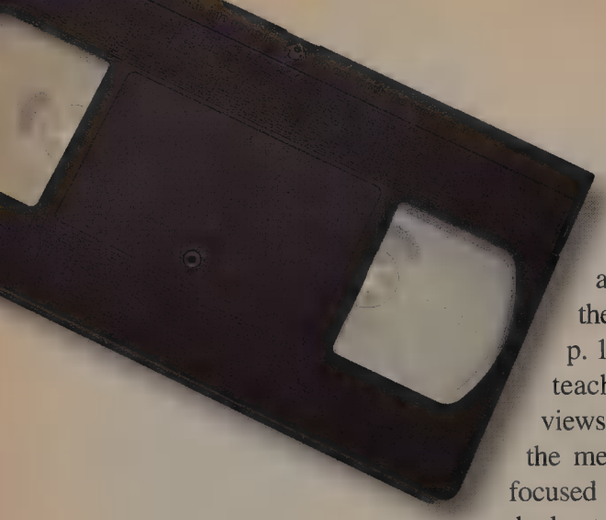
Labinowicz (1985) emphasizes that without an atmosphere of acceptance in an interview, students try to answer with what they think the adult wants to hear, or they become wary in answering. To explore the student's "unadulterated" thinking, the interviewer needs to accept a student's answers without judgment (Reys et al. 2007). Hence, a critical aspect of interview-based assessment is focusing respectful attention on the student's thinking rather than on the interviewer's judgments or expectations. This is a difficult skill to master, as Labinowicz's first-time teacher interviewers reflect:



> I was too quick to evaluate the response as right or wrong. I would then either immediately validate with "OK" or respond with "but" ....
>
> In my insecurity, I often prodded them gently for the answer.... I had an irresistible compulsion to not leave the child with the wrong answer. (p. 34)

The VIBA approach supports teachers in cultivating their attention to student thinking. Because the teacher does not take any notes during the interview, she can focus on her role as observer. Robyn's teacher managed to remain attentive and still while Robyn spent two minutes thinking hard about the "How many more to make one hundred?" task. Robyn, in turn, seemed to thrive on the experience of uninterrupted, focused problem solving.

Watching videotapes of one's own interviews rapidly builds self-awareness of the interviewer role. Munn describes this power of the videotape: When teachers first conduct VIBA interviews, they are most concerned with their own questions and the student's success. When students struggle with a task, teachers generally show tension, offer subtle prompts, and may interrupt students' thinking to stop them from working too long. These natural initial responses generate a social anxiety about students' work and performance. After analyzing their videotapes, teachers come to later interviews with a focus on the student's understanding. They can remain relaxed while a student struggles with a task and can become absorbed in observing the student's thinking (see **fig. 3**). "There is no social anxiety—just involvement with the problem.... [The teacher] lets the children be at their current stage of cognitive development rather than



wanting to cover over the gap and hurry the children through to the desired outcome" (Munn 2006, p. 183). A similar shift occurs when teachers analyze videotaped interviews: "The teachers neither evaluated the merit of the [child's] strategy nor focused on what the child did not know or had not been taught. Instead they tried to understand how the child was thinking and reasoning so that their instruction could be informed by children's perspectives" (Jacobs et al. 2006, p. 279). As teachers develop their skills in attending to students' thinking with VIBA, they can bring these skills to other assessments and teaching.

Inform instruction

"VIBA gives a fantastic starting point for teaching" (teacher B). The detailed insights of a VIBA can make a radical difference to the instruction a teacher offers. When asked, "From seventy-three, how many more to make one hundred?" Robyn answered "Nine hundred and six." Another student, Thalia, answered the same question, "Thirty-six." If this information were all we knew from an assessment, we might conclude that the students did not know the proper subtraction procedure. We might decide to focus our instruction on students learning a correct standard procedure.

After the VIBA, however, we were able to identify that Robyn had not yet developed a mental unit of ten (see **fig. 1**). Her mental strategy, after abandoning counting by ones, seemed to involve manipulating numerals in her mind, so she ended up with a three-digit answer. The teacher decided to focus Robyn's instruction on tasks to construct a mental unit of ten. In particular, the two of them developed mental subtraction strategies from Robyn's intuitions about whole numbers, such as trying the jump-by-tens strategies she used with the ten-strips.

On the other hand, Thalia's VIBA revealed that she was already using a jump-by-tens strategy. We observed that she arrived at the answer of thirty-six after making some counting mistakes, including an error in her forward number word sequence: "ninety-three, ninety-four, ninety-six, ninety-seven, ninety-eight, ninety-nine, one hundred." The teacher decided to focus Thalia's instruction on non–counting-by-one strategies—such as knowing without counting that ninety-three to one hundred is seven—and also to include some number word sequences practice.

In this example, the answer-based assessment and resulting instruction differ entirely from the VIBA and resulting instruction. Assessing only behaviors and answers leads to instruction focused on changing behaviors and procedures. In contrast, VIBA assesses students' strategies and conceptual development, allowing instruction to build on students' current concepts and targeting the edge of the student's sophistication in thinking about number.

Professional development

"We have found that the best and quickest way for teachers to develop a clear understanding of students' conceptual development is for them to analyze and discuss videotapes of students' responses to [assessment] tasks" (Munn 2006, p. 182). The National Council of Teachers of Mathematics (NCTM) Assessment Standards pose that assessments contribute to teacher learning as well as to student learning (see **fig. 4**). Watching assessment interviews on video is a powerful experience for teachers learning about children's thinking and reflecting on their own modes of interacting with students. Using VIBA, teachers develop knowledge that strengthens their use of interview assessment with and without video. Their increased knowledge of students' mathematical thinking also helps teachers develop instructional strategies. "The better interpreter you become of how children see the problem [in an interview], the better you will be at making on-the-spot decisions in the classroom" (Labinowicz 1987). Robyn's teacher, for example, became adept at recognizing and responding to students' tens-thinking in her general teaching. Professional learning is enhanced when teachers collaboratively analyze their videotapes of student

Figure 3

Pointers for conducting interviews

- Select tasks and follow-up tasks that are, in your estimation, at the student's knowledge edge.
- Avoid re-posing the task unless you are sure it is necessary.
- Verbal instructions may be rephrased if it is likely to help the student understand the task.
- Give the student sufficient time to focus on solving the tasks.
- Avoid temptations to talk to the student when he is engaged in solving a task.
- Watch closely for lip, finger, head, and body movements that might provide insight into the student's thinking.
- As a general rule, do not comment on the correctness of the student's response.
- Be careful not to overdo the technique of asking the student how she solved the task.
- Be sensitive to the student's state of ease and comfort during the course of the interview; use motivation and encouragement sparingly.

interviews (Jacobs et al. 2006). When teachers watch assessment videos together, they see other teachers' efforts, think hard about students' strategies, become inspired by students' thinking, and share instructional strategies. Using VIBA as an assessment tool contributes to ongoing professional learning that has much relevance to both classroom teaching and intensive intervention.

Further Reading

The *Arithmetic Teacher* articles (Labinowicz 1987; Long and Ben-Hur 1991) are excellent summaries of the interview method. *Early Numeracy: Assessment for Teaching and Intervention* (Wright, Martland, and Stafford 2006) includes detailed support for the VIBA approach and assessment tasks used in the Mathematics Recovery program. *The Teacher's Guide to Flexible Interviewing in the Classroom* (Ginsburg, Jacobs, and Lopez 1998) supports bringing interview assessments into general classroom practice.

Figure 4

A report card for VIBA using NCTM Assessment Standards

Mathematics Standard: Assesses mathematical sophistication
VIBA: Poses problem-based tasks across a range of mathematics contexts

Learning Standard: Affords an intimate opportunity for students to demonstrate their thinking and knowledge
VIBA: Significantly informs instruction for students' learning
Fosters teachers' professional learning

Equity Standard: Allows for students' multiple interpretations and approaches
VIBA: Cultivates a respectful, attentive setting

Openness Standard: Involves teachers directly in selecting tasks and interpreting responses
VIBA: Engages colleagues and wider audience in viewing assessments

Inferences Standard: Develops professional expertise in making inferences about students' thinking
VIBA: Offers a distinctive source toward multiple-source assessment

Coherence Standard: Records richness of student development
VIBA: Directly informs instruction—and instructional tasks can be used for assessment

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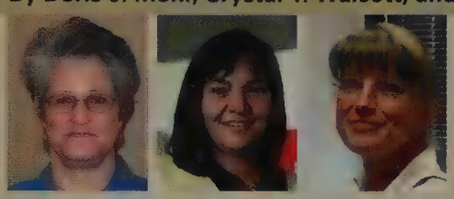
Using Your *Inner Voice* to Guide Instruction

By Doris J. Mohr, Crystal Y. Walcott, and Signe E. Kastberg



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Planning mathematics instruction has become a challenging dilemma. Teachers are asked to address standards and use tools such as textbooks and manipulatives as well as understand and challenge student thinking. At the heart of this planning dilemma are seemingly conflicting tasks: Teachers must (1) design instructional opportunities that provide the potential for children to grow mathematically and (2) plan from the perspective of students' existing knowledge.

Instruction based on what students know might be particularly difficult to plan in content areas

such as understanding shape. Shapes have names and attributes that we want our students to be able to use, identify, and classify. How can teachers support the development of what appears to be a simple exercise in memorization?

Children have words for objects and ideas that may be different from our own. Paley described her practice as becoming "curious about what [students] might think or say" (1986, p. 122). She recognized the distance between her teaching and her students' thinking: "In my haste to supply the children with my own bits and pieces of neatly labeled reality, the appearance of a correct answer gave me the surest feeling that I was teaching.... It did not occur to me that the distractions might be the sounds of children thinking" (p. 122). This new awareness encouraged her to revisit her own teaching practices through a consideration of the children's understandings. When building a practice from children's thinking, teachers might wonder—

- What are my students thinking?
- How might the children's understandings become the center of my instruction?

A variety of ways exist in which teachers may go about gaining understanding of what students are thinking. One of the most basic is to listen to what our students tell us. Davis (1996) identified three categories of listening: evaluative, interpretive, and hermeneutic. The first results from teachers posing questions to elicit an expected response. As teachers listen with an evaluative ear, they focus their attention on hearing a "correct" response, often overlooking important information of student understanding. Through interpretive listening, teachers seek information from students instead of a predetermined correct response. The teacher's communication goal is the primary difference between evaluative and interpretive listening. Instead of listening for a particular response (evaluative), the teacher listens to make sense of students' understanding (interpretive). The third type of listening, hermeneutic, requires the teacher to become part of the learning community, listening and responding to students through actions of mathematical inquiry.

Our recent efforts to make sense of children's written responses led us to draw parallels between the challenges we faced and the challenges that teachers face as they strive to make sense of students' mathematical understandings. In this article, we describe our process, share a tool that teachers can use to make sense of their students' understandings of shape, and provide sample instructional activities that can be used to elicit and build on the ways children think about shapes.

We began our inquiry with a belief that "listening" to the students' written responses would help us make sense of their thinking. As we listen to the children through their writing, our understanding of their thinking changes, much like the changes that take place when teachers attend to their students through hermeneutic listening (D'Ambrosio 2004; Davis 1996). In the practice of hermeneutic listening, the distinct roles of teacher and student dissolve as a community of learners develops in which everyone is engaged in making sense of the concept being investigated. D'Ambrosio believes that this type of listening is used by a teacher who "integrates the voice of the child, the voice of the discipline, and [the teacher's] inner voice in order to build a model of the child's mathematics" (2004, p. 139–40).

Instead of listening for a particular response, *teachers listen* to make sense of students' understanding

"The voices of the children include the ways in which children make sense of an idea.... The voice of the discipline includes ways of thinking, strategies, and understanding of the content that the teacher has acquired in her own learning experiences" (D'Ambrosio 2004, p. 137), as well as her beliefs about the nature of mathematics. When the voice of the discipline and the teacher's beliefs come together, the teacher develops her inner voice, which includes "the sense that the teacher has made of the content, the tools that she believes best illustrate the concepts, and her understanding of the nuances involved in learning the content" (p. 137).

Although our inquiry could not strictly be defined as hermeneutic listening—because no direct interaction took place with the learners in our sample—our inquiry provides structure for the discussion of our process. We share with you how our inner voices developed through listening to the voice of the discipline and the voices of the children.

Inquiring into children's understandings of shape

As researchers in mathematics education, we became interested in uncovering ways in which to listen to the students we study. A unique opportunity to investigate student understandings arose as we

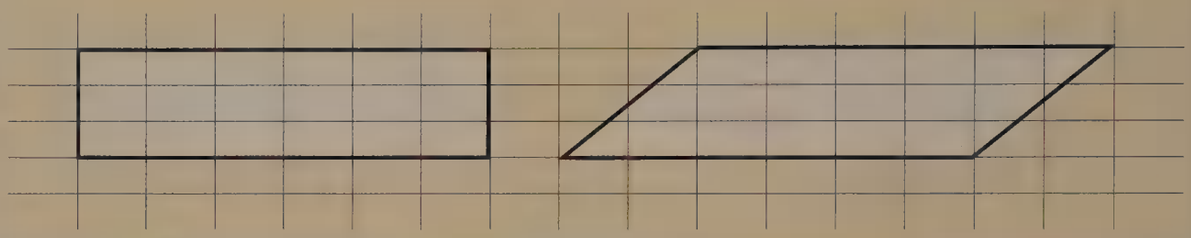
Figure 1

Fourth graders were asked to compare and describe a picture of a rectangle and a parallelogram.

Think carefully about the following questions. Write complete answers. You may use drawings, words, and numbers to explain your answer. Be sure to show all your work.

1. In what ways are the figures alike? List as many ways as you can.

2. In what ways are the figures different? List as many ways as you can.



began studying responses to open-ended mathematics items on the National Assessment of Educational Progress (NAEP). One particular item stood out because of the interesting nature of the student responses: When asked to describe how a picture of a rectangle and a parallelogram are alike (see **fig. 1**), a fourth-grade student responded, "They are kind of both rectangle[s], but one is tilted." We examined and analyzed 900 student responses, drawn from the 1992 and 1996 assessments, and found many intriguing descriptions like this one. We were captivated by the number of students who described the shapes as the same and cited a transformation of one shape into the other as their reason. Initially, we were unable to make sense of the children's approaches to this item and began to search for the logic in their thinking. In fact, when comparing the student work to the NAEP scoring rubric (available for download from nces.ed.gov/nationsreportcard/), we found little in common with the way the item was officially scored and the interesting responses in the student work we examined. Unlike scorers of the NAEP, who were looking for evidence of attention to certain attributes of the two figures, we anticipated that an examination of the children's responses could provide us with insights into their general conceptions of shape. In turn, we used the responses (and the framework we developed to analyze them) to help us think about instruction that builds on conceptions of shape that are different from our own or from those displayed in the NAEP scoring rubric. To shed additional light on the written responses, we turned to existing research.

Cultivating Our Voice of the Discipline

We considered prior research regarding the use of prototypes and visualization to inform our understandings of the children's responses. From that research, we found evidence that some children use prototypes and imposed attributes (such as orientation on the page or length of the sides) to classify shapes (Clements and Battista 1992; Hershkowitz 1989). For example, a triangle whose side is not parallel to the edge of the paper may not be identified as a triangle. In other cases, a long quadrilateral with at least one pair of parallel sides may be identified as a rectangle. Lehrer, Jenkins, and Osana (1998) also found that children's sense-making strategies often relied on the visual appearance of the figures. When asked to identify pairs of similar figures, children tended to identify a pair consisting of a rectangle and parallelogram of similar dimensions to be more alike than a pair of rectangles with obviously differing dimensions.

The ability to visualize plays a role in solving many mathematical problems, particularly when learning is achieved through sense making (Wheatley 1997). Mental images can often be manipulated, transformed from one shape to another. Evidence of morphing a figure was identified in the work of first- through third-grade students. When comparing a rectangle and a parallelogram, 40 percent of the students mentioned "pulling on the 'corners' to morph the first figure into the second or reasoned about the parallelogram as a 'stretched' rectangle" (Lehrer, Jenkins, and Osana 1998, p. 142).

These insights by other mathematics education researchers helped us make sense of the student responses. By integrating the voices of the children and our own understandings of the content, we began developing our inner voices and came to appreciate some of the nuances involved in building models of children's conceptions of shape.

Developing Our Inner Voices

In our research, we "listened" to the children through an analysis of their written responses. The categories we developed as a result of the analysis helped us organize the responses but did not suggest their specific meaning. We drew upon the voice of the discipline and our own understandings and beliefs about mathematics to interpret the evidence. We made the assumption that the children were thinking logically and attempting to make sense of the problem at hand. Our examination of the responses resulted in sorting the student work into two groups. One group listed formal attributes of the shapes as reasons for similarities and differences, or they referred to familiar prototypes or mental images. The second group mentioned some sort of manipulation of one or both of the shapes to explain the similarities or differences.

In the first group, some students claimed that the two shapes were the same or different on the basis of attributes such as angle measure, side length, or orientation of the sides (see **fig. 2**). Others in this group compared one of the shapes to a familiar prototype or mental image they developed from prior experiences. For instance, children identified one shape as a rectangle and the other as a parallelogram (see **fig. 3**). We used our understanding of mathematics to explain the existence of responses that identified the shapes as the same. We know, for example, that both shapes belong to the class of parallelograms. Using this perspective, the shapes are the same. Our experience as teachers of mathematics led us to wonder whether calling the shapes the same is linked to classroom experiences with the area formula. Often students' first experience with the area formula consists of a demonstration of the decomposition of a parallelogram into a rectangle, perhaps an implication that the shapes are the same.

Responses from the second group of students indicated that the two shapes were the same or different because of some type of morphing (Lehrer, Jenkins, and Osana 1998) of particular shape attributes or the mental manipulation of one of the shapes into the other. Students sometimes associated one shape with the other by indicating an action to be performed on one shape to transform it into the other. For example, in **figure 4**, the description suggests the action "slide" could make shape B the same as shape A. In other cases, the students described an action that has already been performed on one of the shapes. (i.e., "twisted" in **fig. 5**). Responses containing references

Figure 2

Some students used attributes to determine if the shapes were the same or different.

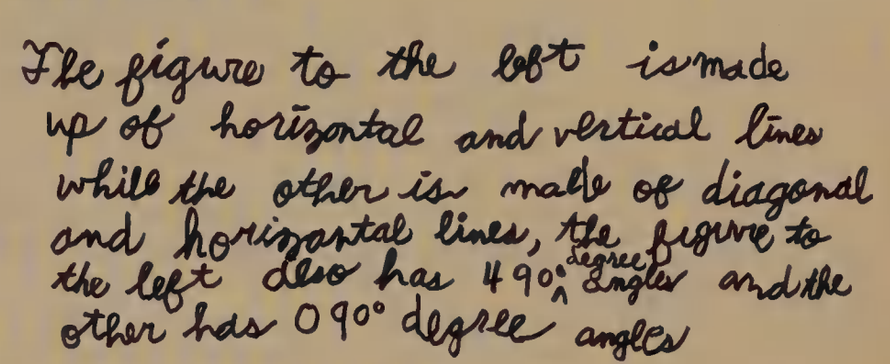

The figure to the left is made up of horizontal and vertical lines while the other is made of diagonal and horizontal lines, the figure to the left also has 4 90° degree angles and the other has 0 90° degree angles

Figure 3

This student compared the shapes to a previous mental image.

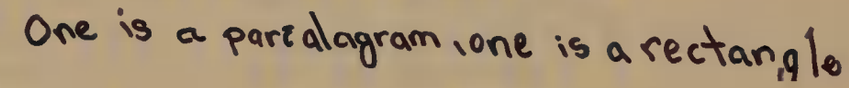


Figure 4

This student found shape similarities by morphing the shape.

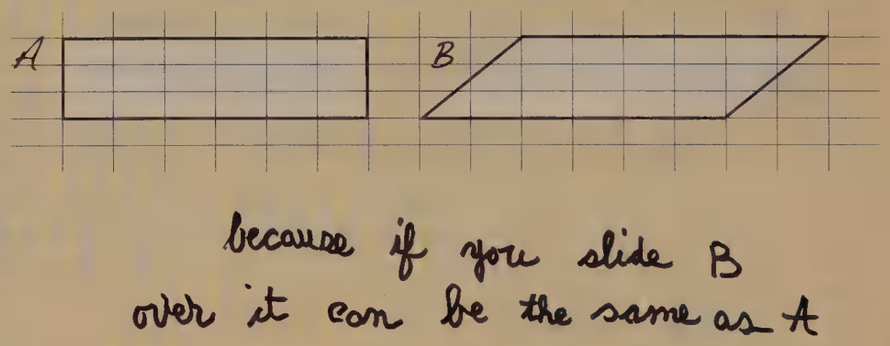
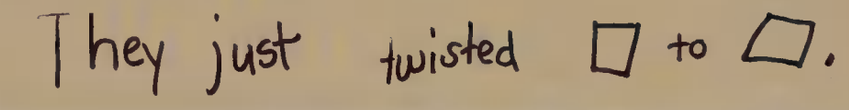


Figure 5

Students attempt to make sense of shapes using strategies they have at their disposal.



to some sort of morphing action on one or both shapes indicate to us that children are trying to make sense of shapes using strategies they have at their disposal, such as the mental manipulation of shapes, as well as prior knowledge gained from exposure to prototypes of rectangles and parallelograms.

Would your students respond in similar ways to this question? How can you use their responses to guide instruction? By calling on your inner voice as you analyze your students' responses, you can begin to make sense of students' understandings of shape and provide activities that will help children build on their existing knowledge. We provide tools to assist you in categorizing your students' responses and in selecting an appropriate activity.

Your inquiry into children's understandings of shape

As already stated, mathematics teachers face a dilemma in designing instructional opportunities that both consider students' existing knowledge and challenge them—particularly in content areas such as understanding shape, where understanding seems predetermined. Understanding of shape involves more than invoking memorized facts. To support the development of new ideas, teachers first must understand students' conceptions. The dynamic nature of teacher-student interaction is an imperative aspect of classroom communication.

As teachers move away from evaluative and interpretive listening and toward hermeneutic listening, they face the challenge of understanding their students on many different levels. One way in which teachers can begin to listen hermeneutically is through formative classroom assessment. Classroom assessment is "designed or used by classroom teachers for making instructional decisions, monitoring students' progress, or evaluating students' achievement" (Wilson and Kenney 2003). Formative classroom assessment incorporates gathering evidence of students' mathematical understandings as well as responding to that evidence by planning instructional experiences based on those understandings.

Before teachers start a traditional textbook geometry unit, they could do an open-ended investigation of students' understanding to identify novel interpretations of definitions and prototypes—much like those we shared from our research—often not elicited in traditional curriculum. The first step in assessment is choosing an appropriate open-ended item with the potential to elicit responses that will meet the assessment purposes and goals. Not all open-ended items serve this function. To ensure the choice of a quality item that represents what we value in mathematics, we should consider the item's alignment with three characteristics: "High-quality, open-ended assessment items should (1) involve significant mathematics; (2) have the potential to elicit a range of responses…; [and] (3) strike the delicate balance between providing too much information … and too little information" (Leatham, Lawrence, and Mewborn 2005). A good place to start is the NAEP archive of released items. Several NAEP open-ended items, including the one we used in our research, are available to teachers online. Along with released items, NAEP provides comprehensive achievement data, a scoring rubric, and samples of student work.

Once an appropriate item is chosen, the teacher's investigation refocuses on collecting data. By asking students to respond in ways that make sense to them, the teacher may assume that children will respond in unexpected, novel ways not addressed as part of the formal scoring rubric. These candid responses provide an opportunity for the teacher to build early understanding of the ways in which students think about shape; in turn, the classroom learning community is strengthened.

As teachers, we form categories of "right" and "wrong" answers. It is often difficult to move away from this mindset and toward a focus on children's sense making. We suggest using a process similar to our research process to identify themes in student thinking. To begin this process, administer the NAEP item (see **fig. 1**), and then read through the student responses. Pay particular attention to the expression

Figure 6

Use a checklist to look for ideas in student responses.

Categories of Ideas in Responses	Number of Responses	Number of Misconceptions
Identifying shapes by name		
Attributes		
Number of sides/angles		
Parallel/perpendicular sides		
Morphing of shapes		
Leaning, bent, slanted, or other movement		
Measurable properties		
Side length/angle measurement		
Perimeter/area		

of common ideas. We compiled themes from our findings into a checklist of ideas likely to be seen in your students' work (see **fig. 6**). If you find responses that do not fit into these categories, add new categories to the list. In addition, some responses may fit more than one category. If so, make photocopies of the response so that you may place it in multiple categories. Once you have sorted the responses on the basis of these categories, note the number of responses in each category in the second column of the checklist. Reread the responses, noting the number of misconceptions you identified within each category.

Instructional experiences based on children's understandings of shape

The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (NCTM 2000) challenges classroom teachers to become active participants in realizing a new vision for school mathematics. To meet this challenge, a learning community that builds on its students' experiences and prior knowledge is essential. In an attempt to make sense of children's mathematics and properly scaffold their students' conceptual growth, classroom teachers engage in observations of students at work, classroom discourse, and formative assessment. Any effort to make sense of a child's mathematics becomes ineffective apart from a teacher's ability to connect with the child's perspective on the task at hand. Activities that promote dialogue about shapes, their properties, and transformations of those shapes can easily be implemented in the classroom. From the sorting checklist, you gained evidence of the aspects of the shapes to which your students attend and the degree of misconceptions associated with those aspects. The following activity descriptions focus children's thinking on the investigation of characteristics of the shapes identified from our research.

Upon reviewing your checklist, if you find that your students failed to focus on naming shapes or identifying formal attributes of shapes, use activity 1 to provide a hands-on opportunity for students to develop their own categories of shape. If you found that several responses from your class tended to focus on dynamic movement (or morphing) of shapes, have students use the online applet in activity 2 to explore the relationship between rectangles and parallelograms. By manipulating dynamic figures, students can investigate which attributes remain fixed under manipulations such as dragging corners and pulling on sides. If you found that your students did not attend to measurable aspects of the shapes, consider using activities 3 and 4 to build a conceptual understanding of the relationship between these properties. These investigations provide opportunities for children to explore the effects of manipulating shapes' attributes on their perimeter and area.

Activity 1—Exploring Attributes: Refer to the shape-sorting and comparison activities offered by Fox (2000). These activities provide an opportunity for children of all levels of understanding to reason about, engage in, and communicate mathematical ideas related to shape. First, children operating at a visual level of understanding may communicate geometric reasoning through a discussion of the resemblance of a given shape to other shapes. Children who reason through resemblance may compare and group the shapes by nonmathematical characteristics of the shape. For instance, children may argue that a chevron resembles a triangle because it is "pointy."

Second, children operating at an analytical level of understanding may communicate geometric reasoning through a discussion of the formal attributes of shape. Children who reason through an analysis of the attributes of shapes may compare and group shapes by formal attributes of the shapes. For instance, children might state that two rectangles are alike because all angles in both shapes are right angles.

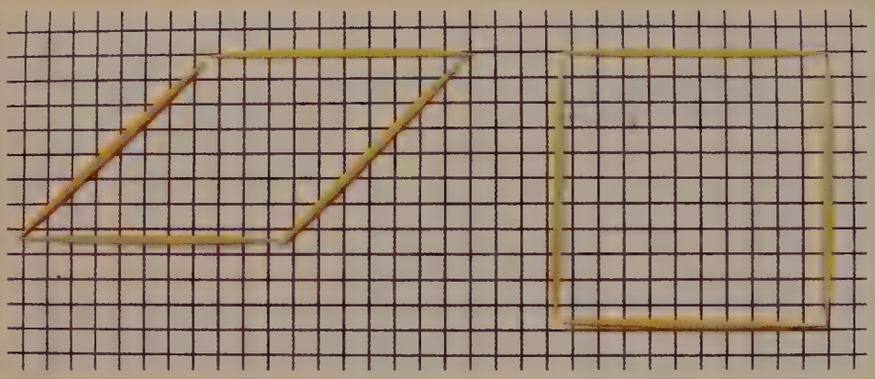
In hermeneutic listening, distinct roles of teacher and student dissolve and a *community of learners* develops

Third, children operating at a deductive level of understanding may communicate geometric reasoning through a discussion of the similarities or differences between classes of shape. Children who reason deductively may compare and group shapes by common properties within a class of shapes. For example, children may reason that squares are special cases of rectangles in which all sides are of equal length.

Activity 2—Using Dynamic Software: An activity using an electronic applet to explore parallelograms and rectangles can be found at illuminations.nctm

Figure 7

Using toothpicks, students constructed a rectangle on a grid and estimated the number of squares enclosed by the toothpicks.



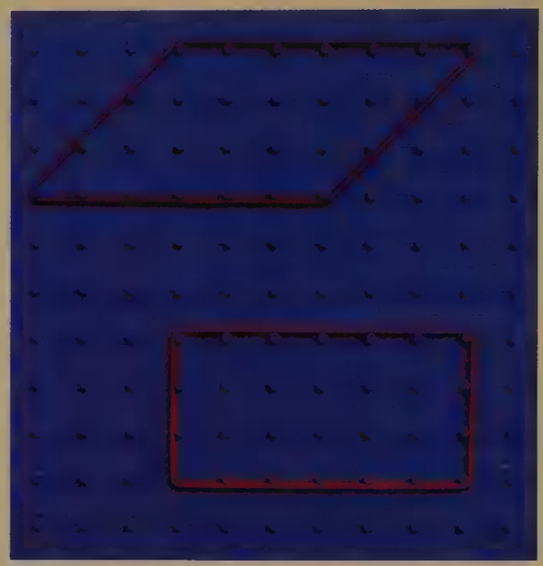
.org/LessonDetail.aspx?id=L350. Children are encouraged to manipulate the shapes by dragging on the vertices (corners) and pulling on the edges (sides). The shapes, however, will maintain their attributes. The activity begins by having the children brainstorm as a group about the similarities and differences of a rectangle and a parallelogram. Then they categorize the attributes according to length and number of sides, measure and number of angles, and so forth. After mentally manipulating the shapes and discussing whether a shape will retain its attributes, children use the applet to test their conjectures. The activity ends with a discussion of their findings. Questions to pose include the following:

- How are all the figures produced by the dynamic rectangle alike?
- How are all the figures produced by the dynamic parallelogram alike?
- What characteristics are common to both rectangles and parallelograms?
- How do they differ?

Activity 3—Constant Perimeter: Using toothpicks, children construct a rectangle on a grid (see **fig. 7**) and count the number of squares enclosed by the toothpicks. Have the children use the same number of toothpicks to construct a parallelogram with slanted sides. By estimating the number of squares enclosed by the second shape, the children can visually ascertain that the area changes but the perimeter remains constant. If they are already familiar with the area formula, focus their attention on the length of the base and the height of each shape. Have the children repeat the activity with a

Figure 8

By moving the base of their rubber bands horizontally on Geoboards and keeping the height constant, students formed parallelograms that had the same area; they counted the number of squares inside the rubber band to convince themselves of the constant area.



different number of toothpicks. As they work, ask questions such as the following:

- Are both shapes parallelograms? How do you know?
- Are both shapes rectangles? How do you know?
- Did the perimeter remain the same?
- Did the area remain the same?

Activity 4—Constant Area: Students form rectangles using Geoboards and rubber bands. Then, by moving the base of their rubber band horizontally (keeping the height constant), they form other parallelograms with the same amount of area (see **fig. 8**). If they are not yet familiar with the area formula, they can count the number of squares inside the rubber band to convince themselves that the area has remained constant. Using rulers to measure the lengths of the sides, the children can now see that the perimeter has changed but the area has not. Again, follow up with questions:

- Are both shapes parallelograms? How do you know?
- Are both shapes rectangles? How do you know?
- Did the perimeter remain the same?
- Did the area remain the same?

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In both activities 3 and 4, focus attention on changes in the attributes of the shape, such as orientation of the sides and angle measure and the subsequent impact on perimeter or area. Ask students, "In activity 3, why did the area change but not the perimeter? In activity 4, why did the perimeter change but not the area?"

While exploring rectangles and parallelograms, some children may identify the two shapes as "the same." This comment can give rise to a discussion of naming conventions and an exploration of what it means for two shapes to be "the same." Whereas both shapes are parallelograms, they are not both rectangles. The discussion of language used to identify shapes will help students build an understanding of the power of definitions while supporting their understanding of mathematical terminology developed through community discussion. The students' construction of a working definition will be more useful than one imposed on them by an external authority such as a textbook or the teacher.

Final Remarks

Through the use of high-quality, open-ended assessment items, you will be able to listen to your students and avoid overlooking, as Paley did, "the sounds of children thinking." Using hermeneutic listening (through dialogue or written responses) engages teachers, as well as students, in making sense of mathematics. This, in turn, leads to pedagogically sound instructional decisions that build on prior knowledge and engage students in the development of further sense-making strategies. The voices of the students and the voice of the discipline merge to strengthen your inner voice as you build a model of your students' understanding of shape.

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Look Who's Talking: Differences in Math Talk in U.S. and Chinese Classrooms

After twenty-three years of teaching, I stepped out of the classroom and into the world of education research. As part of a team of researchers comparing mathematics teaching and learning in the United States and China, I spent many hours watching videotaped mathematics lessons from fourth- and fifth-grade classrooms in both countries. It was fascinating. (To be honest, it was luxurious, since I was not also trying to grade spelling tests while I watched.) After I got past my initial reactions to the foreign setting—including bare walls, desks in rows, and over forty students per class—more substantive features of the differences between Chinese classrooms and what I was accustomed to seeing in U.S. classrooms began to capture my attention.

For example, the Chinese teachers did not appear to talk as much as the U.S. teachers. And, in contrast, the Chinese students seemed to talk much more than the U.S. students did. In the U.S. videos, I saw familiar scenes: numerous manipulatives, creative activities, tables set up with mathematics games—all designed to engage students in the mathematics. In visual contrast, the crowded Chinese classrooms seemed almost bare: a teacher and her students; pencils and paper; thin, paperback textbooks. Yet this was somehow enough; the students clearly engaged with the mathematics. In the Chinese classrooms, the star attraction was the *mathematics,* and the core ideas were most often featured through *student talk.*

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Edited by **Cindy Langrall**, *Langrall@ilstu.edu, a professor in the mathematics department at Illinois State University in Normal. "Research, Reflection, Practice" describes research and demonstrates its importance to practicing classroom teachers. Readers are encouraged to send manuscripts appropriate for this section by accessing* **tcm.msubmit.net**. *Manuscripts should be eight to ten doubled-spaced typed pages.*

The idea of "math talk" was certainly not new to me. I was aware of the importance of purposeful discussion as a major element in mathematics reform. In the early 1990s, when NCTM called attention to classroom dialogue, leaders in the educational community began pressing for teachers to analyze how they currently used conversation in mathematics classrooms and to consider how they could use it more effectively (Ball 1991, 1993). Researchers looked closely at how teachers—through selection of activities, instructional style, and, yes, classroom talk—shaped students as active, verbal lesson participants. Their findings suggested that student achievement and engagement are enhanced when students receive opportunities to explain and justify answers, compare multiple solution strategies, and accept and learn from errors (Kazemi and Stipek 2001; McNair 1998; Rittenhouse 1998; Turner and Meyer 2004; Turner and Patrick 2004; Whitenack and Yackel 2002). These findings were familiar to me; I would have nodded in agreement with anyone purporting the benefits of student math talk. Yet I was still taken aback by the striking differences I saw in the videos. I was convinced that I had not given math talk its deserved place in my own teaching repertoire, and I resolved that should I ever return to teaching, I would work hard to include more math talk, especially from my students.

I shared these observations and reactions with my research colleagues. We were intrigued and decided to quantify my initial impressions through a more formal investigation of teacher and student talk in Chinese and U.S. classrooms. In the end, these and related investigations (Schleppenbach et al. May 2007; Schleppenbach et al. November

2007) not only confirmed my initial observations, but actually revealed greater differences than I originally imagined. This article presents the results of two investigations that look closely at math talk in Chinese and U.S. classrooms and examines the questions these studies raise about how we go about the business of teaching elementary mathematics.

Study 1: Chinese and U.S. Classrooms

As mentioned previously, one of my initial observations of the Chinese mathematics lessons was that the students engaged in a large amount of math talk. Therefore, we began our investigation by directly comparing the math talk in Chinese and U.S. classrooms. By closely examining twenty-eight videotaped lessons from fourth- and fifth-grade mathematics classrooms in both countries, we hoped to capture *how much* math talk took place as well as *who* was doing the talking in these classrooms. (The data were originally collected as part of a larger investigation, the intent of which was to collect and catalog video data of mathematics classrooms for the purpose of understanding successful teaching and learning practices in China and the United States.)

To answer the questions of how much math talk took place and who was doing the talking, we transcribed all the dialogue in the lessons and then split each transcript into two parts, one consisting only of the teacher's words and the other of students' words. From these split transcripts, we compared what the teachers were saying to what the students were saying.

Teachers talked more than their students in both countries, as measured by the total number of words uttered in a lesson (student words plus teacher words in all statements produced in the lessons). However, this difference was more dramatic in the U.S. classrooms, with teachers producing 89 percent of classroom discourse (as measured by number of words uttered) compared to 65 percent in the Chinese classrooms.

We then searched the separate teacher and student transcripts for the presence of *math talk*, which we defined as explanations, declarations of formal principles or procedures, and other mathematical statements. When we narrowed our focus to include only mathematical statements, we found a striking difference: In U.S. classrooms, *teachers* produced the vast majority (79 percent) of the mathematical statements; Chinese classrooms showed the inverse pattern, with *students* producing 69 percent of the mathematical statements (see **fig. 1**).

In sum, Chinese students produced both more total talk in classrooms and more mathematical statements compared to their U.S. counterparts. Some excerpts from the U.S. and Chinese lessons, presented in **figures 2** and **3**, illustrate this contrast further. As you examine these excerpts, pay particular attention to how much the teacher talks compared to how much the students talk.

The investigation results clearly point to differences in the frequency of student and teacher math talk in Chinese and U.S. fourth- and fifth-grade classrooms. Further analyses (Schleppenbach et al. May 2007; Schleppenbach et al. November 2007) pointed to even more striking differences in the *character* of this talk. For example, teachers in our video sample of Chinese classrooms explored errors in greater depth. They also pushed students to evaluate and explain their reasoning, even after responding with correct answers. Predictably, these findings sparked many discussions about the reasons behind the different math talk norms in these classrooms.

Would the U.S. results have looked different if we had examined only classrooms that used a discourse-focused reform curriculum? (None of the classrooms in our initial study used an NCTM-inspired reform curriculum.) Because reform curricula are specifically designed to promote student dialogue (Wagreich et al. 1997), a logical next step would be to apply the same analytical lens to U.S.

Figure 1

Proportion of mathematics statements produced by students

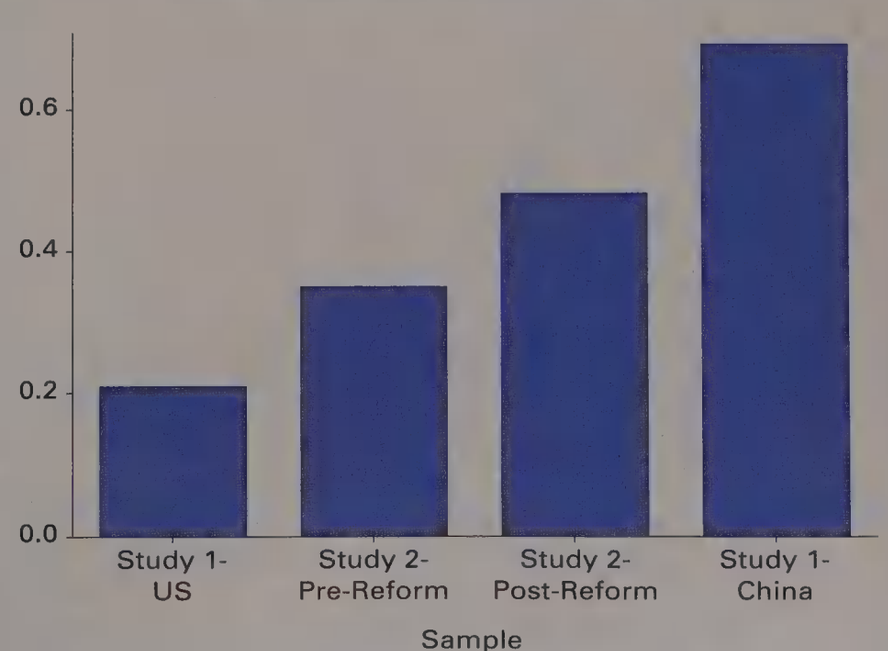

Figure 2

Excerpt from a typical U.S. lesson
(teacher contributions are noted in bold)

T: **OK, today we are going to learn about something called equivalent fractions. And all year we have been talking about how a number can have more than one name or there can be more than one way to express a number. On the test you just took, there was a whole section about the number 27. Do you remember that? And you were supposed to write or express the number 27 in many ways. And you all did a great job, so I know you understand that a number can have many different names, many different ways of expressing it. And the same thing is true for a fraction. A fraction can have many different names and mean the same number or same amount. Kristen?**
S: I have an example of two numbers. Like three-sixths can be one-half because it is half of something.
T: **Good. And that's exactly what we are going to be learning today. Kristen said that three-sixths and one-half equal the same thing, they represent the same amount. And that's what we mean by an equivalent fraction. Now let's begin with a little review. What is a fraction and why do we need fractions? Why do we need to be able to write numbers in fraction form? Andrew?**
S: Because we need, if there's more than the number and less than the other, we need something in between.
T: **Okay. Alright. What do you think, Josephine?**
S: If there's less than a number, then you'll need to write like a half.
T: **Less than which number?**
S: One.
T: **Less than one. Okay? If we want to talk about less than one, we need some way to express it. You already learned one way to do that. What unit did we have recently where we learned a form in which we can write a number when it was less than one whole thing? Corey?**
S: Decimals.
T: **Decimals. We just finished a unit on decimals, so you already know about decimals. And fractions and decimals are very much related to each other. So a fraction is another way to express something that is less than one whole thing. So, if you buy a gallon of milk at the grocery store and you only drink one cup for breakfast, you didn't drink the whole gallon, did you? And so you need some way to talk about less that that whole gallon of milk. If you order a pizza—if it's a large pizza—probably you can't eat the whole thing. And you need a way to talk about what part of that pizza you ate. Now you also need to know that we have special names for the numbers in a fraction. And what do we call the top number? Gina?**
S: Numerator?
T: **It's called the numerator. And this should be review from last year. And Tessa, what do we call the bottom number?**
S: Denominator.
T: **The denominator. Those are important to remember because we'll be talking about those terms a lot. And then I told you what this line in the middle means. And we've talked about that all year too. That little slanted line. You can either write it like this or you can write it slanted. And what does that line mean always? Jacob?**
S: It's the divide sign.
T: **It's the divide sign. It's one way of expressing division. So this really means three divided by six. Ok, and that's how you change a fraction to a decimal, which we're going to be doing later on this morning.**

lessons specifically designed to promote student math talk. Thus, we designed a study to answer the question: Does the implementation of a reform curriculum change classroom dynamics in such a way that U.S. classrooms exhibit participation levels closer to what we find in Chinese classrooms?

Study 2: U.S. Reform Mathematics Classrooms

To conduct our study of talk in U.S. reform mathematics classrooms, we analyzed thirty U.S. fourth-grade lessons (McConney 2003). These data included videotaped classroom observations of three U.S. teachers who used a traditional mathematics curriculum in year one and a reform curriculum, Math Trailblazers, in year two. For each teacher, we studied a set of five lessons from each year. We performed the same analyses of the frequency of student and teacher math talk as in the previous investigation into Chinese and U.S. classrooms.

Our findings regarding word utterances as measured by the total number of words uttered across all lessons (student words plus teacher words) were similar to the U.S. classrooms in the first study. Teachers talked more than their students in both year one (89 percent of all words uttered came from teachers) and in year two (81 percent came from teachers). Although we found a relative decrease in teacher talk from year one to year two, the decrease was not dramatic.

Teachers' production of mathematical statements went virtually unchanged, from a total of ninety-three in year one to ninety-seven in year two. However, when we looked at students' production of mathematical statements, we witnessed an increase of almost double (from fifty-one to ninety) after the adoption of the reform mathematics curriculum. Thus, the relative proportion of student math talk changed to more closely resemble what we observed in the Chinese classrooms. In addition, we found that in year one, teachers produced the vast majority (65 percent) of the mathematical statements. In year two, after reforms were instituted, the students contributed considerably more mathematical statements than in year one (producing 48 percent of all the mathematical statements in year two compared to 35

percent in year one). **Figure 1** shows the amount of math talk produced by students across both studies.

Discussion

After our research group completed these studies, I wondered, as a former teacher, what all this actually means in terms of teaching. Certainly, we can make the claim that reform curricula may be an important aid in improving student participation, but, by itself, it is no panacea. The Chinese teachers did not use what we would consider a reform curriculum, yet student contributions to the mathematical discussions were abundant. Clearly, what a teacher thinks and does plays a major role in shaping classroom discussion norms. Looking beyond the obvious, other important considerations began to emerge.

First, I was struck by what teachers' instructional decisions imply about their underlying convictions about students' proper roles in a mathematics classroom. As I witnessed the abundance of teacher-orchestrated student discourse in the Chinese classrooms, a clear message came through the subtext: Student contributions are crucial to learning. When students' contributions predominate, students appropriate a shared (albeit tacit) belief that they and their peers are responsible for maintaining high levels of classroom discourse and for persistently exerting effort to achieve their own learning. Naturally, I began to wonder about what *my* teaching practices said about my own beliefs. I thought back to all the times I had interrupted a child's explanation with the noble intention of helping him articulate his thoughts more clearly or the times I had quickly moved on to the next question as soon as I received a correct answer. Had my interruptions sent the message that students were incapable of expressing ideas clearly? Did my excited affirmations of correct answers send the signal that accuracy and efficiency were more valuable than debating mathematical ideas or struggling with important concepts? As Pajares noted, the assumptions we make about our students' abilities and how our students acquire mathematical knowledge are likely manifested in our practice (1992). In other words, our actions speak volumes. If we want to encourage meaningful student contributions in mathematics class, we must occasionally step back and consider what our instructional practices say about our beliefs. We must ask, "What do my words and actions say about how I evaluate student contributions in my classroom?"

Second—and perhaps most important—I was struck by my surprise at our research findings. Could I trust my perceptions about the amount of math talk going on in my classroom? When I originally watched the videotaped lessons from China and the United States, it was obvious that the Chinese students were engaged in more math talk, but I was truly astonished when I examined the transcripts that showed the U.S. student talk. "Where's the rest of what they said?" I wondered. "This can't be everything." After so many years of teaching, I expected myself to be a better judge of this phenomenon—yet I had clearly overestimated the frequency and quantity of students' contributions to classroom discourse. Perhaps the frantic pace of my classroom life had robbed me of the chance to step back and absorb important details. Perhaps I had planned so

Figure 3

Excerpt from a typical Chinese lesson
(teacher contributions are noted in bold)

T: **The first one is a quick calculation. Stand up when you know the answer. Let's see who can be the first. (Teacher holds up cards with addition and subtraction of fraction problems.)**
S: 8/11.
T: **OK. Next. Ready?**
S: 3/14.
T: **See if you can be faster.**
S: 7/12.
T: **Make sure to reduce it to the lowest terms.**
S: One.
T: **Good! Sit down, please. Very good job of mental calculation. Remember, in this practice, pay attention not only to speed, but also to accuracy. Now, do you recall how to calculate the addition and subtraction of fractions with the same denominator? Chen Xu?**
S: Add or subtract the numerators while keeping the denominators unchanged.
T: **Is that right?**
S: Yes.
T: **Okay. When you add or subtract fractions with the same denominator, keep their denominators unchanged and add up the numerators. What does it mean to keep the denominator unchanged? And what does it mean to add up or subtract the numerators? Zhang Rui.**
S: Unchanged denominators refer to the unit of the fraction. To add or subtract numerators is to add or subtract the number of fractions.
T: **The fractions ...?**
S: Unit.
T: **The number of fraction units. Very good. Now let's review the following content that we studied in the past. These are completions. Please raise your hand to answer. Zhong An?**
S: 3 ones plus 2 ones is 5 ones.
T: **Quite easy, isn't it? Zhangzhong?**
S: 3 tens plus 2 tens is 5 tens.
T: **Wangmiao?**
S: 3 tens plus 2 ones is 32 ones.
T: **In other words ...**
S: 32.
T: **But shouldn't we just say "3 plus 2 equals 5"? Chengzhaoyu?**
S: No, because the units are different.

many lessons and considered so many concepts that I wrongly assumed that my thinking must surely mirror my students' thinking. As an experienced teacher, it was unsettling to consider that I could not necessarily trust my perceptions. But the undeniable evidence of the transcripts provided a long-overdue reality check that was impossible to ignore.

I encourage you to arrange your own reality check. Record yourself on audiotape (or videotape) for just ten minutes. Or pair up with a colleague and observe and take notes on how much the students are talking. More important, pay attention to the feature over which you have the most control—how much *you* are talking. You may be surprised. (For additional background and practical tips about creating a discourse-rich classroom, see, for example, Amos 2007; Chapin, O'Connor, and Anderson 2003; and Charles 2006).

When I was teaching, I could not have envisioned a classroom where students talked twice as much as I did. But, as I learned from this study, this is exactly what happens in a discourse-rich classroom. I can imagine this now—and perhaps that is the first step. I can see myself as one who directs, not dominates, the discussion. I can hear the lesson unfold in the voices of the students as they share their problem-solving strategies, confusion, or challenges to others' ideas. I am not saying it would be easy—but I believe it would be worth my best effort. As teachers, we can arm ourselves with promising new curricula that promote important math talk among our students, but ultimately it comes down to what we do and say—or do not say. I urge you to think before you speak: Letting our students provide most of the mathematical talk in our classrooms may allow them to own the mathematical knowledge in a much deeper way than if we try to hand it to them.

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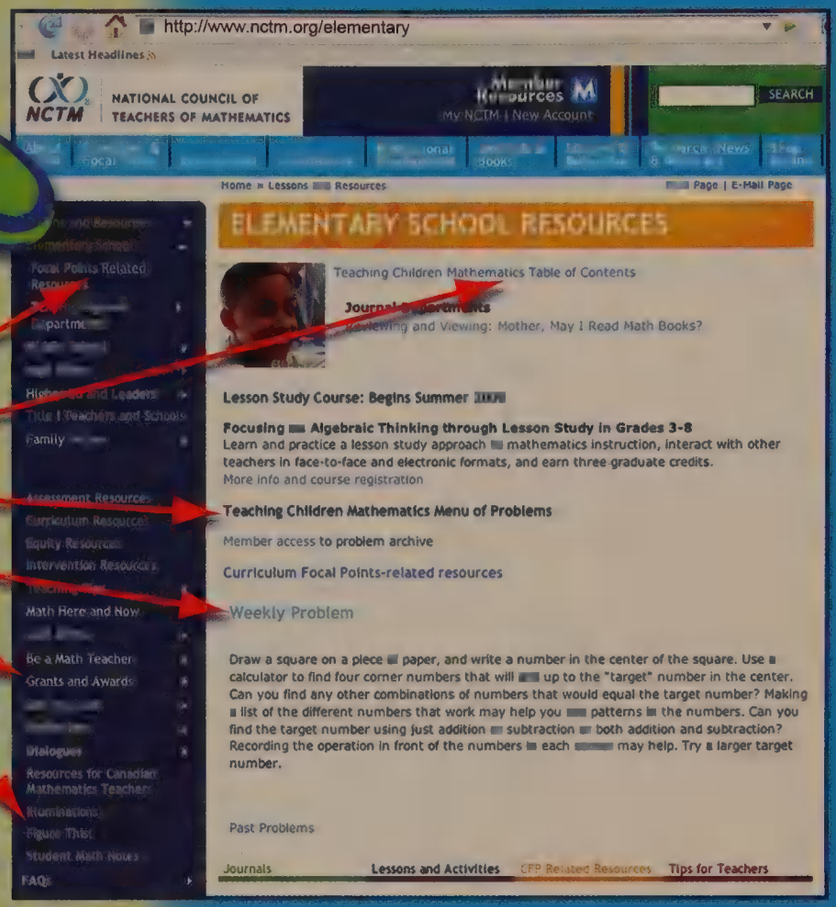
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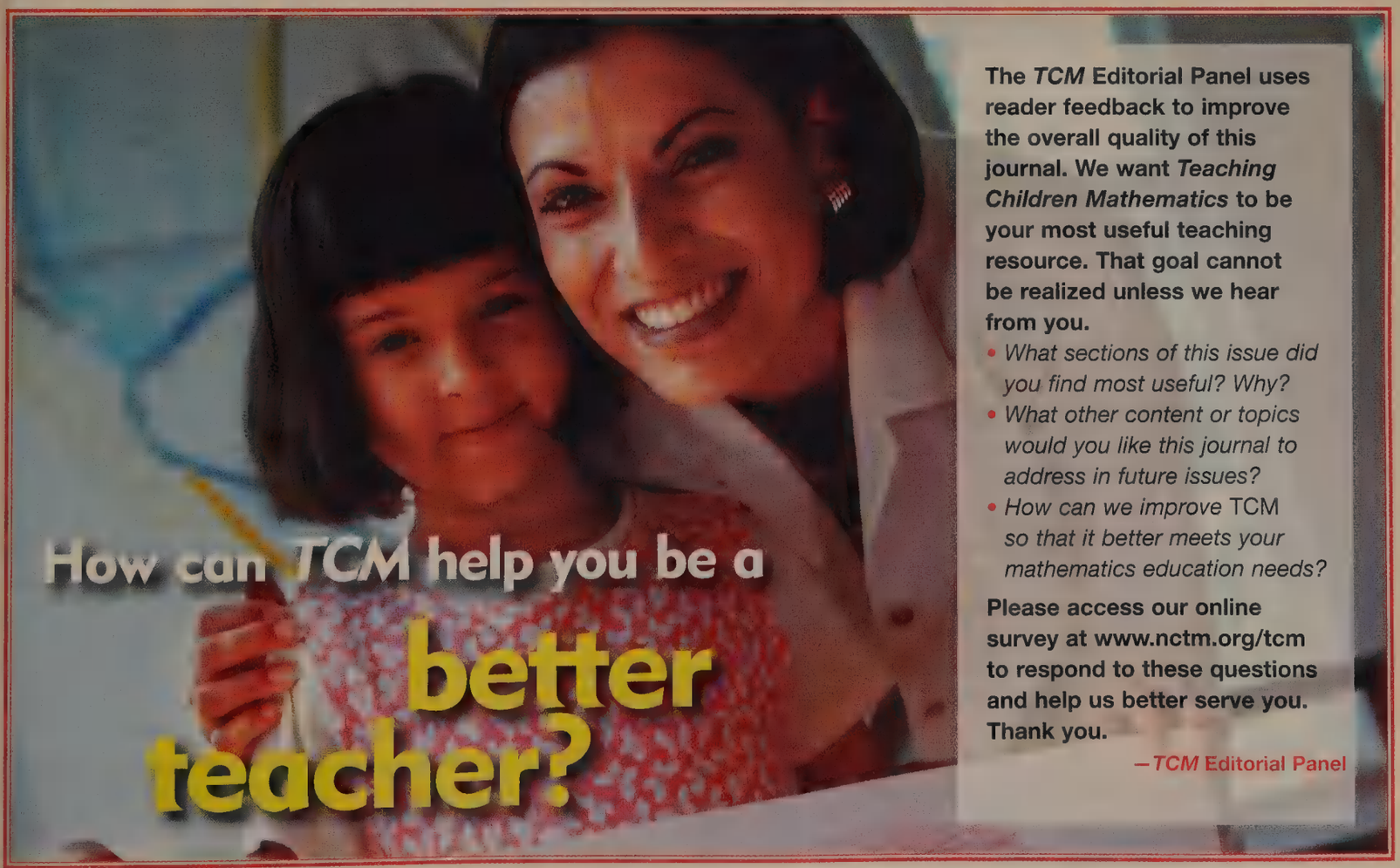

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—*TCM* Editorial Panel

Mathematics Every Day

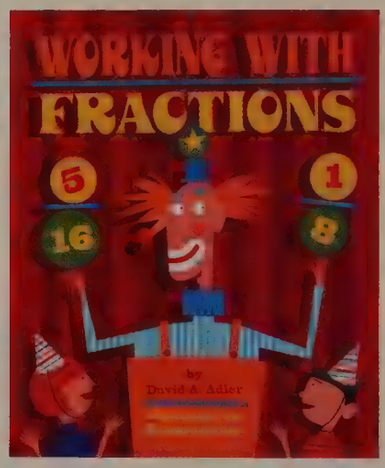
Books

Read-it! Readers: Math Series, *Marcie Aboff, 2008. 32 pp. ea. $19.93 cloth; $79.72 set of 4.* **The Pizza Palace**, *ISBN 978-1-4048-3665-5PK;* **Mike's Mystery**, *ISBN 978-1-4048-3667-9PK;* **The Tallest Snowman**, *ISBN 978-1-4048-3666-2PK;* **The Lemonade Standoff**, *ISBN 978-1-4048-3668-6PK. Picture Window Books Publishers; (877) 845-8392; picturewindowbooks.com.*

Dan and Sue play outside in *The Tallest Snowman*, attempting to build a snowman that is taller than a nearby tree. The author uses customary units for all measurements and provides metric equivalents. In *The Lemonade Standoff*, Jen and her brother Nick pair off with two others for a friendly competition to see who can sell the most lemonade, providing context for practicing the use of tally marks and two-digit addition without regrouping. *The Pizza Palace* is the setting where Flory Glory discusses fractions of both sets and wholes and uses tally marks. *Mike's Mystery* features class members collecting books for their school's book drive. To measure how close they are to achieving their goal, Mike and his friends subtract the number of books they collect from seventy-five. The computation in this book does not extend to regrouping.

Each of these four books from the Read-it! Readers: Math series portrays everyday situations that provide a context to show students how to use mathematical concepts. The texts are on a second- to third-grade reading level with appropriate mathematical vocabulary and include a special code that readers can use at facthound.com to access additional online resources and activities.—*Deirdre Dacus, James Poole Elementary, Big Sandy, TX 75755.*

Working with Fractions, *David A. Adler, 2007. 32 pp., $16.95 cloth. Ages 6–10. ISBN 978-0-8234-2010-0. Holiday House; fax (212) 688-0395; www.holidayhouse.com.*



Another picture book about fractions? Yes! Adler is a well-known children's book author as well as a former mathematics teacher. With illustrator Edward Miller's bright colors and graphic illustrations of a birthday party setting, Adler demonstrates each idea in a real-world situation, explains numerators and denominators, fractions with like and unlike denominators, adding and subtracting fractions with like denominators, and finding equivalent fractions. Pictures and text combine to introduce young elementary students to fractions and then thoroughly explore them as parts of sets and parts of a whole pizza or cake.

Working with Fractions clearly shows each concept and revisits them with additional examples. This is not a book to read to children straight through in one sitting. Lots of information is packed into a mere thirty-two pages. A teacher would most likely read sections of the book to reinforce fraction concepts as students work on each idea. Students could re-create most of the concepts illustrated in the book, for example, groups of children, coins, and book pages. One section uses folded and colored paper to help explain fraction comparisons. Any of these models is easy and engaging for students. The book is also helpful for review. I recommend this book as enrichment and

Prices on software, books, and materials are subject to change. Consult suppliers for current prices. Comments reflect reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

reinforcement for your fraction unit.—*Rebecca Leff, Quest Academy, Palatine, IL 60067.*

Et Cetera

Geometry Works Grade 3, *2007. $375. Grades 3–5. Item #INP75633. ETA Cuisenaire; (800) 445-5985; www.etacuisenaire.com.*

Geometry Works is a supplementary geometry program including a large assortment of manipulatives, a teacher's guide, and student activity pages. The materials can be used to teach geometry lessons that cover many traditional topics, including two- and three-dimensional geometric shapes, congruency, the coordinate plane, transformations, symmetry, perimeter, and area.

Designed to supplement any mathematics program by adding hands-on geometry lessons, the components I evaluated were designated for third grade, but I believe second through fourth graders could use them effectively.

I used the Reflecting Mirrors, Geoboards, Tangrams, AngLegs, and Pattern Blocks materials in a summer program for struggling readers who had completed kindergarten through fourth grade. In mixed groups of low-level readers, every student was able to participate, and all appeared to enjoy the geometric experiences. No matter the students' grade or ability, they all gained some geometric understanding.

I highly recommend the *Geometry Works* materials. Elementary teachers may already have some of these manipulatives, but the assortment and the storage box are extremely well done. In particular, the AngLegs were new manipulatives for my students and me. I thought the materials were very clever and could have many geometric applications at all grade levels. I found student worksheets in the teacher edition very useful; I would have liked more of them. In particular, I would have liked more Tangram and Pattern Block activities.—*Lynn Shevelenko, Math Tutoring Specialist, Crystal Lake School District 47, Crystal Lake, IL 60014.*

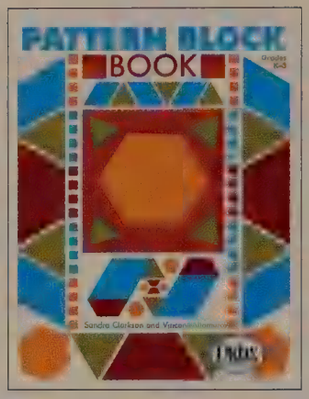
Money & Making Change, *2007. $19.99 DVD. Ages 6 up. ISBN 1-87848-928-3;* **Math Test-Taking Strategies**, *2007. $19.99 DVD. Grades 3–4. ISBN 1-934312-01-0. Rock'N Learn; (800) 348-8445; www.rocknlearn.com.*

I put the *Money & Making Change* DVD to the test with a group of second graders the day before a long vacation. The class was captivated by the songs and drawn into the action. The storyline is thin: Penny (a penny) and Bill (a dollar bill) are preparing to play in a big concert when their amplifier goes on the fritz. Will they have enough money to fix it so they can play in the big concert? This mattered little to the students or to the success of the DVD. Rather, the students became engaged in information on the newest coins as well as a brief history of money. The DVD includes practice in identifying coins and counting with them and can be paused to allow students to practice adding money. The children were unanimous in their favorite part—the shopping! Afterward, I could hear them humming the final song as they went about their activities. *Money & Making Change* is helpful for review, too. The pace is very quick, which is fine for stronger students. I would not use it with students who are not proficient with English. However, the children requested to see it again. The quality of the research makes this a fine video to add to a school or home collection.

With their new *Math Test-Taking Strategies* DVD, Rock'N Learn has jumped on the standardized testing bandwagon. Today's reality is that students are taking standardized tests, and lots of them. This DVD is full of test-taking tips for third and fourth graders—and perhaps second graders. The tips are quick reminders of useful strategies, such as carefully reading questions, drawing a picture, and reviewing vocabulary. The pacing is careful and suitable for students of all levels of proficiency. The mathematics covers a wide range of topics, from operations to data analysis and probability, measurement and geometry, fractions and algebra. Third graders and fifth graders who viewed the DVD enjoyed the jokes and felt that the DVD helped remind them of what strategies they could use in a testing situation. Of note to teachers and parents: Problems are solved in only one way and might confuse students who solve them in another manner. The DVD also uses the word *diamond* to name a rhombus. A link from the Rock'N Learn Web site would have been helpful to use with the test in the video. The DVD could then be paired with the test to practice answering questions in real time rather than in the ever-moving world of video. Even without the test from the DVD, *Math Test-Taking Strategies* would best be used in conjunction with a remote control and viewed over several days, so students could practice the useful

strategies Rock'N Learn cleverly presents.—*Lauren Mattone, Crocker Farm Elementary School, Amherst, MA 01002.*

Pattern Block Book, *Sandra Clarkson and Vincent Altamuro, 2007. 133 pp., $11.95 paper. Grades K–3. ISBN 978-1-58324-276-6. Didax; (800) 458-0024; www.didax.com.*



The *Pattern Block Book* authors present a set of lesson plans covering a wide range of mathematical concepts using the ubiquitous colored blocks, which are often relegated to teaching only fractions or patterns. Neatly divided into sections covering exploration, number and operations, geometry, measurement, algebra and probability, and statistics, the book features lessons blending the familiar with the unusual. Each lesson includes a central study theme. For example, the Making Masks lesson in the Number and Operations section has an algebra link to introduce students to using a table to record and extend number patterns. Also included in each lesson are a materials list, concise instructions, a reproducible student sheet, and extension suggestions. The authors have thoughtfully added a brief assessment checklist to the front of the book, as well as reproducible grid sheets and giant pattern block cut-outs to the end of the book.

The *Pattern Block Book* is for grades K–3, but lessons do not indicate a grade level. Teachers must look at the lessons and decide for themselves whether they meet student needs at a certain grade level. Most of the lessons are intended to be taught to a class, but many would also be fine for independent student exploration and work stations. For example, Just One Square, Undercover, and How Far Around would be nice extensions to area and perimeter lessons. The book does not offer anything new, but it is a nice supplement to a professional library.—*Lauren Mattone, Crocker Farm School, Amherst, MA 01002.*

Sizing Up Measurement: Activities for Grades K–2 Classrooms, *Vicki Bachman, 2007. 192 pp., $23.95 paper. ISBN 978-0-941355-79-7. Math Solutions Publications; (800) 868-9092; www.mathsolutions.com.*

Bachman wrote *Sizing Up Measurement* so that K–2 students can have fun and make sense of what they learn while doing measurement activities. The book connects six measurement focus areas—length, area, capacity, weight, temperature, and time—with literature, art, movement, vocabulary, reading skills, and science. Lessons cover graphing, geometry, problem solving, estimation, prediction, and number sense.

Each measurement strand begins with a short introduction. Easy-to-understand overviews explain the lesson goals. The materials needed for all the lessons are easy to obtain, including the literature connections. Setup and expenses for any activity are minimal, as when my class made Gorp. A glossary with diagrams includes vocabulary words used in each lesson. Instructions for doing the activities are numerated and bulleted for ease in following. A motivational hook at the beginning of each lesson entices students, and each lesson ends with some form of summary, such as a class discussion. The author suggests extension activities and includes blackline masters at the book's end.

Although choosing developmentally appropriate activities is the key to success (and fun) with the book, lessons do not list a grade level. Even so, grades K–3 teachers will enjoy using this mathematically rich resource to supplement required curriculum materials. Parents who homeschool students in these grades will also find the book a valuable tool. I recommend *Sizing Up Measurement* as a worthwhile addition to any K–3 classroom.—*Maxine E. Pincott, Oliver Ellsworth School, Windsor, CT 06095.* ▲

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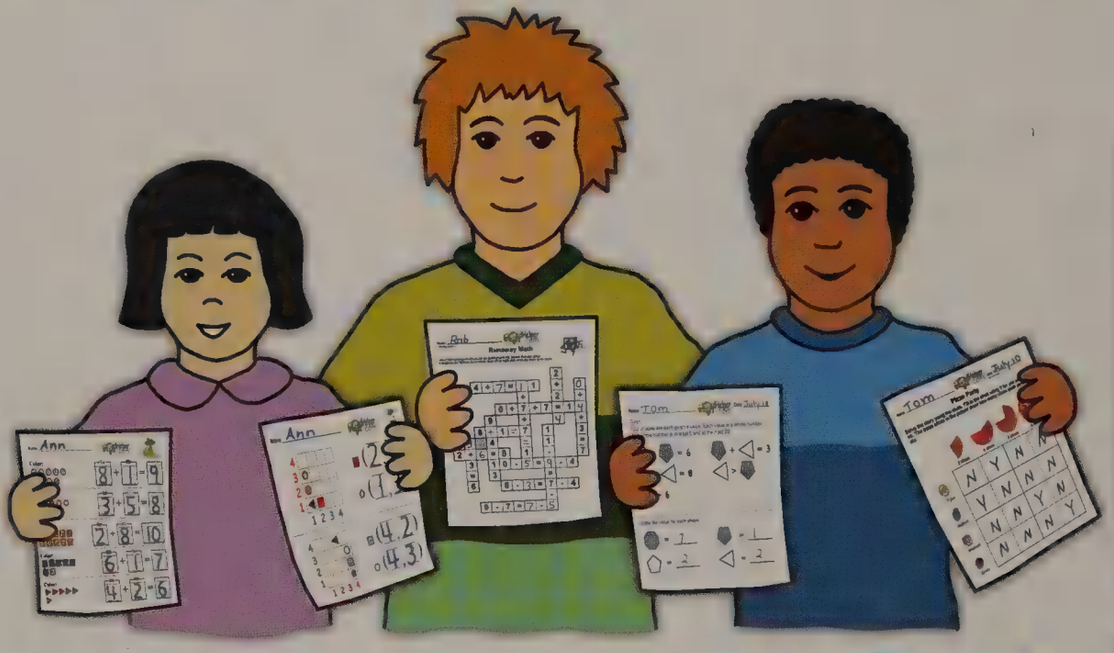

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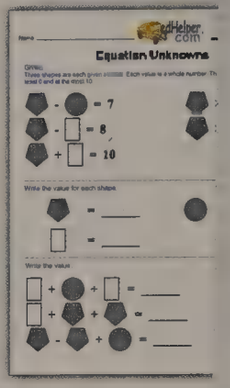

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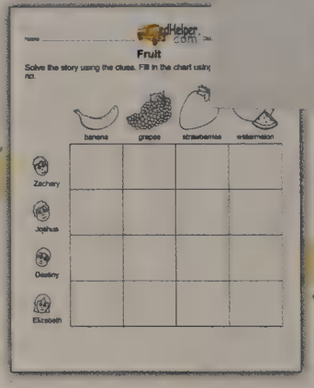

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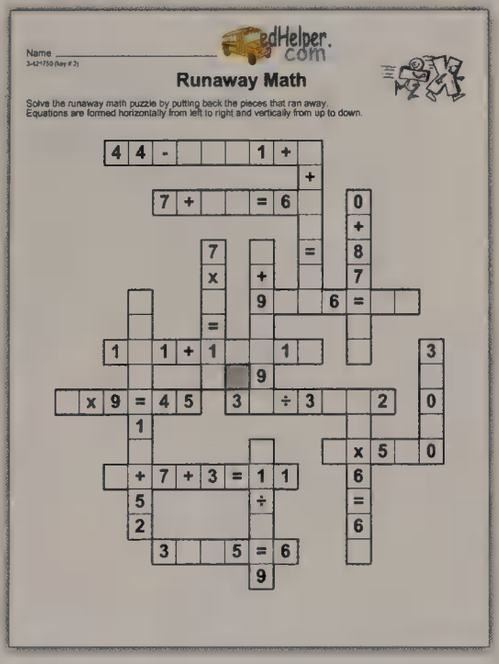
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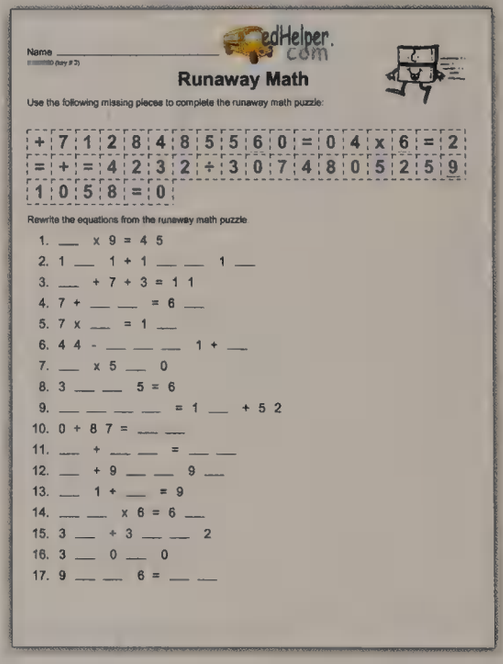
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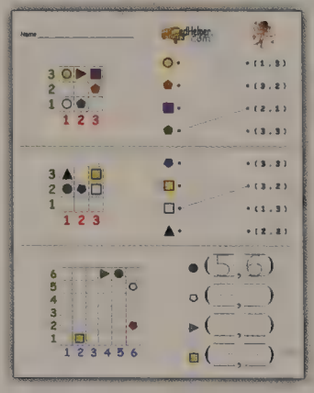

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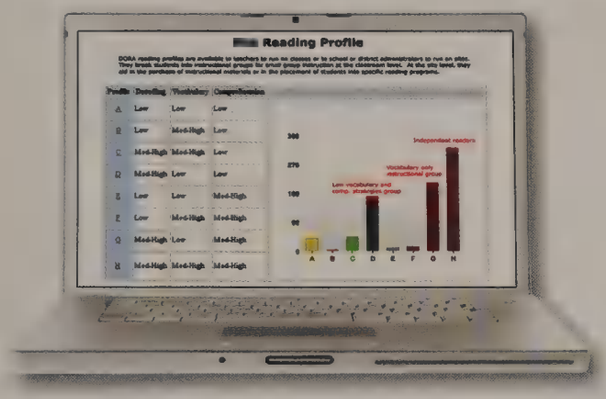

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ON THE COVER

Susan Carter's piece, "Disequilibrium and Questioning in the Primary Classroom: Establishing Routines That Help Students Learn," describes how first graders engage with the concept of *disequilibrium* in a classroom milieu in which all students can learn. Children feel success in mathematics even before mastering a particular concept—*if* they understand the struggle as an expected, essential part of learning. Photograph by Susan Carter; all rights reserved

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Information is available from the Headquarters Office regarding the three **other official journals**, the *Mathematics Teacher*, *Mathematics Teaching in the Middle School*, and the *Journal for Research in Mathematics Education*. Dues support the development, coordination, and delivery of the Council's services. Dues for individual membership are $78 (U.S.), which includes $33 for each *Teaching Children Mathematics* subscription, $16 for a nine-issue *NCTM News Bulletin* subscription, and *ON-Math*. Each additional school journal (*Mathematics Teacher* and *Mathematics Teaching in the Middle School*) subscription is $33. Each additional subscription to the *Journal for Research in Mathematics Education* is $60. **Foreign subscribers** add $18 (U.S.) postage for the first journal and $4 (U.S.) postage for each additional journal. Special rates for students, institutions, bulk subscribers, and emeritus members are available from the Headquarters Office.

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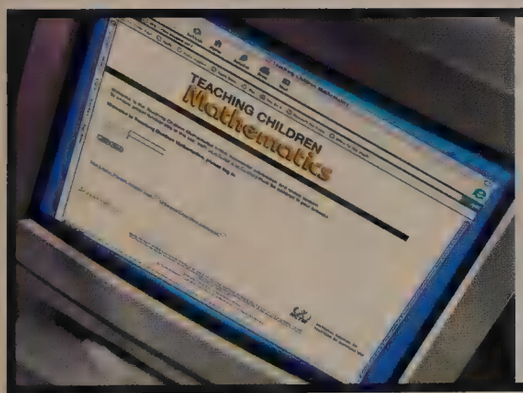

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We look forward to receiving your online submissions. Thank you for writing.

Learning Environments That Support Mathematical Understanding

The first chapter of *Principles and Standards for School Mathematics* (NCTM 2000) presents a vision of the ideal learning environment for all mathematics learners—one that supports student acquisition of knowledge, incorporates learning tools that promote student comprehension, engages students in mathematical tasks that provide opportunities to make sense of mathematics concepts and procedures, and incorporates all the NCTM Process Standards.

How do teachers establish such an environment? In particular, how do we transition toward this environment if we have not previously experienced it? In the call for manuscripts for this Focus Issue of the journal, the *Teaching Children Mathematics* Editorial Panel encouraged authors to discuss approaches that the mathematics education community recommends and uses to establish such a learning environment. Specifically, we sought manuscripts that—

- help readers understand how to establish appropriate mathematics learning environments;
- describe the process or phases that teachers and students experience as they attempt to alter their mathematics learning environment; and
- address issues pertinent to the implementation of various mathematics learning environments.

When reviewing manuscript submissions for this special issue, we found that mathematics educators are clear about features of effective learning environments, but they often have difficulty articulating how to establish these environments. The articles we selected for publication provide insights and perspectives about establishing learning environments that support *all* mathematics learners. We share other educators' approaches and experiences to encourage all elementary mathematics teachers to examine the nature of their current learning environments and engage in activities that support the transition to NCTM's vision for the ideal mathematics learning environment.

Each article in this issue was selected to represent a different perspective about the nature of the learning environment. The stories will take you into exciting, mentally stimulating classrooms and schools. Inventive strategies, created and field-tested by teachers like you, showcase how students' potential for deeply understanding and clearly articulating mathematics concepts can be supported in appropriate learning environments.

In "Disequilibrium and Questioning in the Primary Classroom," Susan Carter describes how a learning environment for primary students can be established to engage students in questioning their own and others' mathematics conjectures and understandings.

"Using TARGETTS to Create Learning Environments That Support Mathematical Understanding and Adaptive Motivation," by Melissa C. Gilbert and Lauren E. Musu, outlines a lesson planning and analysis tool that teachers can use to examine lessons and support adaptive motivation and mathematics learning.

In "Learning to Think and Thinking to Learn," Kate Kline addresses issues to consider as teachers attempt to establish learning environments with a focus on encouraging and supporting student thinking.

Kim J. Thach and Kimberly A. Norman discuss how technology can be used to engage, motivate, and respond to a diverse group of learners in "Technology-Rich Mathematics Instruction."

A process that a school staff used to enhance the mathematics learning environment for all its students is described in "Shifting Roles and Responsibilities" by Pia Hansen and Donna Mathern. This article outlines needed changes in the roles of the teacher, principal, and students to improve their school learning environment.

Malinda F. Apanay and Autumn C. Vavoso discuss revitalizing the workshop cycle to create a classroom milieu for mathematical understanding. "The Workshop Way: Creating a Stimulating Intellectual Environment" posits that mathematical

understanding lies at the intersection of the NCTM Content and Process Standards.

"Equity for English Language Learners in Mathematics Classrooms," by Gina M. Borgioli, provides a framework of examining approaches to promote an equitable classroom for English Language Learners.

The articles we chose represent only a small sample of efforts by mathematics educators, but they highlight teachers, administrators, and researchers who inspire. The mathematics education community must continue to examine issues and share experiences that develop and maintain a supportive learning milieu, provide equitable access to mathematics, and enhance all students' learning experiences. Through various means, the supportive classroom environments featured in these pages nurture learning, motivate students to strive for success, and, above all, embrace the power and elegance of mathematics.

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Gladis Kersaint and Robyn Silbey
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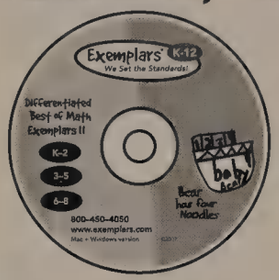

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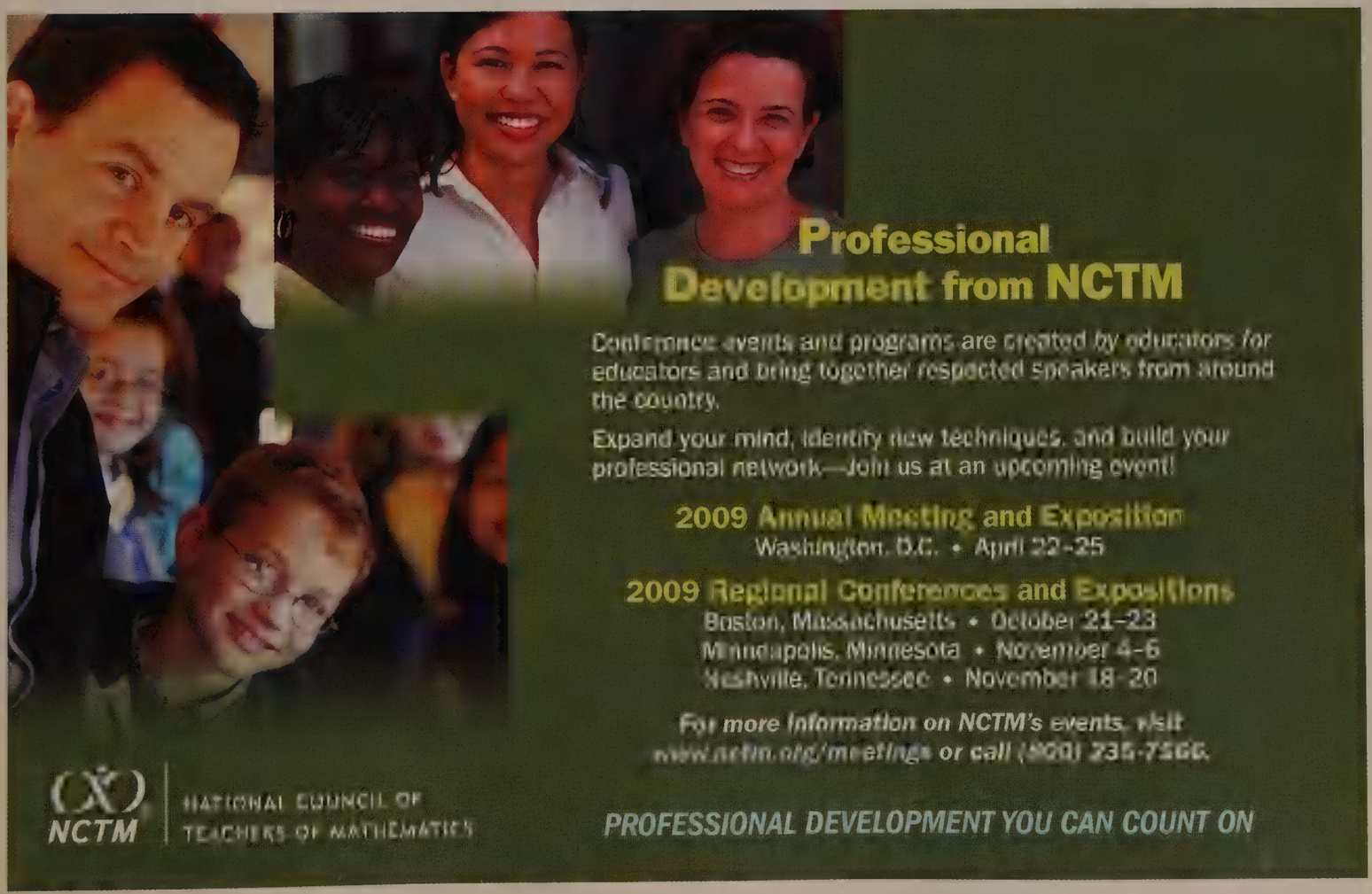
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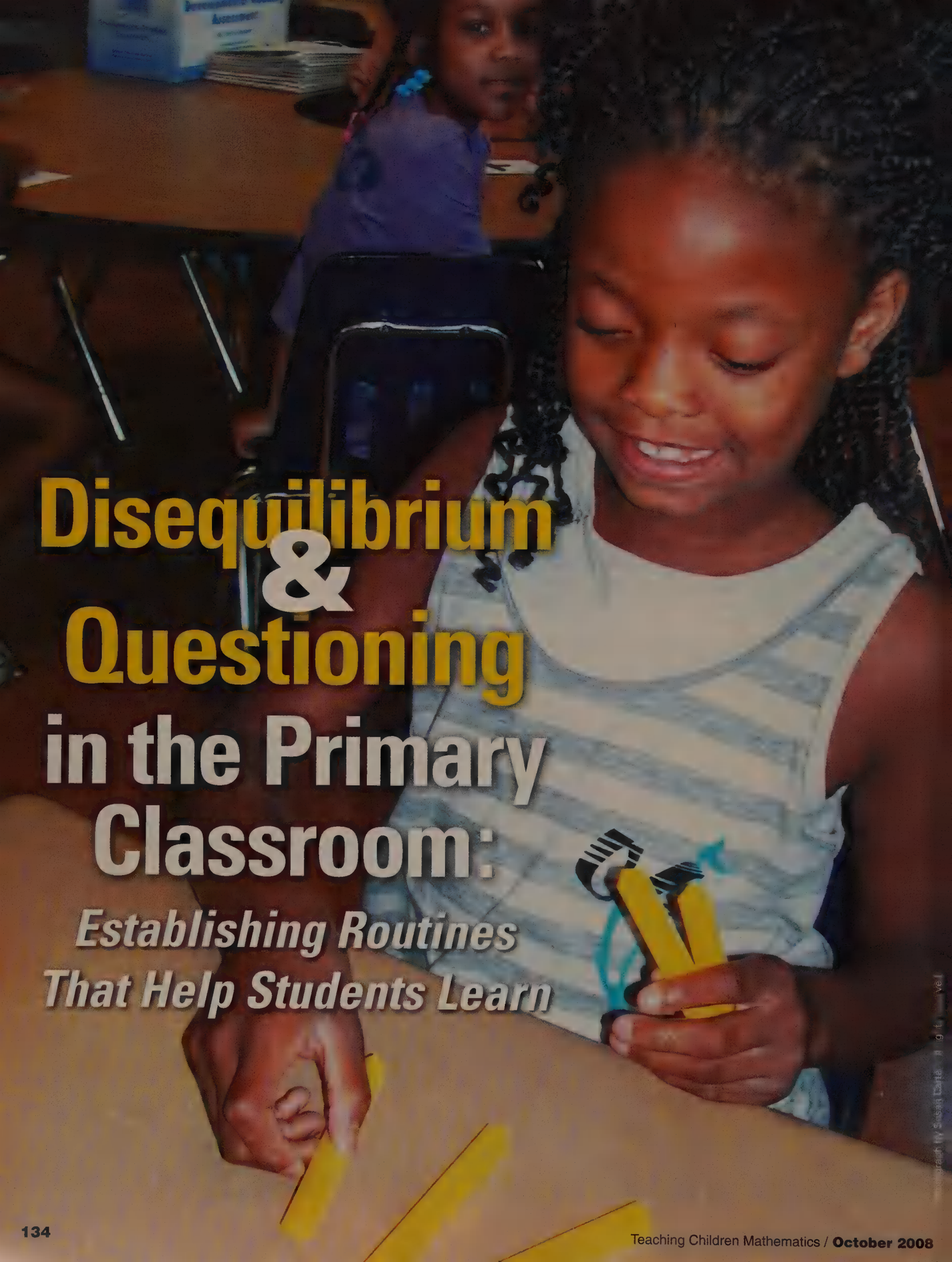

Disequilibrium & Questioning in the Primary Classroom:

Establishing Routines That Help Students Learn

Jean Piaget (1970) defined *disequilibrium* as a conflict between new ideas and current conceptions. Introducing this concept to my first graders improved formative assessment results and the overall classroom climate during mathematics class. Compared to my earlier students, my first graders made more progress with less difficulty. Engaging with a new concept allows students to feel successful as they struggle to integrate it into their current thinking. Although they have not yet mastered a particular mathematical concept, students can feel success if they can understand that the struggle is an expected, essential part of learning. By identifying their own disequilibrium, my students were able to continue to struggle with a concept or problem without "getting it right."

Principals and Standards for School Mathematics (NCTM 2000) asserts that all students, regardless of cultural or mathematical background, deserve rich opportunities to study—and support to learn—mathematics. My students have diverse backgrounds and widely varying ability levels. Allowing, even encouraging, students to grapple with mathematical concepts steered them away from the more familiar feelings of "getting it" or not.

Transitioning our mathematics instruction to a Standards-based curriculum was a challenge for many teachers at my school, including me. I sometimes felt that I was giving up on a child if we proceeded to a new idea before every student had mastered a concept. I had to learn to create an atmosphere of success for me and my students by allowing equal opportunities for all students to struggle with concepts at whatever level they are operating.

How the Teacher Found Balance

When I began teaching mathematics to first graders, I based the success of my lessons on the happiness of my students. I can remember telling my teammates, "That lesson went really well; the kids liked it!" If students became unhappy, got stressed out, or seemed frustrated, I was known to actually stop a lesson and start an activity sheet. I repeated the mantras of ineffective teachers: "This is too hard for them!" or "My kids just do not have the background for this kind of assignment."

By Susan Carter

Susan Carter, scarter@u-city.k12.mo.us, is a National Board Certified Teacher from St. Louis, Missouri. She is interested in improving mathematics instruction and content knowledge for teachers of the early primary grades.

The focus in my classroom was not on worthwhile mathematical tasks but on tasks everyone would succeed with—at varying speeds. (Addition worksheets with holiday motif borders come to mind.) Fortunately, I continued my own professional development and began studying the teaching and learning of mathematics. Through several years of graduate classes and workshops, the current best practices and the reasoning behind them became clear to me. Imagine my heartbreak when I realized the disservice I was doing to my students, especially the ones who needed it most. By substituting a focus on happiness for a focus on engagement with the ideas, I deprived students of what they needed most: worthwhile mathematical tasks and the support to think through them. The more I challenged myself through professional development, reading, and studying the Standards, the closer I moved to an understanding of the necessity of struggle in learning.

At one summer workshop I attended, a presenter told a story about a professor who, after the class had struggled to find the answer to a challenging problem, exclaimed, "We have the right answer, and now the mathematics can begin!" I embraced this concept and brought it right back to my classroom.

Changing the Classroom Climate

As a whole class, we discussed our feelings about and during mathematics class. Students expressed

The first grader shrugged and smiled: "I still have the disequilibrium."

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different levels of comfort and confidence about various topics. I explained the concept of disequilibrium, the feeling of confusion associated with struggling with a mathematics concept. Through several discussions, students began to understand that confusion is something you go through, not a permanent state of being. Typically, when students are frequently confused during mathematics class, they assume they are not good at the subject and stop trying. Understanding that disequilibrium is normal gives students a foundation from which to struggle and move toward understanding.

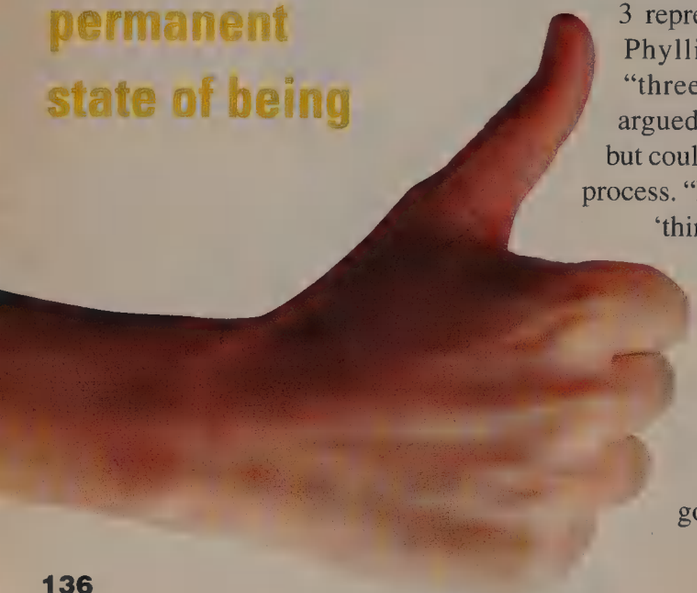
The first step was to embrace this concept and apply it in a practical way. To begin, I needed to help students identify disequilibrium for themselves. So, together we attempted a new mathematics concept: place value. I wrote the numeral 38 on the board and asked students what the digit 3 represented in this problem. Phyllis suggested it meant "three of something." Lucy argued that it meant "thirty," but could not explain her thought process. "It just is, because you say 'thirty-eight,'" she insisted. I asked students if there was a mathematics tool we could use to help us see the problem better. We decided to use place value blocks.

I kept the conversation going by asking questions and trying out solutions. Phyllis counted out thirty-eight units cubes and made a group of three cubes and a group of eight cubes to represent 38. Taylor wondered where the leftover cubes should go. I checked to see how students were feeling about their understanding of the problem. Eventually, they reached a point where no one seemed to be moving along in their thinking or asking genuine questions to help others along. The class was stuck, and most of the children were expressing disequilibrium. Lucy suggested that we had miscounted. Zoey said we should count by tens "so it goes faster." We ended up with a representation of thirty-eight cubes consisting of three rows of ten each and a group of eight. Every couple of minutes, I checked again on how the students were proceeding.

We traded the thirty units cubes for three long tens rods, and suddenly the number began to make more sense for some children. Taylor summarized that the 3 in 38 was for the three long tens rods. Lucy added that the 3 meant 30 because three rods are equal to thirty cubes. I asked Phyllis to show what the 3 in 38 meant using the place value blocks. She pulled three cubes out of the group of eight. "What about the three rods?" I asked. She shrugged and answered, "I still have the disequilibrium." Several others agreed with her. Phyllis smiled. This was an important moment for me as a teacher. Although Phyllis did not yet fully understand the concept we were discussing, she felt successful in her learning because she had permission to struggle. No one was silenced. We had the answer, and the mathematics was just beginning.

From then on, whenever the class worked on a new or difficult concept as a whole group, I reminded students of the value of disequilibrium in our mathematics class. Students assessed their own progress by giving thumbs up if they were thinking, "I get it, and I am feeling very comfortable"; thumbs sideways if they were thinking, "I have disequilibrium, and I am struggling"; or thumbs down to signify, "I am totally lost, and I have no idea what to do next." Giving students this kinesthetic cue gave them permission to struggle and approval for participating in the process. When the majority of the class agreed by showing thumbs sideways ("We are confused and feeling stuck"), I stopped the lesson to discuss disequilibrium. "This is the feeling of disequilibrium," I explained to my first graders, "and now the real learning can begin!" Our motto became "If you are not struggling, you are not learning." Beginning or ending class by checking in with students—using the thumbs sign—allowed me to quickly assess the progress of the class as a group

Confusion is not a permanent state of being



as well as allowed each child to assess himself or herself individually.

Let the Mathematics Begin!

Integrating the term *disequilibrium* into our mathematics class eventually led to a discussion of what to do when you become confused about a mathematical concept. How do you get past your disequilibrium to achieve understanding? Early in the school year, I had introduced the idea that telling someone the answer does not help them learn; in fact, it stops their learning. We had established a classroom routine whereby students helped each other by asking questions to move the thinking along without telling the answers or the "right way" to do it. Now the climate in our classroom relaxed as students recognized that their questions held more value than their answers.

We talked about how questions could help us when we have disequilibrium. As a class, we generated a list of questions that students could ask one another when they felt stuck and were unsure of what to do next in mathematics class. I modeled these questions during class discussions throughout the year:

- Do you know all these numbers?
- Can you read all the words?
- Do you understand the directions?
- Do you see a pattern?
- Is there something you can try that might work?
- Are there any mathematics tools that would help?

I made a visual anchor to display on the classroom wall: a large sun peeking out from behind the clouds. I labeled the sun with the word *understanding* and the cloud with *disequilibrium*. I recorded our questions on smaller clouds and added them to the visual anchor, a constant reminder of the value of struggle, as well as steps for independent problem solving.

Creating a learning environment where students feel comfortable questioning themselves and another increases the learning opportunities. The *process* of learning, not the end result or answer, becomes the focus. My students began to see that their effort moved them through the process. They learned to delve into mathematical concepts because they had no fear of failure. Students started to ask questions of me and of one another that showed they were comfortable engaging with mathematical ideas without having immediate answers. Their questions indicated a growing climate where we were all enjoying the struggle inherent in real learning: "Is that always a rule, or is there a number that does not work like that?" "If we tried it with a different number, would we get the same answer?"

As a teacher, I was able to guide our first-grade mathematics study deeper into content than I had before because I was no longer in a race to complete the curriculum. I could incorporate complex ideas from a variety of mathematical strands. My comfort level had increased as well: I introduced new concepts while some students still struggled with previously introduced material. Teaching my students how to acknowledge and pursue the struggle and process of learning resulted in worthwhile, meaningful mathematical experiences for me and my students.

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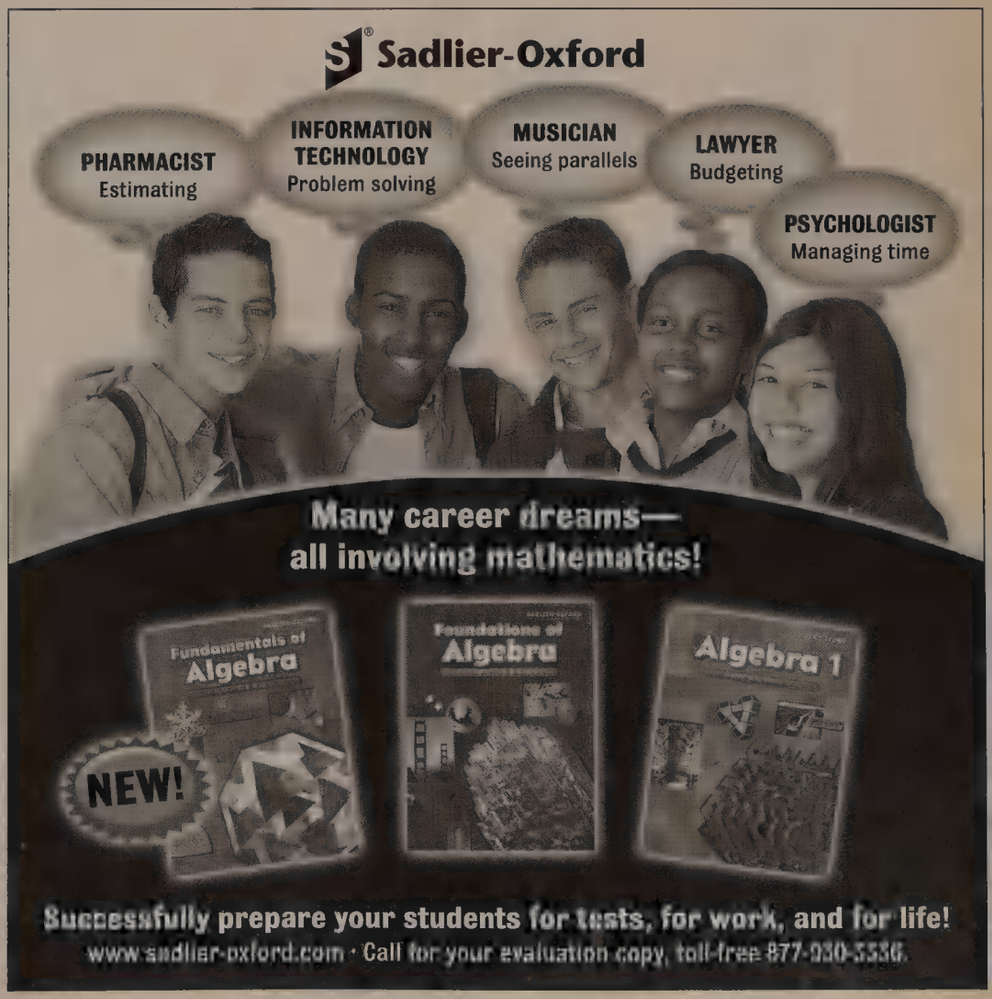

Using TARGETTS to Create Learning Environments That Support Mathematical Understanding and Adaptive Motivation

By Melissa C. Gilbert and Lauren E. Musu

Melissa C. Gilbert, mcgilbert@scu.edu, is an assistant professor of mathematics education at Santa Clara University in California. She is interested in recent research that examines how students' motivation relates to their mathematical proficiency and how teacher beliefs and practices relate to student motivation and achievement in mathematics. Lauren E. Musu, lmusu@umich.edu, is a researcher at the University of Michigan in Ann Arbor. She is interested in how teacher, parent, and student interactions can best foster adaptive student motivation in classrooms.

This article focuses on learning environments that support students' mathematical understanding and enhance their adaptive motivation. Drawing on our experiences as teachers and teacher educators, we first describe adaptive motivation. We then introduce TARGETTS, a lesson planning and analysis tool, and describe ways in which teachers can use this tool to highlight specific instructional strategies that support both adaptive motivation and mathematical understanding.

Adaptive Motivation

Adaptive motivation encompasses a number of constructs (e.g., perceived usefulness, interest in the

subject matter, self-confidence, and achievement goal orientation) that have extensive theoretical and empirical support in the affective literature. This literature well documents that students with adaptive motivation value mathematics (i.e., they feel mathematics is useful and interesting), are confident in their ability to learn mathematics, and approach mathematics class focused on learning and understanding the material (Anderman and Wolters 2006; Turner and Patrick 2004). These students tend to do well in mathematics, both in terms of their course and test performance and their ability to persist in the face of challenging work (Middleton and Spanias 1999; Wigfield and Eccles 2000). Further, they show a more meaningful understanding of mathematics (e.g., the ability to explain their reasoning) when compared to students with less adaptive motivation (Gilbert 2008; Stipek et al. 1998).

Learning environments in which teachers incorporate motivational considerations into their lesson planning and reflection can promote adaptive motivation. We now turn to specific examples of how this can be accomplished using TARGETTS.

TARGETTS

Classroom-based research has identified specific instructional practices that can support or hinder students' adaptive motivation (Patrick et al. 2003; Stipek et al. 1998). TARGETTS is a lesson planning and analysis tool that highlights key findings from this research regarding classroom practices that support students' adaptive motivation. The acronym TARGETTS focuses on the following eight dimensions of classroom practice:

Tasks
Autonomy or responsibility afforded students
Recognition of student achievement and learning
Grouping of students
Evaluation of student work
Time allocation in the classroom
Teacher expectations for student behavior and performance
Social interaction in the classroom

The Tasks dimension focuses on enhancing students' interest in learning mathematical tasks through, for example, real-world connections that illustrate the content's meaning or significance. The objective of the Autonomy or responsibility dimension is to provide opportunities for students to participate so they can take responsibility for their learning. The Recognition dimension refers to *why* students are rewarded and highlights the importance of finding a way to recognize all students, such as through honoring progress in goal attainment. The Grouping dimension recognizes the importance of mingling students in ways that emphasize conceptual understanding and deemphasize social comparison. The focus of the Evaluation dimension is on creating grading and reporting processes that spotlight students' personal growth. For example, we give students opportunities to improve their performance through repeat submissions and test corrections. The Time dimension highlights the importance of planning lessons strategically to address multiple objectives in fewer, richer tasks. The Teacher expectations dimension is more than a catchphrase; communicating clear, consistent expectations for *all* students is a key element to a learning environment that supports mathematical understanding. The Social interaction dimension is the basis of classroom community, and teachers can set the norms early on to facilitate *all* students' participation, especially when they are not sure they have a correct solution approach. **Table 1** provides mathematics classroom examples of each dimension.

TARGETTS and the NCTM Standards

The TARGETTS dimensions support the development and maintenance of mathematics classroom environments that are consistent with the recommendations of *Principles and Standards* (NCTM 2000). For instance, the Task dimension asks teachers to consider enhancing student interest through the use of authentic, open-ended problems. This approach to structuring classroom tasks not only supports adaptive motivation but also promotes development of a variety of appropriate solution strategies and reflection on the problem-solving process as encouraged in the Problem Solving Process Standard.

The TARGETTS dimensions are also relevant to establishing a classroom culture in which students work meaningfully with their peers as emphasized in the Communication Process Standard. For instance, creating appropriate groupings allows students to organize their ideas and learn from one another. Having students share solutions allows appropriate social interaction and helps develop mathematical language and expression, also emphasized in the Communication Process Standard. As these examples illustrate, the TARGETTS dimensions encompass classroom practices that assist students in gaining the proficiencies required by the NCTM Standards.

Table 1

Mathematics Classroom Examples for TARGETTS

Dimension	Examples
Task	• Write word problems that use current events, school data, or statistics. ▪ Substitute students' names and their experiences within a problem. ▪ Use manipulatives (e.g., base-ten blocks, pattern blocks) to provide hands-on and visual experience of content. ▪ Connect lessons to real-life skills, such as percentages with shopping for sale items.
Autonomy or responsibility	▪ Empower students to choose from options (pick two out of five questions on a quiz; select your presentation style). • "Pair-Share" after a pretest; students explain to one another how to solve the problems. ▪ Promote self-evaluation after an exam. ▪ Model for students how to help others without doing it *for* them.
Recognition	▪ Use your students' extracurricular activities (e.g., athletic events, plays, concerts) as a source for class story problems. ▪ Praise in-class progress outside of class (e.g., in the hallway, say, "Great job explaining on Friday's quiz.") ▪ Rather than posting high test scores, highlight class progress by showing class mean improvement from pretest to end-of-unit test. ▪ Help students develop and maintain individual progress graphs (e.g., to show progress with learning number families).
Grouping	▪ Ensure that everyone has a specific task during small-group work (e.g., when solving multistep problems, do a step, pass it on). ▪ Allow group size to vary (e.g., pairs, trios, groups of four) based on the needs of the task. ▪ Establish randomized partners (e.g., use an "appointment clock" where each student has a partner for each clock hour and say, "Work with your 3 o'clock partner today"). ▪ Arrange tables and work stations to support individual, pair, and small-group work.
Evaluation	• Interview students to assess their understanding following a group project. ▪ Use mistakes as teaching opportunities (e.g., "What can we learn from what went wrong here?"). • Provide feedback that highlights student progress (e.g., proficient with two-digit addition without regrouping but not yet with regrouping).
Time	▪ Encourage oral presentations of the subject matter. • Provide opportunities to integrate mathematics concepts and real-life experiences using outside-of-class projects. ▪ Use classroom-based centers so students spend time on what is of interest to and needed for them. ▪ Identify *key* ideas for units to keep up with the pacing chart.
Teacher expectations	• Spend ample time on expectations for students (e.g., "Start slow to go fast"). • Develop and share rubrics for long-term assignments and projects. ▪ Monitor students' participation in and understanding of lessons and activities (e.g., whiteboards, popsicle sticks). ▪ Ask students to demonstrate their solution approaches, not just their answers.
Social interaction	▪ Establish ways to honor students' need to interact (e.g., cooperative groups, Think-Pair-Share). ▪ Emphasize that peers are valuable resources (e.g., for other solution strategies or ways to explain concepts). ▪ Explain acceptable *and* unacceptable behaviors (e.g., no put-downs, politely critique one another's explanations).

Using TARGETTS

How can teachers implement TARGETTS? Analyzing a recent lesson is a good introduction to implementing the TARGETTS framework in your mathematics classroom.

Analyzing a lesson

You can use **table 2** to analyze a lesson and brainstorm possible revisions for subsequent lessons. To illustrate how this might work, we have answered the questions in **table 2** on the basis of the following example:

Imagine a lesson in which you are reviewing material for an upcoming chapter test. Your curriculum provides a two-page practice worksheet with both fill-in-the-blank and multiple-choice problems. You introduce the lesson by reminding students of the big mathematical ideas of the chapter and then ask students to work alone to complete the worksheet. The first two students to finish then compare their answers with the teacher's edition. When everyone is finished, these students read the correct answers aloud. You conclude the lesson by asking if there are any questions.

Table 2

TARGETTS as a Tool for Analyzing a Lesson

How was each TARGETTS dimension addressed in your lesson? What alternative strategies might you use to further support adaptive motivation?

TARGETTS Area	Focus	Teachers Should	Current Method	Alternative Strategies
Task	How learning tasks are structured—what the student is asked to do	Enhance student interest in learning tasks	Students complete a textbook worksheet, providing answers only	Modify worksheets so students explain answers; make problems open-ended: __ + __ = 5 not 3 + 2 = __
Autonomy or responsibility	Student participation in learning/school decisions	Provide opportunities for student choice in learning activities	Teacher decides who reads the answers from the teacher's edition	Students choose the best way to represent the problem: in words, pictorially, etc.
Recognition	The nature and use of recognition and reward in the school setting	Find a way to appropriately recognize *all* students	Only students who finish first are recognized; they get to read answers	All students are recognized and participate in sharing answers
Grouping	How and for what purposes students are grouped	Create groups that emphasize conceptual understanding and deemphasize social comparison	There is no grouping; students work alone	Students spend time working alone, in pairs, and sharing with the whole class to help one another learn
Evaluation	The nature and use of evaluation and assessment procedures	Create grading and reporting practices that focus on personal growth	Students hear the correct answers and are graded on how many items are correct	Students share reasoning with a peer, then the whole class; are evaluated for problem-solving efforts more than answers
Time	The allocation of time to different classroom tasks and activities	Plan lessons that address multiple objectives in fewer, richer tasks	Students spend most of the class time working independently	Students share with peers and the whole class to reinforce understanding in preparation for the upcoming test
Teacher expectations	Beliefs and predictions about students' skills and abilities	Communicate clear, consistent expectations for all students	Teacher interacts only with students who have questions about the worksheet	Teacher monitors all students' progress and encourages all to share their thinking with the class
Social interaction	The nature of teacher-student and student-student relationships	Focus on creating a classroom community where all students will participate	Few students shared during the whole-class presentation	All students take turns sharing answers and solution approaches

Now use **table 2** to consider how the dimensions of TARGETTS could help you modify this lesson to support the mathematical goal while also enhancing students' adaptive motivation. Start by considering the task itself. Are there ways to modify the worksheet so that students' mathematical thinking is more transparent? Perhaps students could write a brief explanation for the multiple-choice items stating why the choice they selected is right and why each of the other choices is wrong. Moving to the Autonomy dimension, how might you incorporate student choice into the review of the worksheet? Instead of two students reading the answers, each student could choose how to present a problem to the class (e.g., pictorially, graphically, in words). This lesson modification addresses another dimension: Students who finish fastest are no longer privileged over their peers; instead, all students are recognized for their effort. Consider grouping students in pairs to review their answers and explanations after they have worked alone and before the whole-class discussion.

Are there ways to modify the original lesson to probe and evaluate students' understanding? Perhaps you could ask students to share their solutions and explanations with the class and give credit for effort, rather than correct solutions only. You could give students more time to work with peers to explain answers and help one another. This could be particularly beneficial in this lesson, given the goal of preparing students for a test. With your expectations in mind, did you demonstrate to all students that you were interested in how they solved the problems? Inviting student participation and seeking their answers and solution approaches could send this message more clearly. And finally, concerning their social interaction, were opportunities for students to interact with you and with one another limited in the original lesson? If so, consider ways to modify the lesson to have a more extensive whole-class discussion.

Planning a lesson

Similarly, the TARGETTS dimensions are useful when you are preparing to teach a lesson. We recently presented a teacher workshop that focused on how to support adaptive student motivation in the classroom. Many of these teachers had seen the TARGETTS dimensions in a workshop we had offered the previous year. While focusing on how reform practices can be used to support adaptive student motivation, a male teacher shared his experience of planning a lesson in the classroom using the TARGETTS dimensions.

His first step was to determine the task that students would be asked to perform. He decided to have them make posters to share their solutions to an open-ended problem, enhancing student interest by encouraging them to represent their solution in multiple ways. Students were grouped to help stimulate ideas for the thought-provoking problem, and each student was allowed to choose his or her role in the group (to promote autonomy). Students shared their final posters with the class in a brief presentation, which allowed for appropriate social interaction.

To evaluate his students' progress, the teacher employed an innovative technique that allowed him to recognize individual students' progress while also evaluating the group's work. After the group presentations, he interviewed students individually and found that although some of the final answers on the posters were incorrect, the students demonstrated a robust understanding of the underlying mathematical ideas. This approach saved instructional time because he realized he needed only to address calculation errors rather than conceptual gaps. Embedded throughout this lesson was the teacher expectation that each student could collaborate with peers and solve the problem.

This teacher's experience illustrates how

Celebrate National Metric Week

Break out your rulers; it is time to celebrate National Metric Week. NCTM started the tradition during the week of May 10, 1976, the year after the Metric Conversion Act of 1975 was enacted. Many states issued official proclamations in support, and the peak was in 1984 with 33 states. In 1983, NCTM moved National Metric Week to the week containing October 10—the tenth day of the tenth month. This year, October 6–10 is set aside to celebrate the importance and convenience of the measurement system based on tens.

Go to
www.nctm.org/metric-week.aspx
for metric-related Web sites.

TARGETTS supports the use of Standards-recommended practices to foster students' adaptive motivation. As a colleague of this teacher commented, "I like TARGETTS! It was good to see that [our reform-oriented, Standards-recommended] practices coincide with these goals."

Conclusion

TARGETTS provides mathematics teachers with a useful, research-based lesson planning and analysis tool consistent with the recommendations of NCTM Standards. The classroom examples provided throughout this article illustrate how to implement TARGETTS by first analyzing and then planning a lesson. Taken together, the Standards and TARGETTS offer teachers a framework and tool, respectively, for enhancing students' adaptive motivation and their mathematical understanding.

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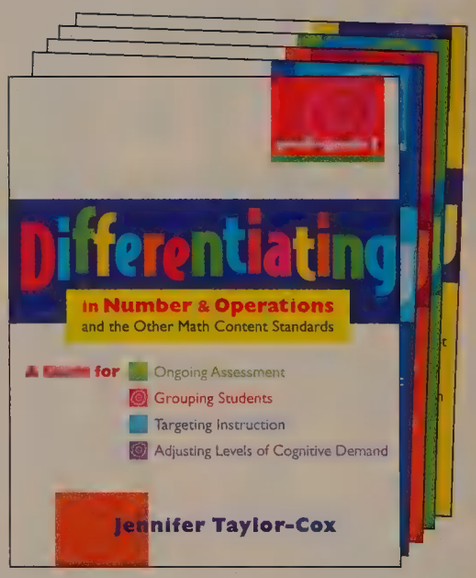

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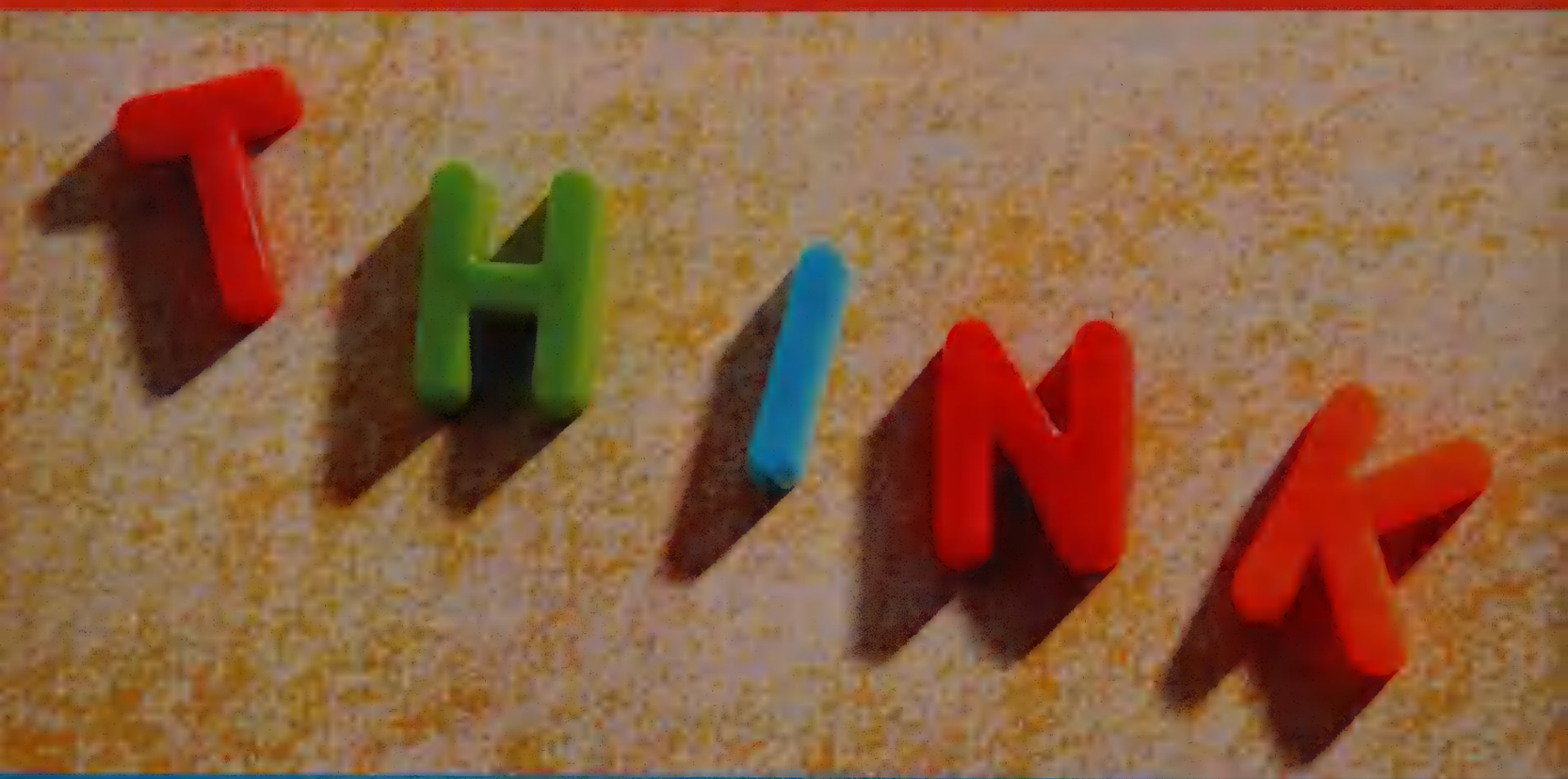

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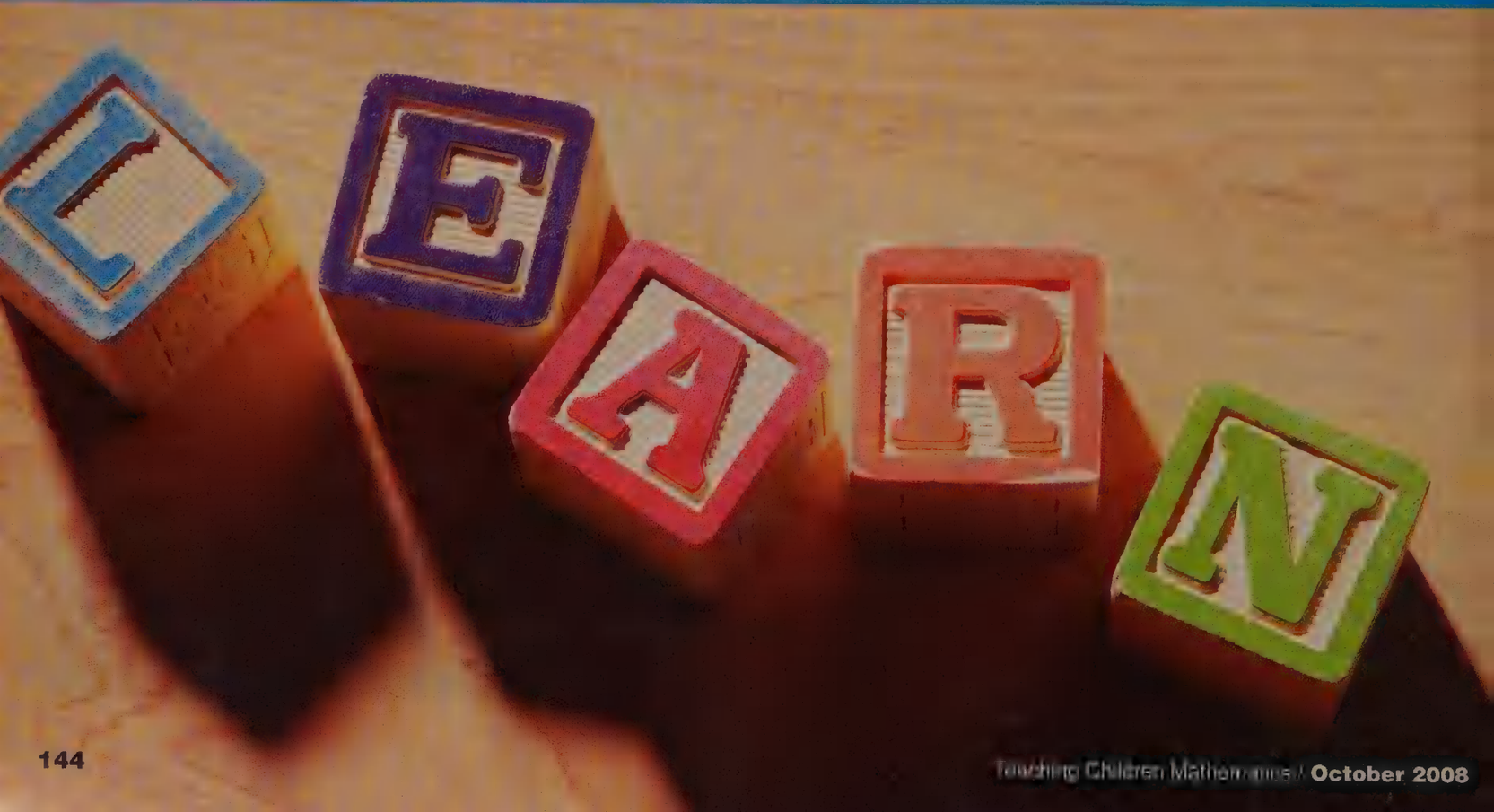

Encouraging children to think—independently and publicly—is possibly one of any teachers' most challenging and rewarding responsibilities. Young children's less-developed language and listening skills further complicate teachers' work at the early elementary level. Yet, I've been fortunate to work with many teachers who have embraced this challenge and use young children's unique attributes, such as their unmitigated curiosity, to center mathematics teaching on thinking and reasoning.

How do teachers create a classroom environment where thinking is primary—where thinking matters just as much as reciting facts, where facts are learned through thinking, and where they are used as the basis for further thinking? I have worked with hundreds of teachers in a long-term professional development project that included classroom visits, monthly discussion groups, and a collection of nearly one hundred instructional videotapes. Throughout this work, several themes emerge in establishing a classroom environment where mathematical thinking is the focus. This article discusses issues to consider when establishing a tone that encourages children to think during whole-group discussions, including addressing children's diverse thinking approaches and using their incorrect solutions. I incorporate examples of exchanges that may occur as teachers interact with students at work and suggest ways in which these interactions impact children's developing notions of what it means to do mathematics.

Facilitating Whole-Group Discussions

There is no question that much of the work of establishing a productive classroom environment happens during whole-group discussions. Many different stories from inside classrooms (e.g., Ball and Bass 2003; Heaton 2000; Lampert 2001), describe how students can be encouraged to share their solution methods, listen and ask questions, grapple with misconceptions, and probe and extend their thinking.

By Kate Kline

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Addressing diversity of thinking

To establish a routine conducive to all learning styles when thinking is at the core of discussions, consider the ways in which students process information, that is, the differences between so-called introverts—most commonly characterized as shy and taciturn—and extroverts—characterized as outgoing and gregarious. These can be accurate portrayals, but they do not quite capture the issues related to their main personality distinctions: cognitive processing (Myers 1995). Extroverts tend to process and think while they are talking. In other words, an extrovert is able to think out loud. Introverts, on the other hand, must think carefully before speaking. This is often why introverts have difficulty participating in group discussions—they are processing ideas. Just when they are ready to contribute, the discussion may have moved on.

Extroverts think aloud; introverts think before speaking

Every student is likely to possess aspects of both introversion and extroversion and to fall somewhere on a continuum, some more extreme than others. Cognitive processing certainly has ramifications for facilitating whole-group discussions and providing opportunities for voices to be heard. One technique that teachers use to deal with this diversity in processing information is to encourage the class to work collaboratively. Here is the way one second-grade teacher talked with her students about her expectations:

> We are going to spend a lot of time talking about our ideas. Sometimes, it may seem like the same students do all the talking. But it is important that we hear everyone's ideas as much as possible. It will help us all think harder about what we are doing. So, I'd like us all to work together to make sure that everyone is sharing. If you think you've said a lot, try to look around and encourage some of your classmates who have been quiet, and I will do the same.

Teachers suggest giving quieter students more time to process their thoughts by letting them know during the students-at-work phase that when the class comes together for a whole-group discussion, they are expected to share their solution method or idea. Another possibility is to pair quiet students

in the beginning of the year with students willing to present the shared findings to the class and then gradually encourage the quieter students to take on that responsibility.

Using incorrect solutions

One of the single most powerful ways to make an impact on young children's thinking is by accepting incorrect answers or ideas as a natural part of doing mathematics and pursuing them in the same ways as correct solutions. Many of the teachers I worked with found this particularly challenging to do and were concerned that facilitating a discussion around an incorrect solution or idea might actually create more confusion. However, they soon realized that the key is to use the same discussion techniques regardless of the student's response. In other words, when they use the same questions, such as "Why do you think that?" or "What do the rest of you think?" whether a student offers a correct or an incorrect solution, the responsibility for determining correctness then falls on the students. For example, the dialogue that follows occurred in a first-grade classroom where students were working informally on multiplication and division word problems. In this case, they were asked to think about the solution to this problem:

Discussion encourages children to think more deeply

> There are twelve cookies on the table. Three children want to eat them. How many cookies can they each have?

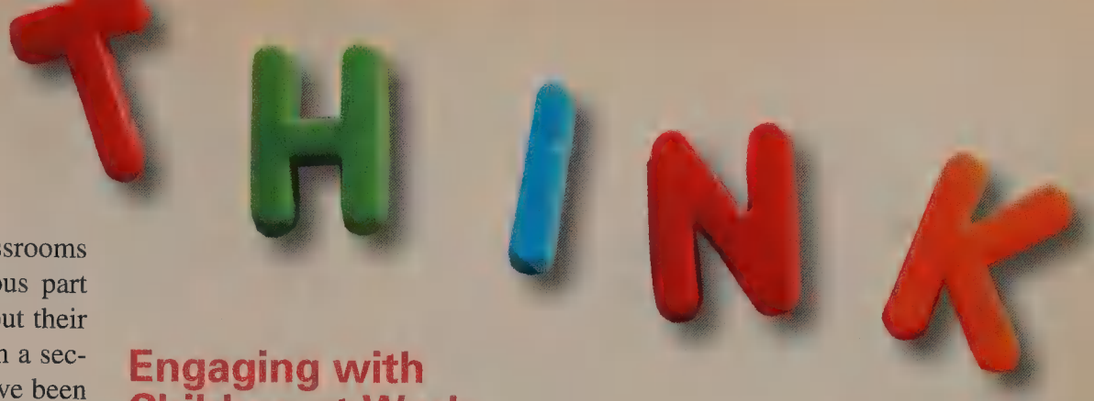
After using Unifix cubes and working with partners on the problem, the students brought their recording sheets to the rug for a discussion, which began in the following way:

Teacher: I am impressed by the thinking you all did to solve this problem. It seems like we have some different ideas about the answer that we should talk about. Jesse, you were thinking really hard about this problem, but there was something a little confusing about it. Could you share with us what you were thinking?
Jesse: I got three cookies, but something isn't right.
Teacher: Well, tell us what you were thinking first so we can all think about it with you.
Jesse: I'll show you. Here, I have my cubes, my cookies … I have three, three, three, and three.
Teacher: And how many children are sharing?
Jesse: Three. Oh, yeah, so, let me take this away [*removes one group of three*].
Students [*simultaneously*]*:* No, you can't do that.
Teacher: Okay, Mai, so what are you saying?
Mai: You have to use all the cookies—all twelve of the cookies.
Teacher: So, then, do you agree with the way Jesse had them in three, three, three, and three?
Mai: No. There are three kids. He has three cookies, and we got four cookies.
Teacher: What do you mean?
Mai: Here [*showing her picture*]. There are four, four, and four, because me, Jennifer, and Elaine got them.
Teacher: So, we have two ideas here, and I want to thank both Jesse and Mai for sharing their thinking with us. This will happen a lot when we are working on our problems. Now, here is the important part—we have to work together to figure out what the answer is or if there might be two answers that are okay, like with our crayons problem the other day….

It may seem surprising that such young children were working on multiplication and division, but research has shown that they approach such contexts in many intuitive ways, given that they are certainly faced with sharing situations in their daily lives. This work is helpful for building important foundations for these operations. (See references on the Cognitively Guided Instruction project, such as Carpenter et al. 1993, for further discussion.) This case is compelling because the teacher began the discussion with an incorrect solution but did not indicate her own thoughts about it. Rather, she used the discussion to encourage students to think more deeply about the problem's meaning. She recognized that Jesse and, most likely, other students in the class were grappling to make sense of one of division's fundamental ideas—distinguishing between measurement (how many threes in twelve) and partitive (how many in three groups) interpretations. Using Jesse's incorrect solution allowed the teacher to begin a discussion that would deepen children's understanding in ways that may not have been possible had she simply asked a student to share a correct solution.

Questioning one another's solutions

What are the best ways for teachers to establish an inquiry approach during whole-group discussions? The most productive discussions around



mathematical ideas seem to happen in classrooms where questioning is an almost spontaneous part of the way children talk to one another about their work. Consider the following example from a second-grade classroom where the students have been encouraged to develop their own procedures for computing. In this case, the students were working on the problem 35 – 17 = ___. Toward the end of the discussion, one student, Kiesha, wanted to share her solution method, and she wrote the following on the board:

35 is 30 and 5	30 – 10	=	20
17 is 10 and 7	5 – 7	=	–2
	20 – 2	=	18

As one can imagine, her classmates were intrigued with this strategy and posed a series of questions to Kiesha:

- If you are putting the twenty and the minus two together, why did you subtract the two?
- How did you know to take away two? Why couldn't it be any other number?
- How did she know to take away the two? Why didn't she add it?
- Why did you split up the thirty-five? The problem is saying to start at thirty-five and then take away seventeen.
- How did she know to take away the five and the seven together?

Some of these questions were necessarily related as the students worked to make sense of this strategy. What was impressive about these questions is that students were able to target places in the solution method that required justification. They were truly curious about this strategy and had a sincere desire to understand why their classmate chose to split the thirty-five (when most other students did not do so) and how to deal with a deficit (negative two) when subtracting.

When I asked the teacher about this exchange, she explained that she models this behavior in the beginning of the year by questioning students each time they share a solution method. Then she asks other students if they have additional questions. She then discusses with them the value of this questioning and how it helps everyone more deeply understand one another's methods. Gradually, students take over this responsibility and become questioners without much prompting, as was evident in the previous discussion.

Engaging with Children at Work

All the previous examples highlight ways to establish the learning environment during whole-group discussion where thinking is placed at a premium. Keeping these same goals in mind can capitalize on moments to extend thinking when engaging with students while they work.

Suggesting a strategy

One of the challenges teachers face in engaging with students as they work is making those on-the-spot decisions regarding students who seem to be struggling. Decisions about how to deal with students' frustrations have lasting implications for setting a tone for doing mathematics. Consider the following example, where second graders are working on creating and recording as many equations as they can to make twenty-two by adding two groups of cubes. They have about ten correct equations recorded on their paper, but they are wondering about one of them in particular: 15 + 4 = 22. The students request the teacher's help. As she approaches, the following discussion ensues:

Nick: Is this right? Are these right?
Teacher: Well, I want you to check them and make sure they're right, okay? Use your cubes and count them and see [*focusing on 15 + 4 = 22*]. Take fifteen and add four to it. How else could you figure out if that was right instead of taking all the cubes? What's that strategy we talked about?
Nick: Count …
Teacher: Yeah, counting …[*motioning upward with her hands*].
Nick: Numbers?
Teacher: Counting …
Nick: Umm …
Teacher: Counting up.
Nick: Up.
Teacher: So, if I already have fifteen, I only have to count four, don't I? So what would it be? Sixteen, seventeen, eighteen, nineteen. So, is that [*pointing to 15 + 4 = 22*] correct?
Nick: Yeah.

Many of us typically respond this way when trying to assist struggling students. We attempt to alleviate their frustration over a puzzling solution

by suggesting a previously discussed efficient strategy. However, this approach may have some negative ramifications on the learning environment. First, one important aspect of doing mathematics involves healthy struggle, the kind that occurs when students try to figure out something not immediately apparent. Capitalizing on such moments reinforces their value, gives students opportunities to work through the issues themselves, and establishes an environment where struggle is accepted—indeed, embraced—as part of doing mathematics. Second, suggesting strategies prevents the teacher from learning how students were thinking in the first place.

Consider the message that the exchange might have sent if the teacher had followed through with her original prompt to have students check the accuracy of their equations. The following exchange is between a teacher and a pair of second graders working on the problem 27 – 19 = ___. Students in this classroom are encouraged to develop procedures for computing that make sense to them rather than first learning the standard algorithm involving borrowing across place values. As the teacher approaches, students discuss their different solutions:

Productive math discussions happen when children question spontaneously

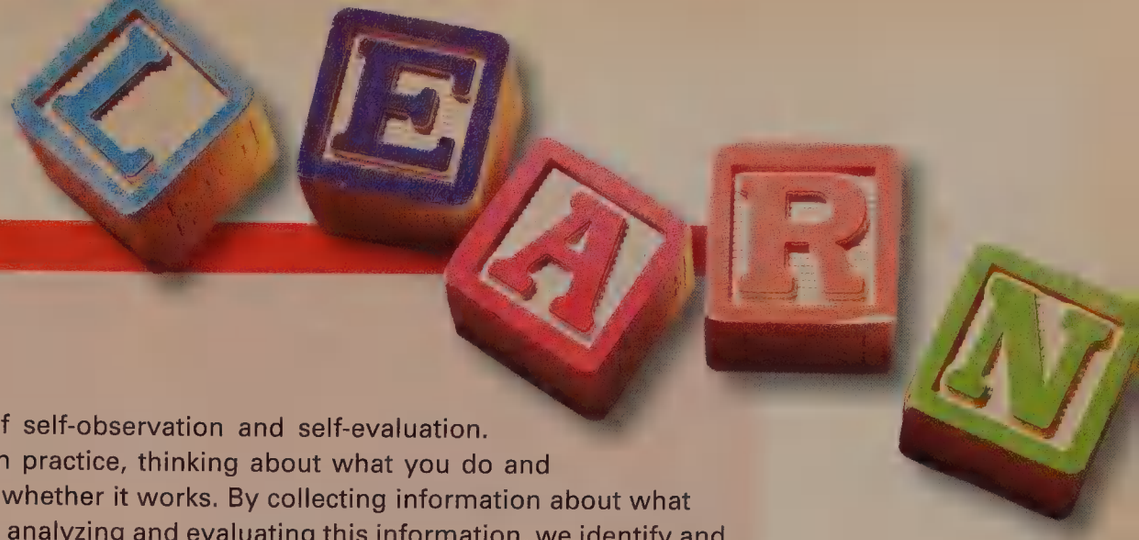
Teacher: So, what did you come up with?
Tyler: I have six.
Aleah: I have seven.
Teacher: Okay, explain your thinking to me.
Tyler: Well, I added one to nineteen to get twenty. So, then I did twenty-seven take away twenty and got seven. But I added one, so I needed to take one away from the seven, and I got six.
Teacher: What do you think of that, Aleah?
Aleah: That's not what I got.
Teacher: Yes, I know that. But what do you think of Tyler's explanation?
Aleah: Well, it can't be right, because I just counted up. I added one to nineteen to get twenty and then added seven more to get twenty-seven. So, I counted eight altogether. Six can't be right.
Teacher: Tyler, what do you think of Aleah's explanation?
Tyler: That makes sense, too. I should have counted.
Teacher: So, do you think both answers are right?
Tyler: No.
Aleah: No. If it was twenty-seven minus twenty, the answer would be seven, because you count up seven. So, if it is nineteen, it has to be eight.
Tyler: Oh, wait. I see something. I did get the seven …. See [*pointing to his work on the paper*], I got the twenty-seven take away twenty is seven. But then … I see … it's twenty-seven take away nineteen. I took away twenty! I took away too many, so I have to add one to the seven. I get eight just like Aleah!
Teacher: Do you understand what Tyler just said? [*Aleah nods yes.*] Can you explain again in your own words how Tyler changed his strategy and why it works?

In this exchange, the teacher worked to determine what the students were thinking and encouraged them to negotiate their different solutions themselves. She asked questions to solicit the students' explanations and consistently reinforced her expectation that they must listen to one another. She did not let the discussion rest on the description of Aleah's correct strategy and solution. Young children attempt to informally disprove others' ideas often by simply explaining that they got a different solution and describing their alternative strategy. And if the alternative strategy is somewhat easier to conceptualize, as was Aleah's strategy, it might be tempting to end the conversation. However, to more fully center the discussion on thinking, the students needed to revisit Tyler's strategy, suggest adjustments in his reasoning, and consider ways in which the two strategies rely on different subtraction interpretations (take away and distance).

Allowing time to develop understanding

When engaging with students while they work, how can teachers help students who are having difficulty in explaining their thinking or making sense of the task at hand? To illustrate the issues, consider the following two examples from video clips I have used many times. In the first example, first graders work on a story problem solution: "Twelve squirrels are playing on the ground. Find how many are left when four of them run up a tree." The teacher comes up to talk with Julie, who has the numeral seven written on her paper:

Teacher: Okay, Julie, what is your answer?



Reflect and Discuss

Learning to Think and Thinking to Learn

Reflective teaching is a process of self-observation and self-evaluation. It means looking at your classroom practice, thinking about what you do and why you do it, and then evaluating whether it works. By collecting information about what goes on in our classrooms and then analyzing and evaluating this information, we identify and explore our own practices and underlying beliefs.

The following questions related to "Learning to Think and Thinking to Learn," by Kate Kline, are suggested prompts to aid you in reflecting on the article and on how the author's idea might benefit your own classroom practice. You are encouraged to reflect on the article independently as well as discuss it with your colleagues.

- **Reflect on the question that Kline poses at the beginning of her article: "How do [you] create a classroom environment where thinking is primary—where thinking matters just as much as reciting facts, where facts are learned through thinking, and where they are used as the basis for further thinking?"**
- **Which strategies do you use to establish a safe learning environment where students are comfortable with expressing their mathematical thinking? Kline suggests strategies for encouraging introverted students to participate in whole-group discussions. What other strategies or techniques could you use to help introverted students find their voice?**
- **Kline recommends using incorrect answers to deepen students' thinking. What ideas do you have for helping students become comfortable with sharing answers even when they think they may be incorrect?**
- **Under the subheading "Allowing time to develop understanding," Kline introduces us to Ann as she wrestles with recording her count of eight cubes in a bag. Ann exhibits a gap in her thinking that indicates she is not developmentally ready to tackle such problems. How do you identify developmental gaps? What do you do to help students develop the needed conceptual understanding?**
- **Rather than continuing to push Ann, the teacher asked her to date and file her work in her mathematics folder. How do you manage your classroom—to allow time to work with individual students or to revisit concepts at a later date when other students have grasped the concept and have moved on?**
- **Choose some strategies that your students might use to solve mental computation problems. Locate places in the solution methods where you would like your students to ask questions in order to deepen their understanding of the strategies.**
- **As you engage with your students while they work, consider what kind of thinking might result if you suggest a particular strategy at that moment. What other choices could you make, and how would those choices impact your students?**

You are invited to tell us how you used "Reflect and Discuss" as part of your professional development. The Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Letters may be submitted to Teaching Children Mathematics *at* **tcm@nctm.org**. *Please include "Reader's Exchange" in the subject line. Because of space limitations, letters and rejoiners from authors beyond the 250-word limit may be subject to abridgment. Letters also are edited for style and content.*

Julie: Seven.
Teacher: How did you get that?
Julie: [*no response*]
Teacher: I want to know how you were thinking. Can you explain that?
Julie: I did it in my head.
Teacher: Okay, so how did you do it in your head? Can you do it again for me and talk out loud, so I can hear how you were thinking?
Julie: [*no response*]
Teacher: Did you count on your fingers? Can you show me?
Julie: [*no response*]

Accept—indeed, embrace—healthy struggle in mathematics

Teacher: Okay, let's get out the cubes. Now what was the problem asking?
Julie: There are twelve squirrels.
Teacher: Okay, show me twelve squirrels with the blocks. And then what happened?
Julie: Four ran up the tree.
Teacher: Okay, so let's move four away like this [*the teacher removes four blocks*]. Now, how many squirrels are left?
Julie: One, two, three, four, five, six, seven, eight [*Julie counts each block*]. Eight.

This is a familiar scenario when working with young children who are struggling to explain how they arrived at their solutions. The kind of assistance the teacher offers is fairly common as well.

Now consider a different example from a kindergarten classroom. Students are counting the number of objects in an inventory bag and then recording on paper what they have found. As the teacher approaches one student, she finds that the student is having difficulty recording her findings. The child says there are eight cubes in the bag, but she has drawn six cubes on her paper. The teacher engages her in conversation:

Teacher [*pointing to picture of six blocks*]*:* Is that eight?
Ann: No.
Teacher: So, what do you need to do?
Ann: Write six.
Teacher: Well, we're not going to change the number of blocks. You have to show me the number of blocks in the bag. What should you do? Did you draw a picture of eight cubes?
Ann: Yes.
Teacher: Count them and see.
Ann: One, two, three, four, five, six. I keep on messing up! [*Her partner suggests that she needs to draw more blocks.*]
Teacher: What about that, Ann? Do you want to try that? Let's see. We've got one, two, three, four, five, six. What if you [draw] one more? [*Ann draws one more block.*] How many do you have now?
Ann: Eight.
Teacher: Count them again.
Ann: One, two, three, four, five, six, seven.
Teacher: Do you have eight yet?
Ann: Yeah.
Teacher: You do? Are you sure?
Ann: Yeah.
Teacher: You're going to leave this just like this and put it in your math folder. [*The teacher dates it and walks on to other students.*]

Stop for a moment to think about your reactions to these two exchanges. I have used these video clips many times to initiate conversation with teachers about tone setting when engaging with students at work. For the most part, teachers initially show a positive reaction to the first exchange and a bit of surprise at the second. However, when we begin to contemplate possible ramifications of both interactions on students' developing notions of what it means to do mathematics, some additional ideas emerge.

First, recognize that students will be working on similar problems throughout the entire year. Doesn't it make sense to allow them time to develop their understanding of the problem types? Second, try to support children's growing beliefs in their own ability to determine the correctness of solutions and ideas. In some cases—in Ann's case, perhaps—showing a student the correct solution may not be useful if she is not developmentally ready to do the thinking necessary to understand it. Finally, allowing children time may suggest to them that many mathematical ideas take time to develop; they would not be problems if they did not. I am not suggesting that guiding students to a solution when they are struggling is never beneficial. However, I do think that carefully considering such exchanges is important in learning how to more consistently support children's attempts to think for themselves.

Concluding Remarks

What most positively impacts learning environments? The extent to which we center children's instruction on thinking for themselves, use their struggles, encourage ownership of their learning, and embrace their natural inquisitiveness. The ideas I present for facilitating whole-group discussions and engaging with children while they work help establish conditions where thinking is valued as the avenue toward learning. I have worked with many teachers as they incorporate reflection into their repertoire, mainly through collaborative video clubs that meet to discuss teaching by analyzing one another's whole-group discussions and exchanges with students at work. (See Archer, Grant, and Kline [2006] for a description of one teacher's journey.) Maintaining a consistent focus on thinking is not easy; it requires formal, deliberate reflection on the impact of specific instructional moves. However, by combining such analyses with a commitment to learning through thinking, we can make major strides in creating classrooms where children are willing to take risks and think their way through whatever challenges they encounter.

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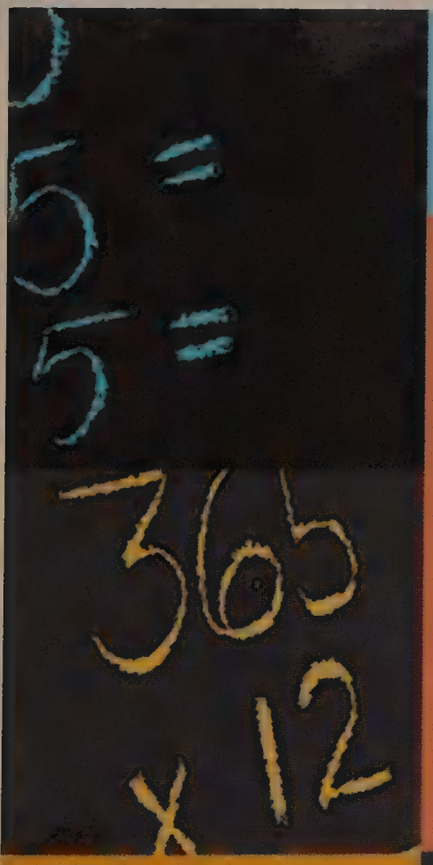

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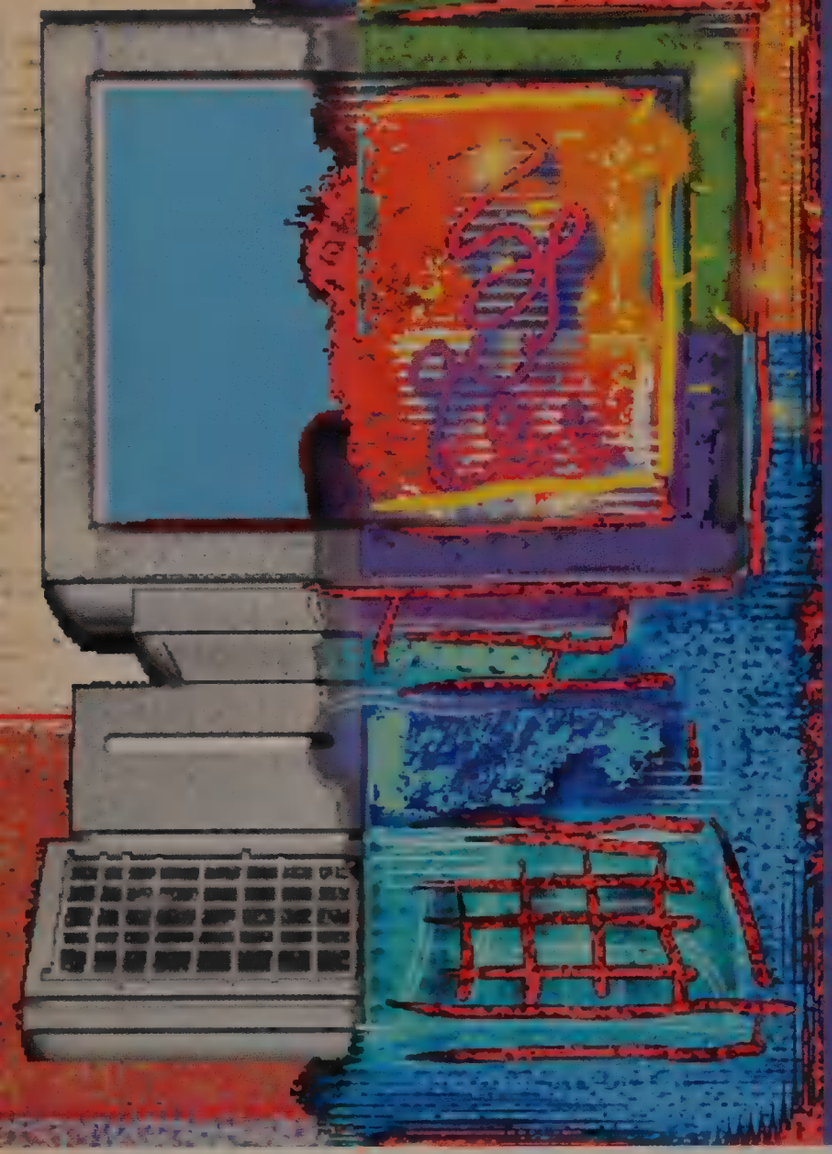

Technology-Rich Mathematics Instruction

Over the past decade, technology—particularly computer technology—has become more accessible to students in public school settings. From 2000 to 2005, the ratio of students to computers with Internet access improved from approximately seven students per computer to four students per computer (Wells and Lewis 2006). Although access continues to increase, one of the remaining challenges is the successful integration of technology as a tool for facilitating learning across the curriculum.

In mathematics education, technology has been widely recognized as a significant teaching and learning tool; it "influences the mathematics that is taught and enhances students' learning" (NCTM 2000, p. 24). The National Educational Technology Standards also identify technology as a tool that promotes learning in other subject areas (ISTE 1998). The recently released standards (ISTE 2007) reinforce this concept by shifting the emphasis from *learning-to-use* technology to *developing-skills-and-strategies-through-the-use-of* technology (e.g., communication, research, critical thinking, decision-making, and collaboration).

Students use computer technology to present ideas and represent concepts, manipulate variables and objects in ways that are difficult to accomplish with paper and pencil, and explore problems in greater depth. For example, using dynamic geometry visualization software, such as The Geometer's Sketchpad (Key Curriculum Press 2002), enables students to construct and investigate mathematical models, objects, and graphs, thus allowing them to understand more sophisticated and abstract problems. Other technology-based problem-solving tools, such as electronic spreadsheets, enable students to examine fundamental concepts and problems to enhance understanding of variables, representation, and mathematical models. The ease and flexibility enable students to pose and pursue their own questions about mathematical concepts in areas such as algebra and statistics. This is consis-

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tent with the NCTM Problem Solving Standard that suggests that "students be given the opportunity to learn important content through their exploration of the problems and to learn and practice a wide range of heuristic strategies" (NCTM 2000, p. 341).

When computer technology is incorporated into mathematically rich curricula that require complex thinking, application of concepts, inquiry, and problem solving, then student learning increases (Hyun and Davis 2005; Bessier 2006) at both the upper elementary level (Tiene and Luft 2001–2002) and in the primary grades (Hyun and Davis 2005).

This article uses one of the authors' classroom experiences to explore how teachers can create technology-rich learning environments that support upper elementary students' mathematical understanding of algebra and number and operations. We describe a unit that presents a common financial problem (the use of credit cards) to engage sixth graders in an exploration of linear equations. Using electronic spreadsheets, students generate and pursue questions to understand how changing one variable affects other variables. Additionally, the unit addresses how to select appropriate methods and tools for computing purposes, shows the use of models to solve problems, and reinforces multiple-step arithmetic operations.

Lesson 1: Understanding Credit

To begin the project, I recounted a recent trip to the shopping mall, where I used my credit card to purchase a pair of shoes. Students' responses included compliments on my purchase, groans from those who think I have far too many shoes, and beliefs about spending. One student asked, "Aren't credit cards bad for you?"

To focus their attention on credit, I asked: "What do you know about credit cards?" Several students responded that their parents and older siblings have credit cards, while others explained that their parents use only cash. A few recalled seeing television commercials about solving credit problems. Some expressed that using cards was a "bad idea" but were unable to articulate a reason why. Most believed that credit cards function as ATM cards: "Money is added to and subtracted from the card" and are used "when you don't have cash." In general, their knowledge centered on when and where credit cards are used. They did not demonstrate conceptual understanding of credit or interest or how these concepts transfer to real-life contexts.

Figure 1

Strategies to understand the concepts of credit and interest

1. Students prepare for small- and large-group discussions on their prior knowledge about credit and interest by responding to these questions in their learning logs:

- Describe what you know about credit cards.
- What does it mean to purchase an item by using a credit card?

2. Students consider scenarios in which a company loans money to an individual with the understanding that the total amount will be repaid with an additional charge for the privilege of borrowing the money. The financial transaction is compared and contrasted with informal loans that students make to friends and family.

3. The concept of interest rate is introduced as the additional charge for the privilege of borrowing money. The charge, in the form of interest, is a percentage or portion of the purchase. Students discuss the concept and generate solutions (using standard and nonstandard algorithms) for calculating the additional charge:

- What does it mean to pay an additional 15 percent for the item purchased?
- How could you determine the additional charge?

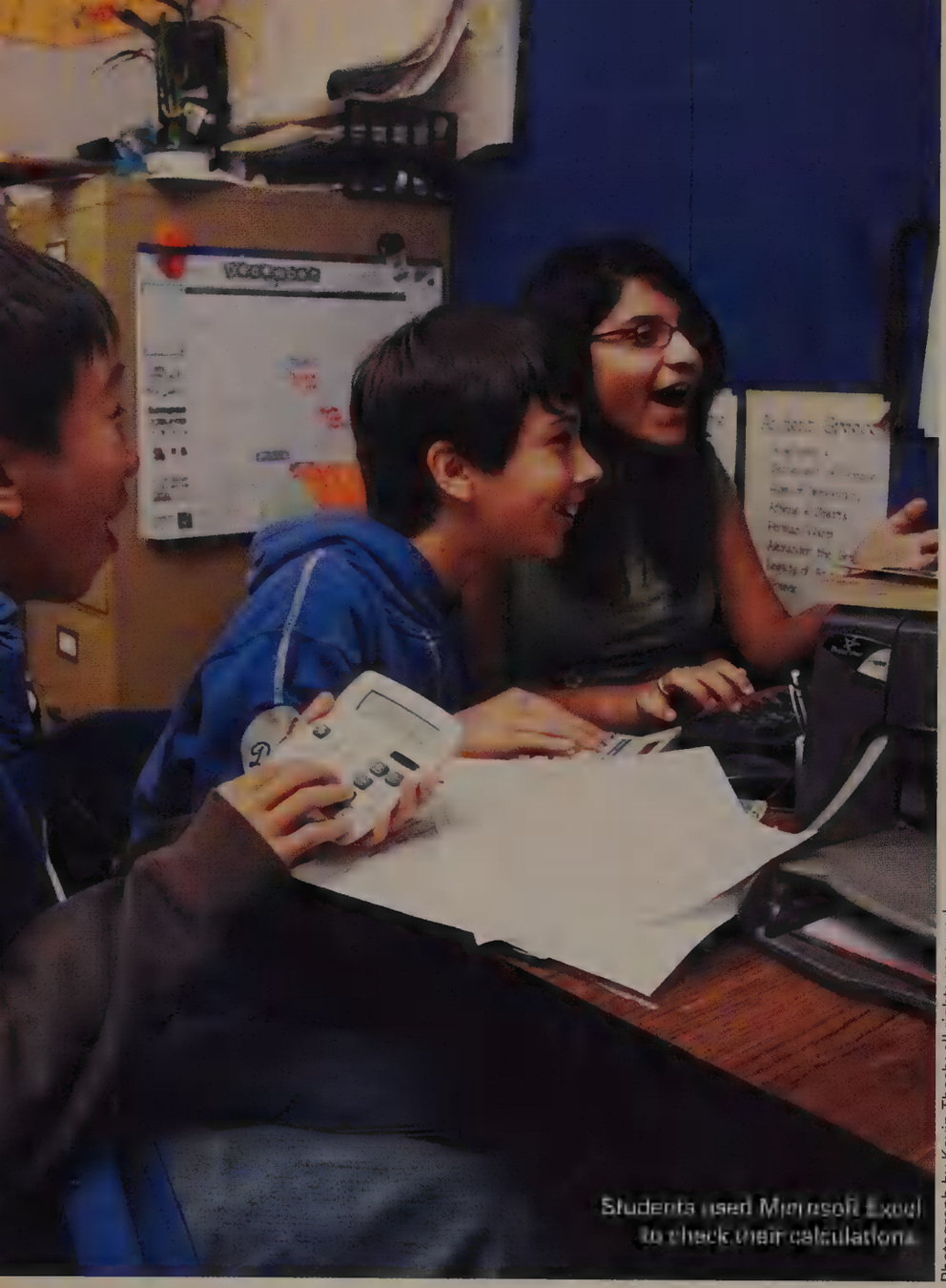
Some student responses follow:

- I don't know.
- Change the percent to a decimal.
- I think that means you have to pay $15 more if they said you have to pay 15 percent more.
- If I cut the $50 in half, it would be 50 percent, so I would cut it in half again for 25 percent. If 50 percent is $25, then 25 percent is half of $25, or $12.50.
- They [Master Card] are going to make you pay 15 percent of the $50 you already owe them. You have to multiply like this [demonstrating $50 × 15, resulting in $750]. It can't be that much. Oh, $7.50 [the decimal was erroneously inserted in the solution only].

4. Tap into students' familiarity with store sales (e.g., 30 percent off) to calculate a percentage. Have students discuss what it means when a store sign reads, "SALE—15% off." How much would you save? How could you figure it out? (This is a modification of questions in #3 above).

Figure 1 shows the strategies used to access my students' prior knowledge and develop the concepts of credit and interest. Students considered what it means to purchase an item by using a credit card, and they generated solutions for calculating interest charges. After multiple experiences such as these, we returned to the context of shoes to further explore the concept of interest.

Using the simple example of purchasing shoes, we examined the difference between buying the shoes with cash versus using a credit card. I explained that the price of the shoes was $70 and that my bank charges a 19 percent annual interest rate. Credit card companies calculate interest charges on a daily basis; however, because compa-



Students used Microsoft Excel to check their calculations.

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Figure 2

Student's credit card spreadsheet

Balance × 1.58% = monthly interest
(Note: 1.58% = 19% APR/12 months)

Date	Balance	Interest	Amount Owed	Payment	New Balance	New Charges
01/01/06	$700.00	$11.06	$711.06	$40.00	$671.06	
02/01/06	$671.06	$10.60	$681.66	$40.00	$641.66	
03/01/06	$641.66	$10.13	$651.79	$40.00	$611.79	
04/01/06	$611.79	$9.66	$621.49	$40.00	$581.49	
05/01/06	$581.49	$9.18	$590.67	$40.00	$550.67	
06/01/06	$550.67	$8.70	$559.37	$40.00	$519.37	
07/01/06	$519.37	$8.20	$527.57	$40.00	$487.57	
08/01/06	$487.57	$7.70	$495.27	$40.00	$455.27	
09/01/06	$455.27	$7.19	$462.46	$40.00	$422.46	
10/01/06	$422.46	$6.67	$429.13	$40.00	$389.13	
11/01/06	$389.13	$6.14	$395.27	$40.00	$355.27	
12/01/06	$355.27	$5.61	$360.88	$40.00	$320.88	
01/01/07	$320.88	$5.06	$325.94	$40.00	$285.94	

nies require payments to be made monthly, we used a simplified formula to calculate how much would be owed for the first month. Dividing the 19 percent annual interest rate by 12 months, we determined that the monthly rate is 1.58 percent, which would be an additional $1.11 in the first month.

Once the students had an understanding of the concept of credit, I introduced the spreadsheet (see **fig. 2**) so that students could track payments as well as understand how interest charges build over time and how making a payment reduces the balance owed. Together, we worked through examples to further develop understanding of the following formula:

> current amount owed = [previous month's balance + (previous month's balance × monthly interest rate)] – payment.

To explore individual cases, students selected a slip of paper that identified a hypothetical credit card transition. The dollar amounts ranged from $300 to $900, thus providing different scenarios and giving students opportunities to discuss varied results. Students recorded the purchase amount on their individual spreadsheets and, using $40 as the minimum monthly payment, worked through the calculations to determine how long it would take to pay off their debt. Then they compared their results to the cost of purchasing an item outright. Through the use of equations, students observed that the number of payments and amount of time could significantly affect an item's cost. Once they had demonstrated an understanding of the credit card calculations and could explain the process to their peers, I asked them to draw another transaction from the bowl, record the additional charges on their spreadsheets, and adjust their balances accordingly.

Over the next two days, students updated their spreadsheets and demonstrated an understanding of the concept and the ability to successfully perform the calculations. As anticipated, some students began to tire from performing calculations that were becoming increasingly tedious. Their growing awareness of calculation errors and frustration with the slow progress of completing the assignment created an opportunity for me to introduce technology designed to address some of these issues.

Lesson 2: Using Calculators

The purpose of the second lesson was to demonstrate the efficiency and accuracy of calculators and to show their application in another context. A brief

demonstration using the interactive calculator on our SMART Board provided another opportunity to review the underlying mathematical concepts of credit calculations. Students discussed what it means to multiply the balance by the interest rate, the implications of adding a specific amount of interest to our balance, and the newfound ease in carrying out the calculations. The tool enabled them to refocus their attention on the purpose of the mathematical problem: the interest accrued over time and its effects on the balance. For the remainder of the session, students were absorbed in their work, revisiting their previous calculations before moving forward.

Lesson 3: Introducing the Spreadsheet Application

At this point in the unit, students had demonstrated conceptual knowledge of credit and interest, the equations and operations used to calculate monthly accrued interest, and the effects of interest on the balance. The day before the assignment was to be completed, the class moved to the computer lab for an introduction to Microsoft Excel (2003), an electronic worksheet that handles mathematical computations quickly and accurately. Making a connection to the previous lesson, I noted that Excel can perform the same functions as a calculator. To demonstrate, I opened a credit card template identical to the one that students had been using (see **fig. 3**) and explained that the previously used formulas are built into the worksheet. Within seconds of my entering a student's data into the template, the computer completed the calculations.

We focused on the speed and accuracy that technology offers in terms of performing calculations, giving students the ability to manipulate variables in ways that would be impractical with paper and pencil. For the remainder of the session, the students entered their original data, ran the analyses, and reflected on their findings. The ease with which they entered the data amazed the students, and the noise level in the computer lab skyrocketed when they received results almost instantaneously:

- It took me a year to pay everything off.
- Mine took twenty-eight months! How much did you pay each month? What was your interest rate?

Students demonstrated their ability to communicate their mathematical ideas clearly and coherently, using formal language (e.g., monthly

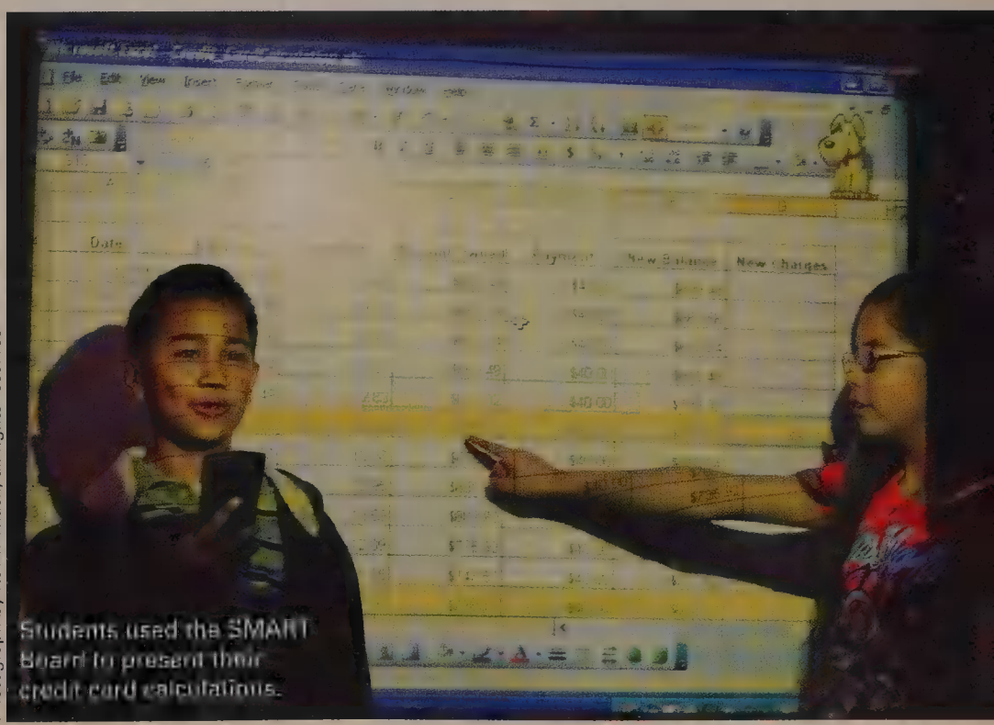
Figure 3

Credit card spreadsheet computed by using Excel

Balance × 1.58% = monthly interest
(Note: 1.58% = 19% APR/12 months)

Date	Balance	Interest	Amount Owed	Payment	New Balance	New Charges
1/1/2006	$700.00	$11.06	$711.06	$40.00	$671.06	
2/1/2006	$671.06	$10.60	$681.66	$40.00	$641.66	
3/1/2006	$641.66	$10.14	$651.80	$40.00	$611.80	
4/1/2006	$611.80	$9.67	$621.47	$40.00	$581.47	
5/1/2006	$581.47	$9.19	$590.65	$40.00	$550.65	
1/1/2008	$443.59	$7.01	$450.60	$40.00	$410.60	
2/1/2008	$410.60	$6.49	$417.09	$40.00	$377.09	$6.14
3/1/2008	$383.23	$6.05	$389.28	$40.00	$349.28	
4/1/2008	$349.28	$5.52	$354.80	$40.00	$314.80	
5/1/2008	$314.80	$4.97	$319.77	$40.00	$279.77	$97.00
6/1/2008	$376.77	$5.95	$382.73	$40.00	$342.73	
11/1/2010	$19.74	$0.31	$20.05	$20.05	$0.00	

Total Interest Paid	Total Payments	Total Charges	Total Time
$303.91	$2,340.05	$2,036.14	4 yr 11 mo



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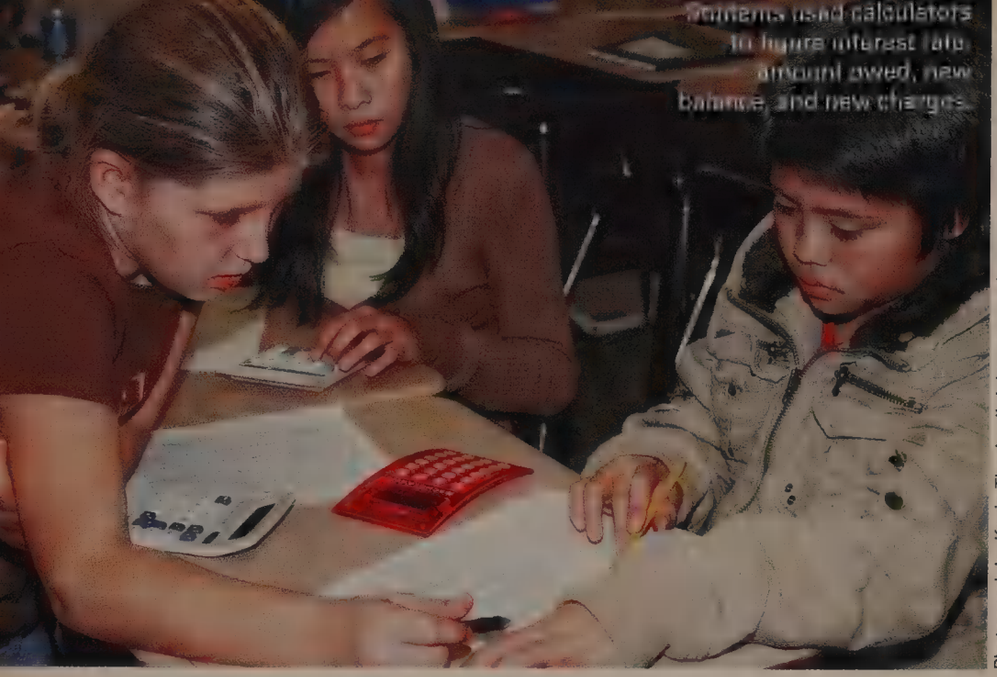
Students used the SMART Board to present their credit card calculations.



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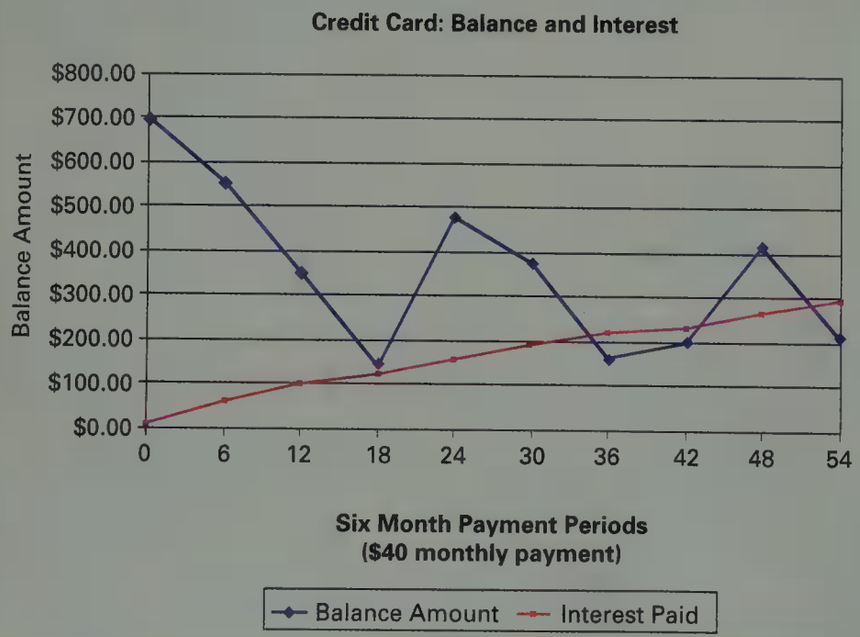
payment, balances, interest, and chart wizard) in the process. The context-embedded mathematical problem and the exploratory setting provided students with a natural situation to work on the skills that are part of the NCTM Communication Standard (NCTM 2000).

Sharing discrepancies between results generated by Excel and students' manual calculations was a profound experience and afforded students the opportunity to explore why those differences may have occurred:

- [One month] I forgot to add my monthly interest to my balance.
- I forgot to subtract my monthly payment.
- I pressed the wrong key [on the calculator].

Figure 4

Graph comparing changes in balance and interest paid over time



To conclude the lesson, students graphed the balance and interest paid at six-month intervals using the chart wizard available in Excel (see **fig. 4**).

Lesson 4: Reflecting, Discussing, and Exploring Deeper

In the final lesson, students reviewed their spreadsheets and graphs and reflected on their learning by responding to the following prompt: In a paragraph or two, describe what you learned about credit cards and interest.

After small-group discussions, I read the reflections aloud anonymously, soliciting comments on the similarities and differences in experiences. One student observed that he made payments for three years after his classmate's debt was paid off. Other observations included—

- I wonder why people don't pay more each month.
- It took forever to pay off the debt; what would happen if we paid more than $40 per month?
- My interest was almost half of my total charged. How is this possible? Did I take too long to pay the bill off?
- I'm going to use my credit card only when I have to.

These observations led to a discussion of factors that influence how much to pay. Students proposed that the following variables impact the total amount paid: interest rate, initial balance, and monthly payment.

The next day, we returned to the computer lab, armed with students' questions and insights, to explore the results of manipulating variables. Electronic spreadsheets such as Excel simplify exploring the impact of changing one variable. This type of exploration is critical to understanding equations and the concepts of credit and interest. Although we could have divided the queries among us and compared results, the mere calculations would have been time consuming and left little time to achieve the intended learning. Excel allowed students to interactively manipulate the variables of payment, percentage of interest, and time as a function of monthly payment.

After seeing a demonstration on how to change variables on the spreadsheet, some students increased the amount of their monthly payment to observe the impact on the amount of interest paid and the amount of time required to pay off

the debt. Others adjusted their initial balance and interest rate. Students explored questions such as the following:

- What happens to the overall payment when you change the monthly payment?
- What happens when you change (increase or decrease) the interest rate?
- Which one has the greater influence on changing the total amount paid? (or graphically, Which one has the greater influence on the slope of the line?)

Students observed how changing key variables impacts overall interest paid, which became more significant to them than their initial concern about the amount of time required to pay off the debt. From a financial literacy perspective, it is critical that students do not view credit as simply good or bad; they must understand its underlying mathematics and logic so they can make sound financial decisions.

Conclusion

To help students achieve the knowledge, skills, and dispositions envisioned in *Principles and Standards* (NCTM 2000) and *Technology Standards* (ISTE 1998, 2007), technology should be an integrated component of the mathematics curriculum. Technology not only reinforces skills and strategies but also can facilitate open-ended exploratory experiences that deepen mathematical understanding. In this unit, technology enabled students to explore complex problems in greater depth and sophistication than would have been feasible with traditional modes of instruction. With understanding and control of the software, they were able to generate and pursue their own questions, further developing their understanding of algebraic equations.

Integrating technology and mathematics instruction enables all students to engage in more complex activities at earlier levels. Reflect on ways you use technology in your teaching and learning environment (see **fig. 5**). Web-based resources can support your integration efforts with teaching suggestions, lesson plans, tutorials for students and teachers, and multimedia tools (see **table 1**).

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Figure 5

Reflecting on technology and mathematics practices

1. How does technology improve the learning process for your students? How would your students describe the role of technology in the learning process?
2. Could the same instructional goals be met without the use of technology?
3. In what ways do students use technology to present ideas and represent concepts?
4. How often do students use technology to explore problems, pursue their own questions, collaborate with peers, and use higher-order thinking skills?
5. How is student expertise demonstrated?
6. Do students have opportunities to reflect on the use of technology? Have they analyzed the use of various technologies and the information derived from them?
7. How do you monitor the use of technology to ensure that all students have access to technologies, develop essential skills and strategies, and feel self-efficacious about their technology use?
8. What opportunities do students have to *choose* whether, how, and which technologies might be used? How much *choice* do you have in planning and implementing the curriculum? Do you give yourself freedom to explore and take risks in integrating technology into the curriculum?

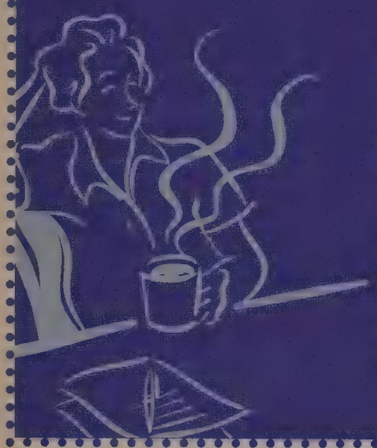
Table 1

Resources Available on the World Wide Web

Illuminations illuminations.nctm.org	NCTM's interactive Standards-based resources for mathematics instruction (pre-K–12) Includes lesson plans, student activities (interactive games, simulations, Java applets), Standards correlation, and external links
BrainPop www.brainpop.com	Standards-based animated educational movies (K–8) Subject Areas: Science, English, Math, Social Studies, Technology, and Health Simple, clear, and kid-friendly language Includes lesson plans, Standards correlation, interactive quizzes, experiments, comic strips, timelines, and printable activity pages Subscription fee required
National Library of Virtual Manipulatives (NLVM) nlvm.usu.edu/en/nav/index.html	Uniquely interactive, Web-based virtual manipulatives or concept tutorials (K–12) Aligns with NCTM Standards Includes lesson plans, descriptions of concepts
Gizmos www.explorelearning.com	Interactive, Web-based simulations Math (6–12) and Science (3–12) Aligns with state standards Subscription fee required
Internet4classroom www.internet4classrooms.com/on-line_excel.htm	Interactive tutorials for working with Microsoft Excel (K–8) Students practice skills in Language Arts and Mathematics
Harcourt Math Glossary www.hbschool.com/glossarymath2/index_temp.html	Animated and illustrated definitions of math terms for elementary school students Organized by grade level and alphabetically
Math, Number, and Counting www.computerlab.kids.new.net/math_sites.htm	Offers many resources for students to practice math skills in an interactive format

How can we help you be a better teacher?

The TCM Editorial Panel uses reader feedback to improve the overall quality of this journal. We want *Teaching Children Mathematics* to be your most useful teaching resource. That goal cannot be realized unless we hear from you.



- What sections of this issue did you find most useful? Why?
- What other content or topics would you like this journal to address in future issues?
- How can we improve TCM so that it better meets your mathematics education needs?

Please access our online survey at www.nctm.org/tcm to respond to these questions and help us better serve you. Thank you.

TCM Editorial Panel

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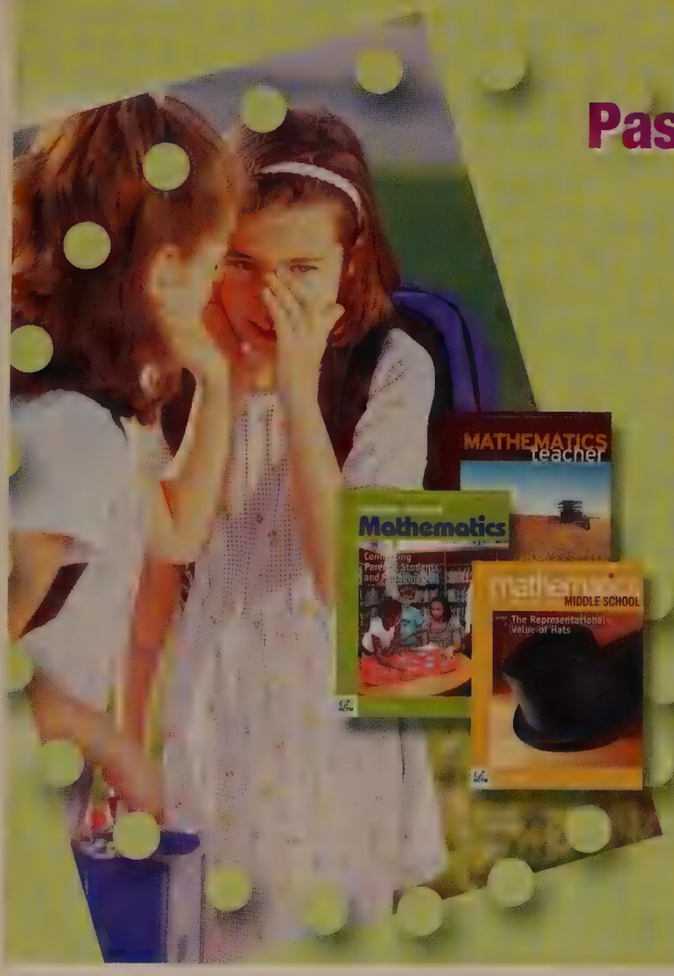

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CALL FOR MANUSCRIPTS:

Links to Literature Department

Children's literature is an important part of the elementary experience, but can reading a book encourage students to do some mathematics? Yes! Many books provide a valuable context for mathematical experiences, explorations, and problem solving. The *Teaching Children Mathematics* (*TCM*) "Links to Literature" department is searching for classroom-tested lessons that link literature and mathematics.

Your manuscript should include—

- the book's title, author, and ISBN number;
- a brief summary of the book;
- a clear description of the mathematics addressed in the book and in the lesson;
- questions posed to the students;
- classroom dialogue between students and the teacher;
- student work samples; and
- ways of differentiating the lesson to accommodate English Language Learners, special needs students, and gifted and talented students.

The following information would also be useful to our readers:

- How did you use the lesson to assess student understanding? How did you use the knowledge you gained about your students?
- Describe any misconceptions that you uncovered as a result of the lesson. How did you address these misconceptions?
- Give details of your students' reactions to the lesson, including their responses to the book and to the mathematics.
- Convey your reflections on the lesson.
- Explain the next steps that you envision taking in this process.
- Show photographs of your students involved in the lesson.

Submit your manuscript to the "Links to Literature" department by accessing **tcm.msubmit.net**. Your manuscript should be 1500–2000 words. References, figures, and relevant student work samples are not included in the page limit. The *TCM* Writer's Packet has information about *TCM*'s departments, tips for preparing manuscripts for publication, and contributors' guidelines that provide procedural information such as how to include photographs and figures and how to use resources and quotations. The packet can be viewed on the Web at my.nctm.org/eresources/submission_tcm.asp.

Christine Gallego, Dora Saldamando, Grace Tapia-Beltrán, and Karolyn Williams

Fun on the Farm

"Math by the Month" activities are designed to engage students to think as mathematicians do. Students may work on the activities individually or in small groups, or the whole class may use them as problems of the week. No solutions are suggested, so students look to themselves for mathematical justification and develop confidence to validate their work.

The following activities are designed to help you and your students enjoy the fall season through mathematical investigations involving data collection, number sense and operations, measurement, communication, and representation. ▲

WEEKLY ACTIVITIES

FUN ON THE FARM: K–2

OCTOBER 2008

0

Jack-o'-lanterns. Ask each of your classmates to draw a picture of their favorite jack-o'-lantern face on a 4" × 4" piece of paper. Collect the finished pictures and sort them into groups. What rule will you use to sort? How many different ways can you sort the pictures? For example, can you sort them by size? Can you sort them by the shapes of the eyes? Will you sort using different attributes? With your classmates, agree on a way to sort the pictures and then post them on the wall as a graph.

13

Hayrides. Countryside Farm is having a hayride for its visitors. Each wagon holds 12 people. If 24 students and 4 adults go on a field trip to the farm, how many wagons will be needed to take only the children on a hayride if they all go at the same time? How many wagons will be needed if both adults and children go? Use manipulatives, pictures, numbers, and words to determine how many wagons would be needed to take all the students in your grade at your school on a hayride.

20

Hens and their eggs. If most hens lay about 5 eggs each week, how many eggs does the average hen lay in one month? How many chickens would be needed to produce 60 eggs in one month? How many chickens would be needed to produce 80 eggs in one month? If 40 eggs were produced in one month, approximately how many chickens were needed to produce them?

27

A-maizing. Countryside Farm is hosting a haunted cornfield maze. Each person pays $5 to enter the maze. The farm collects $150. Use at least two different strategies to show how many people pay for the maze.

All members of the writing team teach at Pueblo Gardens Elementary School in Tucson, Arizona. Christine Gallego teaches first grade; Dora Saldamando teaches fourth grade; Grace Tapia-Beltrán teaches fifth-grade mathematics, and Karolyn Williams teaches a third- and fourth-grade multiage class.

Edited by **Dana Islas**, *who teaches kindergarten at Pueblo Gardens Elementary School in Tucson. Readers are encouraged to submit problems to be considered for future "Math by the Month" columns to the department editor at* **dana.islas@tusd1.org**. *Receipt of the problems will not be acknowledged; however, those selected for publication will be credited to the author. Readers are also invited to submit creative solutions and adapted problems to* **tcm@nctm.org** *for potential publication in "Reader's Exchange." Please include "Reader's Exchange" in the subject line. Because of space limitations, responses beyond the 250-word limit are subject to abridgment. Submissions are also edited for style and content.*

FUN ON THE FARM: 3–4

OCTOBER 2008

6 13 20 27

Pumpkin bread. Pretend that your class is making pumpkin bread for a bake sale and must adapt the recipe to make enough loaves. Work with your group to double the recipe and write it on a piece of paper. Share your recipe with another group. Discuss any differences; make any necessary adjustments. Work with your group to triple and quadruple the pumpkin bread recipe.

Pumpkin Bread

3 1/2 cups flour	2 teaspoons baking soda	1/2 teaspoon nutmeg	2/3 cup water
1 1/2 cups light brown sugar	1 1/2 teaspoons of salt	4 eggs	2 cups mashed pumpkin
1 1/2 cups white sugar	1 teaspoon cinnamon	1 cup salad oil	(fresh, frozen, or canned)

Preheat oven to 350° F. Grease 3 loaf pans; sprinkle with brown sugar. Combine all dry ingredients and fashion a well in the center. Add 4 eggs, salad oil, water, and pumpkin. Beat thoroughly. Pour into pans. Bake 1 hour and 15 minutes.

Loaves and slices. Our original pumpkin bread recipe yields 3 loaves. If your group quadruples the recipe, how many loaves will you bake? You decide to sell your pumpkin bread by the slice. Each loaf may be cut into nine slices. If you slice every loaf that your group bakes, how many slices of pumpkin bread will you have? If the entire class bakes 48 loaves, how many slices does the entire class have? The class decides to work one more day and make 48 additional loaves. How many loaves will that be for both days? How many slices?

Pumpkin bread bakery. We are having a pumpkin bread bake sale to raise $200 for a class trip to a farm. If each loaf sells for $2, how many loaves of pumpkin bread must we sell to raise enough money for the trip? If each loaf can be sliced into nine slices, and each slice sells for $.25, how many slices of bread must we sell to raise enough money for the trip? Will we earn more money by selling the pumpkin bread in loaves or by the slice? Explain your thinking.

Legs, legs, and more legs. When our class went to the farm, we saw some chickens and some sheep. We counted 42 legs. What are the possible combinations of chickens and sheep we saw? Use pictures, numbers, a list, words, or any other way that makes sense to you to show your thinking. What is the greatest number of chickens you could have seen? The least number? What is the greatest number of sheep you could have seen? The least number? Is it possible to have seen an even number of chickens? Why, or why not? What patterns do you notice?

WEEKLY ACTIVITIES

FUN ON THE FARM: 5–6

OCTOBER 2008

6 13 20 27

Farm fencing. A farmer wants to build a new coop for his chickens. He decides that the enclosure must be square or rectangular with an area of 100 square feet. Which different configurations could he build? How many feet of fencing will he need for each possible design? Use graph paper to draw all the possible rectangular or square designs. Be sure to include the number of feet of fence required for each. Include a key to tell how much each unit on the grid paper equals. Which fence would you recommend that the farmer builds? Why? Compare your ideas to a classmate's.

Mooing with fractions. A farmer pays you to help him sort his 24 cows by their color. Of 24 cows, 1/4 are black, 2/8 are brown, and 3/6 are black-and-white. How many black cows does he own? Brown cows? Black-and-white cows? Make a graph to show the farm's groups of cows. If the farmer pays you $2 for each black cow, $3 for each brown cow, and $4 for each black-and-white cow, how much money will he pay you altogether? The farmer wants to buy 12 more cows. He wants each group to be equal. Write a letter to him explaining how many of each color he should buy.

Feeding sheep. A 150-pound sheep should eat 2 percent of its body weight in feed each day. Assuming that your sheep's weight remains constant, how much sheep chow should you feed your 150-pound sheep each day? Each week? Each month? How long will it take the sheep to eat its total weight in sheep chow? A 50-pound bag of sheep chow costs about $11.40. How much will it cost to feed your sheep for one year if he stays at 150 pounds? How do your calculations compare with a classmate's? Besides making a computational error, explain why your results might vary.

Milk. A farmer has 80 cows; each produces an average of 7 gallons of milk each day. One gallon of milk weighs about 8.6 pounds. It takes about 10 pounds of milk to make 1 pound of cheese. The farmer can sell his milk for $3.29 per gallon, or he can make it into gourmet cheese and sell the cheese for $19.75 per pound. Which way will the farmer earn more money? Explain your thinking on paper and then share with a classmate. Do the two of you agree or disagree?

Shifting Roles and Responsibilities

to Support Mathematical Understanding

"Matt, I hear what you are saying, but I respectfully disagree with you. I have another way to think about that."

This discourse occurs daily in learning environments that support mathematical understanding. Over a period of three years, we developed an innovative building and classroom culture by examining the Teaching, Process, and Content Standards (NCTM 1991, 2000), buying the standards-based mathematics program *Bridges in Mathematics* (Marzano, Waters, and McNulty 1999–2007), and leaving behind old practices and beliefs. Today, our teachers have high expectations for all our students, and our students are actively engaged in diverse thinking, questioning, and reasoning strategies. Moving in this direction was new territory for our school staff; succeeding enhanced our school community in ways we never expected. We transformed the classroom environment and the roles of the teachers, principal, and students.

The Teacher's Role

Before 2004, teachers moved students through textbook lessons and stand-alone units, assuming or ignoring previous student mathematical understanding and experiences. They turned the page to the next lesson and talked their way through another skill. Some provided students opportunities for using manipulatives, gave them time for discovery, and then showed students how to use the manipulatives procedurally. Teachers thought they were teaching to the Standards because small groups worked together with mathematics tools during these investigations. However, as in many schools, teachers continued to use a direct-instruction approach and drill from worksheets for the bulk of their mathematics instruction.

When our students did not make Adequate Yearly Progress (AYP) on the state standards exam or show average growth on the Northwest Evaluation Association Assessment (NWEA), we began to seriously examine our practice. *Professional Teaching Standards* (NCTM 1991), particularly the part related to the teacher's role in discourse, suggests that we actively listen to students' ideas, pose questions that invite students to clarify and justify their ideas orally and in writing, and thoughtfully attach mathematical notation and language to students' ideas. We understood the Content Standards and were beginning to use a Standards-based curriculum, but we were not making progress because we were not yet willing to leave our old teaching practices behind. Coupled with this challenge, our staff also believed low-income students would do best with a program that had a low-level, repetitive structure. We later discovered that *all* our students thrive on genuine problem-solving tasks and far exceed our expectations when presented with authentic challenges. By

By Pia Hansen and Donna Mathern

Pia Hansen, phpmath@gmail.com, spends her days modeling lessons and coaching and mentoring K–8 teachers in mathematics. She is interested in developing content knowledge and pedagogy practices that increase teacher efficacy and student achievement. Donna Mathern, Donna_Mathern@ncsd.k12.wy.us, is the principal at Evansville Elementary School near Casper, Wyoming. She is pursuing her doctorate in educational leadership and is especially interested in second-order change.

sharing our student work at staff meetings, in hallway displays, and at parent conferences, we made our students' mathematical learning public.

The role of the classroom teacher shifted slowly. Teachers had to trust that students would experience and understand the mathematics through an interactive process of knowledge construction, that when students share ideas with others, engage in thinking together, and consider viewpoints other than their own, their understanding increases. The teacher is not the *sole* source of knowledge: the entire classroom community is the source of knowledge. Teachers began asking why, what, and how questions, and students enjoyed taking the floor to share their thinking. One powerful strategy that our teachers employed was a think-pair-share structure: Teachers posed a thoughtful question and gave everyone a period of time to think about the idea privately (silent think time). Then pairs of students discussed their ideas with one another. Each partner had a turn to share his or her idea. Finally, teachers chose three or four students in the class to share their ideas with everyone.

The think-pair-share strategy increased students' accountability to their peers and established interdependence as a community. Talking provided students with practice in using mathematical language, which prepared them for the written work required on classroom assignments and in assessment situations. Teachers realized that students needed to talk about the mathematics before they could write about it. As one teacher noted, this structure was particularly helpful with English Language Learners and Special Education students:

> My entire perspective on teaching in all subject areas has changed. The students have become more active participants in the teaching. There is much less teacher talk, more think-pair-shares, respectful disagreements, and this leads to powerful discussions. This gets to a deeper understanding. It's also important for students to feel safe. We respect all thoughts, explanations, ideas, and answers.

In a learning environment that supports mathematical understanding, teachers facilitate (not dictate) learning. Teachers *educe*, draw out, the mathematician within students by engaging them in experiences in which they explore mathematical concepts and procedures in sensory-rich settings. Their findings are reported to peers and are valued and accepted as valid. Teachers in such classrooms nurture students' confidence and autonomy by—

Photograph by Donna Mathern and Pia Hansen; all rights reserved

- providing a variety of manipulatives and choices about how to use them;
- representing students' thinking with models, sketches, and diagrams;
- using notations to make ideas transparent for other students in the class;
- using probing and genuine questions (How could you solve the problem? How can you be sure that ...? Are there any patterns here? Why didn't that work? What do you notice? Can you convince us? What if ...?) to draw out and challenge students' thinking about concepts and procedures;
- listening closely for understanding and misconceptions in students' responses;
- giving nonjudgmental feedback and carefully responding to wrong answers; and
- fostering collaboration among students.

This shift in pedagogy means not knowing the direction a lesson may turn, monitoring and adjusting each day's learning, and being metacognitive and reflective as a practitioner. Teachers have to understand the underlying mathematics, as well as their students' thinking, to make instructional decisions about next steps, which points to spend time on, and which paths to take in the discourse:

> Every day I look forward to coming to school and teaching math. I have a learning goal in mind, but my "ahas" come frequently [regarding] how the students respond, what they say, how they interact, what they do, and how they come up with what they come up with! Each student teaches me something new.

Table 1

Leadership Responsibilities for Second-Order Change

Positively Correlated	Associated Practices	My Actions
Knowledge of curriculum, instruction, and assessment	Is knowledgeable about instructional practices Provides conceptual guidance for teachers regarding effective classroom practice	Read NCTM Standards and handout information and purchased copies for library Arranged for and participated in curriculum mapping workshop Read current literature about math reform and assessment practices
Flexibility	Adapts his or her leadership behavior to the needs of the current situation Portrays a positive attitude about the staff's ability to accomplish substantial things	Arranged and participated in school-wide retreats to build a professional learning community Left observation "quick notes" (evidence of change in practice) Provided evidence of increasing math scores on NWEA and PAWS
Change agent	Consciously challenges the status quo Is comfortable with leading change initiatives with uncertain outcomes	Provided observation feedback to teachers based on six-block observation guide Observed and interacted with students and shared my difficulties in being too directive
Ideals and beliefs	Communicates and operates from strong ideals and beliefs about schooling Shares beliefs with staff about schools, teaching, and learning	Handed out articles about the experiences of others as they engaged in both learning and teaching Placed positive achievement messages on the school walls
Monitor and evaluate	Monitors and evaluates the effectiveness of curriculum, instruction, and assessment	Met monthly with grade levels to discuss achievement goals, and then analyzed data to determine if goals were met
Intellectual stimulation	Ensures faculty and staff are aware of the most current theories and practices and makes the discussion of these a regular aspect of the school's culture	Participated in eight *Lenses on Learning* (Grant et al. 2003a, b) sessions and shared information with staff Arranged for and participated in monthly observations and in-services with math content coach
Optimize	Inspires and leads new and challenging innovations	Hired a math content coach and supported her work in the building Started a math coaching cohort of building teachers to begin leading change Sent faculty to workshops within and outside the district to continue learning about researched best practice and instructional strategies

Adapted from Waters and Cameron (2006)

The Principal's Role

Knowing that changes were needed in our mathematics instruction was one thing; identifying the principal's role in these changes was another. As an instructional leader, I first directed my energy toward the seven areas positively correlated with second-order change initiatives, in this case, "creating learning environments that support mathematical understanding" (Waters and Cameron 2006; see **table 1**). Second, I confronted my own lack of mathematical background knowledge. Through the *Lenses on Learning* (Grant et al. 2003a, b) courses, I began to "see" mathematics as pictures, numbers, and words and experience disequilibrium in inquiry and problem-solving situations, which helped me appreciate changes our teachers would be making.

Finally, I enlisted the support of a mathematics consultant to further our content and pedagogy knowledge. She had a deep conceptual understanding of mathematics, command of best practices, and appreciation for the needs of adult learners. We initiated a book study group, discussed professional readings at staff meetings, and released teachers to observe peers in our building and in the district. The consultant modeled lessons for teachers, team taught in some classrooms, and observed our students and staff at work. Grade-level teams met to examine student work and review upcoming lessons and units with the consultant (see **table 1**). This ongoing and sustained implementation effort, with classroom-based coaching, feedback, and administrative support, positively impacted student achievement.

To implement the NCTM Equity and Learning Principles, we focused on teacher actions during instruction. Our consultant modeled numerous lessons in the classrooms while the teachers and

Table 2

Six-Block Classroom Look Fors

Teacher's Name _____ Date_____

Communication Use think-pair-shares to engage students in discussions. Model clear and precise mathematical language. Teachers talk less; students talk more.	**Questioning** Teachers ask a variety of questions including some "What is this?" (one correct response) and "How?" and "Why?" (higher-order thinking)
Student Engagement Students actively *do* mathematics. Teachers facilitate students sharing strategies with partners or the class.	**Visual Models/Manipulatives** Teachers ensure that students have access to a variety of models and manipulatives to support their communication and reasoning.
Monitoring Student Learning Teachers are aware of grade-level benchmarks, actively listen for misconceptions and mastery, and provide appropriate challenge and support.	**Community of Learners** Projects and posters are displayed around the classroom to communicate the mathematics culture. Student comments and observations are posted. Students respectfully listen and engage in discussions about mathematical ideas.

Additional Notes:

I observed. During these lessons, the consultant acted as a learning facilitator; she posed a task or problem and then turned over the learning to the students. Students were more actively engaged with the materials and one another, willing to take risks and share their thinking with others in the classroom, and they continually astonished us with their discoveries and new understanding.

As principal, my most rewarding days were spent observing teachers with the content coach. We developed two observation tools for taking notes during these visits. The first was a simple T-chart, to record the teacher actions and comments and student actions and responses. Our goal was to provide a visual look both at teachers' and students' actions and at the ratio of teacher talk to student talk. Early on, it was obvious we were still entrenched in teacher-directed instruction, with students giving one- or two-word responses to low-level questions. The notes revealed teachers telling students the answers or telling them how to solve the problem.

The second year, we developed a "Six Blocks to Look For" document based on research on instructional strategies that increase student achievement (Marzano, Pickering, and Pollock 2001) and shared the form with the staff. Teachers knew what we were looking for when we came into the classrooms, and they knew why we were looking for these strategies (see **table 2**). Following the observation periods, we shared the results with them during honest, goal-setting conferences.

Teachers posted probing question prompts on the wall to remind themselves to ask open-ended questions. They sat down during a lesson, in a student chair, to keep from controlling the dialogue in the front of the room. They practiced wait time, knowing that higher-level questions require more time to think. Gradually, we began to see a shift as teacher responses became less prominent and student responses became greater.

The Students' Role

The first week of school, most teachers post rules for being safe, kind, and respectful. These rules only begin to describe the students' role in the classroom environment. Students in a learning environment that supports understanding have a far greater

responsibility to one another: To be successful, students must establish relationships at school. Classroom discourse depends on students making conjectures, connections, and conclusions about big ideas in mathematics. Initially, our students were reluctant to share an answer and their solution because they assumed the teacher was questioning an incorrect response. Now they realize that right and wrong answers are investigated in the same way, and they verify, revise, and discard claims using manipulatives and visual models as mathematical proofs.

Our teachers modeled being a respectful audience for one another's solutions and strategies, and soon the students caught on to this social expectation, too. Most of our classrooms post a Community of Learners chart at the beginning of the year that includes such behaviors as listening to one another, asking questions when you do not understand, helping one another, and showing your work. Teachers and students refer to the chart throughout the year and add additional expectations as the community develops.

Maintaining the community is as important as establishing it. During the school year, new students are assimilated into the culture. Many of these students come from traditional, teacher-directed mathematics programs and are not accustomed to making sense of mathematics. Within a couple weeks, they embrace the routines and structure. In our classrooms, students share their thoughts with peers, and they honor other people's thinking. They frequently follow up a student presentation by agreeing or respectfully disagreeing as a community of learners.

All students learn and retain concepts more quickly when they use mental models. Our students are particularly successfully because mathematics concepts are introduced and explored with visual models and manipulatives. These models are necessary for shared understanding, and they enhance students' communication skills. Fifth graders use base-ten area and linear pieces, grid paper, quick sketches, and standard and student-generated algorithms to solve long-division problems. They examine multiple ways to solve a problem using efficient, flexible, and accurate strategies. Contrary to what we originally thought, abstract constructs and processes taught in a real-world context are more effective than isolated and repetitive practice. In daily problem-solving situations, students experience planning, labeling, and sorting important information. Students use rich mathematical language with partners and peers to build background knowledge and develop a sense of "we." These structures, in addition to goal setting, made the biggest impact on our student achievement. Several times during the school year, students set goals for reading and mathematics that are reasonable, rigorous, and based on standards set by our district and state. Students track their own progress toward these goals, recognizing how important education is to their future.

When we asked a group of students to tell us what they liked about mathematics learning communities, here is what they said:

- We get to talk about the problems.
- We respect [one another] and learn together. We help each other and do our best job.
- I like to ask plenty of questions, and they will be answered. I have gotten smarter.

Even kindergartners ask, "Do you want to know what I noticed?" with great expectation that we *do* want to know.

The Environment's Role

Learning environments that promote mathematical understanding require three components: a viable, guaranteed, standards-based curriculum; sacred time allocated to mathematics learning; and classroom accommodations including desks or tables, display boards, and a variety of tools for mathematical thinking. Standards-based curricula build on the Learning Principles, specifically, activating prior knowledge and promoting inquiry. Cognitively demanding tasks ask students to make conjectures based on their previous understanding, explore big ideas, make connections to other topics in mathematics and the real world, and analyze the results of their investigations. This social activity encourages all students to engage in the synergistic process of thinking together, considering other viewpoints, and broadening their own perspectives.

However, simply adopting a program is not enough. Teachers must commit to working through their own cognitive dissonance and to examining the roles of teacher and student in the process of teaching to the Standards (see **table 2**). Implementing these curricula with fidelity requires a commitment of time. The teachers at our school blocked out an hour and a half to two hours per day for mathematics instruction. Inquiry and analysis does not occur in a forty-five minute period, and teachers recognized that when they were pressed for time, they went back to a direct-instruction format.

We reconfigured the rows of desks and chairs in our classrooms into pods of four to six desks,

allowing students to talk and work together side by side or across from one another on a regular basis. Students now consider mathematics materials as tools for thinking and communicating mathematically and can access them in tubs on shelves or in "tool kits" for investigations.

Conclusions

Teachers and principals can shift the roles and responsibilities of students, teachers, and the learning environment in their schools with the following strategies:

- Examine the NCTM Content and Process Standards (2000) and the NCTM Focal Points (2006) for alignment among current teaching and assessment practices.
- Select and employ a rigorous, standards-based mathematics curriculum that encourages the use of manipulatives and visual models.
- Provide ongoing professional development to be sure the curriculum is implemented comprehensively and with fidelity.
- Look at a variety of data to determine strengths and weaknesses of the program materials and adjust instruction accordingly.
- Study the changing roles of teachers and students by encouraging teachers to observe one another and identify who is talking and who is learning during the lesson.
- Confer about instructional strategies that increase student achievement, including the use of open-ended questions, think-pair-shares, and rich mathematical language.
- Read and discuss articles and books to increase the staff's mathematical content and pedagogy knowledge.
- Set goals with students.
- Celebrate mathematics learning with families.

This journey made our school a professional learning community of members willing to monitor and adjust their thinking about teaching and learning, create an atmosphere that celebrates diversity, and move away from mathematical mechanics to help students build deeper, richer understanding of their own thinking. Student mathematics achievement (measured by state and NWEA assessments) now meets and exceeds our district and state averages. Spring 2006–2007 NWEA Growth Scores for grades 3–5 showed that 76 percent of our students made expected or high growth in mathematics.

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Beyond these measures, we have empowered our students and teachers to see themselves as mathematicians.

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The Workshop Way:

Creating a Stimulating Intellectual Environment

By Malinda F. Apanay and Autumn C. Vavoso

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How can you show your thinking? The question resonated in our heads as we walked over and glanced at Cassi's blank paper (see **fig. 1**). Years ago, we might have felt we were missing the mark as her teachers. But because we had worked so hard to establish the intellectual environment in this class, we now had the foresight to interact

with her during the work period. We stopped long enough to hear her plan to "act it out" as a way to represent her problem-solving strategy. Cassi began her dramatization. Besides capturing the audience's attention, she showed the class how she used the hundreds chart to help her solve the problem.

Moments like these affirm that bringing the Reading and Writing Workshop into mathematics class allows for the intellectual environment we are hoping to create.

The Workshop Cycle

As mathematics educators, we spend a lot of time considering how to arrange the physical space of our classrooms, but how much time do we spend planning the intellectual space? The setting we highlight in the illustration above and throughout this article is the intellectual environment—the thinking that exists in the minds of both teachers and students during a mathematics lesson. Our goal is to provide an intellectual environment for students that will cultivate mathematical understanding. By using the Workshop as our instructional framework and bringing the NCTM Process Standards to the forefront of our mathematics instruction, we have witnessed substantial growth in mathematical understanding, which has translated into student achievement. Furthermore, our students now realize that mathematics is about more than just the numbers or the answers. Mathematics is also about the process of thinking.

The Workshop frames an environment for process thinkers and provides the most effective setting for this intellectual environment, moving us from teacher-based practices to true student-based processes. We agonize over how to organize the content around big ideas so that students leave our classroom understanding key concepts. Our planning takes hours and is a true rehearsal of what we might say or do as we anticipate what the students could say or do.

The Workshop, made popular in literacy instruction (i.e., Writing Workshop and Reading Workshop), puts students at the center of their learning. The authors of *Writing Workshop: The Essential Guide* observe that the "writing workshop puts kids on the spot and makes them responsible for their learning" (Fletcher and Portalupi 2001, p. 3). We can adapt this practice to mathematics. Just as students learn to read and write *by reading and writing,* they learn mathematics *by doing mathematics*. When we say *doing mathematics*, we mean solving real-world problems with no prescribed means of solving them.

Figure 1

A student problem

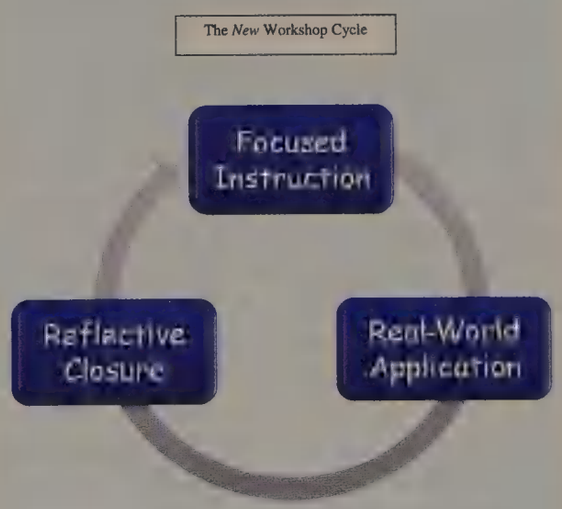
Name_____

Larry and Jerry, two coal truck drivers, are having a contest to see how many loads of coal they can deliver in a month. The winner gets a $100 bonus if he delivers at least 12 loads more than the loser. Larry delivered 31 loads! Jerry was proud of his 17 loads. Who won the contest? By how much did he win? Did the winner get his bonus?

Use the space below to show what is happening in the problem, how you solved it, and the answer.

Figure 2

The arrow represents focused instruction seamlessly threaded through the updated Workshop Cycle phases.



We have clarified our own thinking by now referring to this instructional framework as the *Workshop Cycle*, a phrase that more clearly indicates how one workshop flows into the next and indicates that learning exists on a continuum. We now use the terms *Focused Instruction*, *Real-World Application*, and *Reflective Closure* to clearly articulate the purpose of each Workshop Cycle phase. We use a narrow focus as the thread that fluidly moves through each phase and binds the intellectual environment together (see **fig. 2**).

Table 1

Getting Started With the Workshop Cycle

Week	Goal	Lesson Ideas
1	Students will understand the process of the Workshop Cycle and the expectations of each component.	▪ Establish meeting and work areas. ▪ Discuss procedures and routines for using materials and supplies. ▪ Discuss purpose and function of work teams. ▪ Discuss expected noise level and work ethic. ▪ Discuss responsibility for creating understanding.
2	Students will become familiar with ways to process the mathematics and be responsible for their own thinking and understanding.	NCTM Process Standards ▪ Problem Solving ▪ Reasoning and Proof ▪ Communication ▪ Connections ▪ Representation
3	Students will demonstrate their knowledge of the Workshop Cycle by actively participating in each phase. Students will apply NCTM Process Standards while working through familiar content.	• Learn how to choose a manipulative or mathematics tool. ▪ Explore how to use different Process Standards when working with the same Content Standard or problem. ▪ Own and articulate choices you make when working through mathematics problems. ▪ Find mathematics in daily life.
4	Students will demonstrate and articulate their thinking, decisions, and understanding.	▪ Evaluate progress of work teams. ▪ Evaluate progress of choosing and using mathematics tools and manipulatives. ▪ Evaluate progress of articulating understanding. • Evaluate participation levels and ownership in each workshop cycle phase.

The intellectual environment is made possible when it is framed by a setting that allows *thinking* to take center stage. When thinking takes center stage, understanding naturally follows. Throughout our own process of reflection and in replicating this Workshop Cycle in classroom after classroom, we note key steps in creating an intellectual environment.

Creating an intellectual environment

To create the intellectual environment within the Workshop Cycle, you must devote time to teaching students the process and procedures of the cycle itself. During the first weeks of implementing the cycle, focus on establishing an environment that supports mathematical thinking. We have found the teaching sequence depicted in **table 1** to be most effective. By taking the time to teach the NCTM Process Standards early in the school year, teachers provide students with the thinking tools necessary to understand the NCTM Content Standards.

The intellectual environment begins to take shape when you establish the cycle framework and students take on greater responsibility for their thinking. The most significant shift you will make is handing over the responsibility of thinking to your students. Working in different classrooms revealed to us that teachers do most of the thinking work in mathematics class. We recognized that we had done a good job of training students to receive knowledge and follow mathematical procedures. We had not done as thorough a job of training them to understand the thinking behind the mathematics they were doing. When we began to release the responsibility of thinking to students, it felt awkward both to us and to our students. Students may initially resist this change in expectations. Be patient. They will come around.

Developing each phase

Once the basic framework of the cycle is up and running in your classroom, you will shift your time and attention to developing each phase of the cycle.

Focused Instruction. When designing the first phase, you will want to consider several questions

Table 2

Key Questions for Each Workshop Cycle Phase

Workshop Cycle Phase	Questions to Consider
Focused Instruction	▪ Which Standard will drive today's focused instruction? ▪ What do I know about my students? • Where does this lesson fall on my Unit of Study map? ▪ What content do I need to teach to move my students forward? ▪ How can I use an NCTM Process Standard to deepen content knowledge? ▪ Do my questions allow students to create their own meaning? ▪ How deep is my own understanding of what I am about to teach?
Real-World Application	▪ How can I design authentic practice so it reflects the heart of the discipline? ▪ How can I make the practice relative and meaningful for my students? ▪ What are some shifts I could easily make from "school only" practice (worksheets, board work, etc.) to real-world application? ▪ What have I witnessed during other application sessions that I should consider as I plan for today's work session? • What do I want students to accomplish during today's real-world application? ▪ What misunderstandings do I anticipate? ▪ How will I keep the focus of the lesson embedded in the practice and frame my students' thinking?
Reflective Closure	▪ How will I bring the workshop to a close? ▪ How can I bring students back to the essential question and focus of the lesson? ▪ How will students connect *what they did* to *what it means*? ▪ What did I notice during real-world application that needs to be addressed? ▪ Who will share or celebrate during today's reflective closure? ▪ What misconceptions need to be addressed today? ▪ Have I moved the students toward a deeper understanding today? How can we highlight this during today's reflective closure? ▪ Where will I take my students tomorrow? ▪ What adjustments should I make to tomorrow's workshop cycle?

(see **table 2**), all of which help craft a meaningful lesson that harbors this intellectual environment. Begin by considering the NCTM Standards or the state standards to which you are held accountable. Aligning your instruction with established goals ensures that students are thinking about ideas that will lead them to more sophisticated mathematical understandings. In the earlier lesson describing Cassi's solution, the NCTM Representation Process Standard drives the focused instruction. Establishing a narrow focus gives us the necessary permission to stay on track. We want to avoid "confusing the help," as Arthur Hyde puts it in *Comprehending Math* (2006, p. ix). Our responsibility is to keep the students, and ourselves, focused on the thinking surrounding the lesson objective.

With a clear focus in mind, we can determine what to watch for during the lesson. Observe how students use both process and content to work toward mathematical understanding throughout the workshop. To guide your reactions throughout the workshop, carefully consider your knowledge of each student, the content continuum, and where this lesson falls on that continuum. During focused instruction, you know the specific moves you need to make but rely on the students' actions and reactions to tell you *when* these moves are necessary.

Using essential questions is another way to stay true to your lesson focus and ensure that students think about the lesson's most important features. Questions are essential to the big idea you are linking to and are worth revisiting over time as thinking and understanding develop. "These are the questions that are not answerable with finality in a brief sentence—and that is the point" (Wiggins and McTighe 2005, p. 106). A well-posed essential question guides students' thinking throughout the focused instruction phase and into the real-world application phase. The questions also make students aware of their responsibility to make sense of the mathematics.

Allowing students to grapple with essential questions gives us insight into their intellectual environment. In our opening vignette, Cassi was asked to consider these questions:

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- How can you show your thinking?
- How did you solve that?
- How do you know your answer is correct?

She showed us how she used different models to help her solve the problem; she could prove that her answer was correct. She revealed to us her intellectual environment, her *thinking* about the mathematics. In this way, she demonstrated mathematical understanding.

A well-developed and focused instruction phase opens the door to the intellectual environment for each Workshop Cycle. Your narrow focus, framed carefully by essential questions, invites contemplation and leads students naturally to explore their thinking in the real-world application phase.

Real-world Application. During the real-world application phase, students test their emerging ideas regarding the lesson focus. This is vital to the authenticity of their intellectual environment. The ideas they test must come from them, not from us. When planning the real-world application phase of the Cycle, first consider how the mathematical focus of your lesson connects to life outside school. Real-world application requires students to engage in the type of practice and mathematical thinking that would occur outside the school setting. In *Engaging Readers and Writers with Inquiry*, Wilhelm tells us that "cognitive science has demonstrated that one's purpose drives motivation and what one attends to, remembers, and then applies. Without purpose, significant learning is difficult if not impossible to achieve" (Wilhelm 2007, p. 8). A well-orchestrated, real-world application session allows students the freedom to work through the mathematics as individuals, to struggle to make sense of their ideas, to pursue the answers to their questions as well as ours, and to make meaningful connections.

To maintain students' focus on their intellectual environment, you must continually ask them to explain their thinking and engage in the same processes that professional mathematicians engage in. Mathematicians continually work through problems that they do not know the answers to—just as writers revisit certain pieces time and again, knowing that writing is never finished. Craftsmen in both disciplines use an appropriate process that allows them to focus on their content. Students, too, must rely on a process to help them solve problems throughout the real-world application phase; and they must be encouraged to explain their process to others. This gives way to deeper understanding of the content.

Many teachers, feeling the pressure to cover content and prepare students for high-stakes testing, focus their teaching on rote memorization of facts, procedures, and algorithms. Such shortcuts leave students with the impression that mathematics is about following rules and procedures and has little to do with thinking. This is not conducive to an intellectual environment that requires students to be responsible for their thinking. Allowing students to test their ideas and articulate their thinking holds them accountable to their job as learners.

Reflective Closure. This phase of the cycle spotlights the intellectual environment. Teachers facilitate as students make sense of what they did during previous phases of the Workshop Cycle. Although teachers and students continually reflect throughout the cycle, using the essential question to frame their thinking, here is where participants devote a specific amount of time and attention to bringing the day's workshop to a close. During this phase, teachers guide students back to the workshop's essential question and focus. The goal is for students to connect what they have learned during focused instruction and real-world application and to reflect on how this deepens their thinking about the essential question. Students have the opportunity to stretch their thinking beyond "what I did in math today." Now is the time to integrate what they did, what it connects to, and what it means to them. They put their thinking out in the open to be examined by themselves, their peers, and their teachers. Just as readers synthesize to create personal meaning for a text, mathematicians

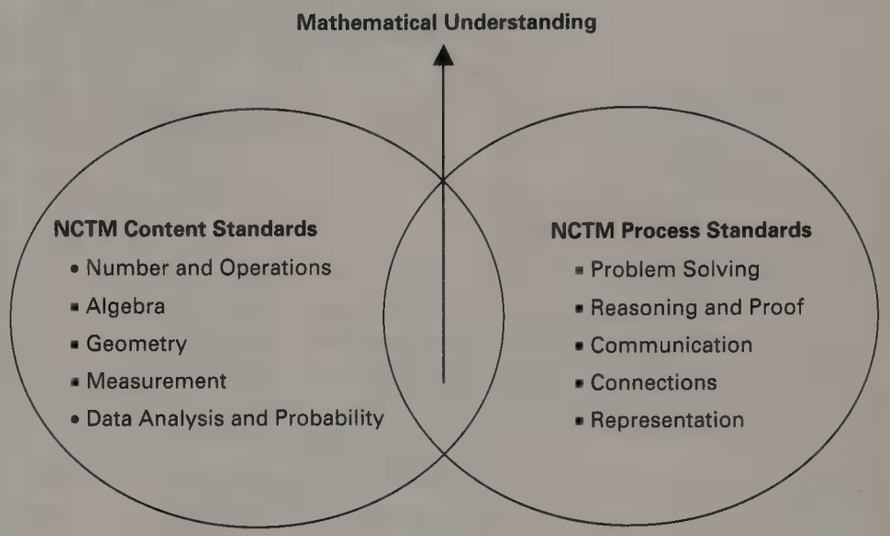
synthesize to create meaning for their work.

The teacher may also use reflective closure to set up future lessons or big ideas for students to consider over the next few days. This can be a time for celebrating mathematical understanding or progress toward an established goal. The teacher reflects on what she is learning about her students and the knowledge they are gaining, as well as what she is learning about the content or process that she is teaching.

During the reflective closure phase, the intellectual environment is laid out in the open for everyone to consider. Students come to anticipate that they will be asked to share more than just "the answer" or prescribed procedures to get an answer. They come to understand that mathematics is about our process of thinking about the numbers, too. They learn that their job is to understand.

Figure 3

True mathematical understanding lies at the intersection of the Content and Process Standards.



What We Learned

Intellectual environment thrives when it is fused with the Workshop Cycle. Each phase is integral for the thinking required in an intellectual environment. Throughout this process of reflection on the intellectual environment within the Workshop Cycle, we discovered that deep mathematical understanding exists at the intersection of the NCTM Content and Process Standards (see **fig. 3**). In the opening vignette, when Cassi showed the rest of the class how she solved her problem, we learned that she had a grasp of both process and content. Her dramatization revealed that she clearly understood the concept of subtraction. Cassi was able to use the hundreds chart as a model to subtract, and she was able to determine the difference between two numbers. Had she been asked to provide the answer only, we would not know for sure if she truly understood the problem or the concept.

Wiggins and McTighe's statement that "understanding is a mental construct, an abstraction made by the human mind to make sense of many distinct pieces of knowledge" (2005, p. 37) implies that we create understanding; it cannot be received. If this is the case—and we believe it is—then we must allow students to create their own meaning of mathematics. We strive to strategically design physical and intellectual space so that students create their own intellectual environment that will enable mathematical understanding. Cassi's representation of the mathematics proved that she had created an understanding.

As we clarified and applied our new understanding of the Workshop Cycle in classrooms, students reaped the benefits. By using the NCTM Process and Content Standards, our daily instruction has become more coherent. As a result of our shift in thinking and practice, student achievement levels skyrocketed far beyond what we could have imagined. Less reteaching is required from year to year because students actually remember what they learn. Students are more engaged in their learning and take ownership of it. They think at much higher levels and can articulate what they did, why they did it, what they learned, and where they might use it again. However, questions and comments such as "Is it math time yet?" and "Math is my favorite subject!" are what made us most proud.

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Eileen Fernández

Taking Advantage of Everyday Activities to Practice Math Every Day

When my son started kindergarten, I discovered that many everyday activities at home allow us to practice the mathematics he was learning (and would learn) in school and encourage mathematics use in his life. Because "children learn through exploring their world … everyday activities are natural vehicles for developing mathematical thinking" (NCTM 2000, p. 74).

The activities my son and I discovered are everyday in two senses: They are part of our *everyday* routines, and they can be done *every day*. The former ensures that the activities get done (we need to do them anyway!) and gives them meaning, while the latter provides opportunities for learning and extending ideas as we use them over and over. The following activities are built on school mathematics experiences and activities.

Kindergarten and pre-kindergarten teachers, whose goal is to encourage mathematical explorations and practice, can share these activities with parents, caregivers, and other adults in children's lives and encourage them to join together in support of a child's mathematical development by having conversations about the many facets of mathematics in our everyday lives.

Eileen Fernández teaches mathematics and mathematics education courses at New Jersey's Montclair State University. Her research interests include curriculum transformation and exploring students' innovative takes on mathematics teachers' knowledge.

Edited by **Andrew M. Tyminski**, *atyminski@purdue.edu, an assistant professor of mathematics education at Purdue University in West Layfayette, Indiana; and* **Signe E. Kastberg**, *skastber@iupui.edu, an assistant professor of mathematics education at Indiana University-Purdue University Indianapolis. "Early Childhood Corner" addresses the early childhood teacher's need to support young children's emerging mathematics understandings and skills in a context that conforms with current knowledge about the way that children in prekindergarten and kindergarten learn mathematics. Readers are encouraged to send submissions to this department by accessing* **tcm.msubmit.net**. *Manuscripts should be double-spaced and must not exceed 2000 words.*

Skip Counting

"Young children are motivated to count everything … and through their repeated experience with the counting process, they learn many fundamental number concepts" (NCTM 2000, p. 79). Kindergarten introduces skip counting by twos and fives—an activity my son found more difficult than his preschool counting by ones and tens.

Counting is about more than memorization. For example, skip counting can provide opportunities to observe and predict patterns that may not be seen if counting is treated as a rote activity. To keep the meaning in skip counting by twos, take advantage of nightly reading routines and page numbers in certain children's books (such as *Curious George* and *Hop on Pop*). When you are finished reading the text (or before you read), use either left- or right-positioned page numbers in your book to practice counting by twos. As you count, ask your child what patterns she notices in the page numbers. Observing the recurring patterns of the numbers, specifically, the repeating pattern of 0, 2, 4, 6, and 8 in the units' digits, helps children become more curious about numbers. Visual reinforcement of number patterns is a powerful representation. If a page number is omitted, encourage children to guess the missing number.

Another way to keep the meaning in skip counting by twos is through counting aloud by ones while alternating saying the odd numbers quietly and the even numbers loudly. That is, count, "One (softly), **two** (loudly), three (softly), **four** (loudly)," and so on. For practice counting by fives, try using fingers and toes or making groups of five pennies. Again, ask children to describe patterns they see in the numbers. This includes the alternating 0, 5 pattern in the units digit and the pattern within the tens digit (1, 1, 2, 2, 3, 3, …). Use index cards and markers or a word processor's color feature to help children describe and extend the noted patterns (5, 10, 15, 20, 25, 30, 35, 40, …).

In counting-by-twos activities, counting by ones serves as a backdrop (either visually or verbally), thereby giving meaning to the relationship between counting by ones and twos and demonstrating how one is "embedded" in the other. Extending patterns gives children practice making predictions and using inductive reasoning, both important aspects of mathematical reasoning used in school and in life. If your child asks (as mine did), "Why do I need to count by twos and fives?" point out activities in which counting by twos and fives makes life easier, including counting change with nickels, counting five-minute intervals on a clock, counting headlights on cars, or counting people's eyes. And (as with all the activities suggested here), take advantage of your child's thinking and where it might lead. Interestingly, in first grade, my son observed that counting by twos and fives was, respectively, like "adding twos and fives." His observation enabled me to use this important connection to help him evaluate certain sums.

Monthly Calendars

On the surface, a monthly calendar may seem to be only a simple counting tool. Calendars, however, provide enriched opportunities to apply ideas about how numbers are positioned in relation to one another. Also, because a calendar contains rows and columns, children practice locating positions in two dimensions. Calendars also introduce an opportunity to think of numbers as more than counting tools because dates can be associated with special events in children's lives. A calendar helps us measure real time as days approach (in anticipation of an event) and days gone by (when an event has passed).

Daily opportunities exist for children to practice the ideas above. To make the calendar a central object, keep a school lunch schedule in your kitchen, your child's bedroom, or some other place where you both have the opportunity to look at it each day. Make a point of crossing out days as they go by. Create opportunities to read dates and locate positions; make sure to practice using days and numbers. Label special days with stickers or words and have your child count the days until events, such as the weekend, a party, a birthday, or a vacation. Ask your child questions about the calendar: "How far apart are the first day and last day of the month? How many days are between Mondays?" Then use the dates (on the first and last days or on the Mondays) to probe how far apart the numbers corresponding to these dates are. Additional questions can include, "How many days until the party? Is that a long time from now? How many days have passed since the weekend? Has it been a short time since (some event) happened?" Follow up on all the questions to see if the length of time "felt" short or long to your child.

Children may experience some confusion when counting on a calendar. For example, when they are counting from Wednesday the ninth to Friday the eighteenth, children may experience confusion in assigning the counting numbers, "1, 2, 3, 4, 5, 6, ..." to the dates "9, 10, 11, 12, 13, 14, ..." One way to avert this confusion is to ask your child to circle the dates and focus on counting the circles. Shifting between rows also presents challenges as children move forward and backward depending on the question they are investigating—for example, when you ask, "How many days ago was (a certain event)?" If counting backward entails moving from the present row to a preceding one, this will involve moving left (to the beginning of the present row) and then right and up (to the end of the preceding row). This movement may seem unnatural because it contrasts with the reading motion. Be open to your child's strategies and personal development in addressing struggles. My son answered questions about past events by moving forward from the event date to the present date instead of backward from the present date to the past one.

The Weather

Weather is a big part of children's everyday lives and provides many opportunities for adults to help students develop number sense. One way to engage in a discussion of relationships between numbers and weather is by watching a weather report with your child every day. The broad spectrum of information given in weather reports—for instance, temperature, rainfall, and wind gusts—and the variation in the units—degrees, inches, and miles per hour—presents another ideal setting for assigning meaning to numbers and introducing more sophisticated ideas than those associated with the calendar. Moreover, the day-to-day numerical observations do not always change sequentially, and negative numbers can arise.

As you talk about air temperature with your child every day, be sure to include the concept of degrees. It was my son's idea to go outside to feel the ambient temperature, thereby providing a direct experience with this measure (see **fig. 1**).

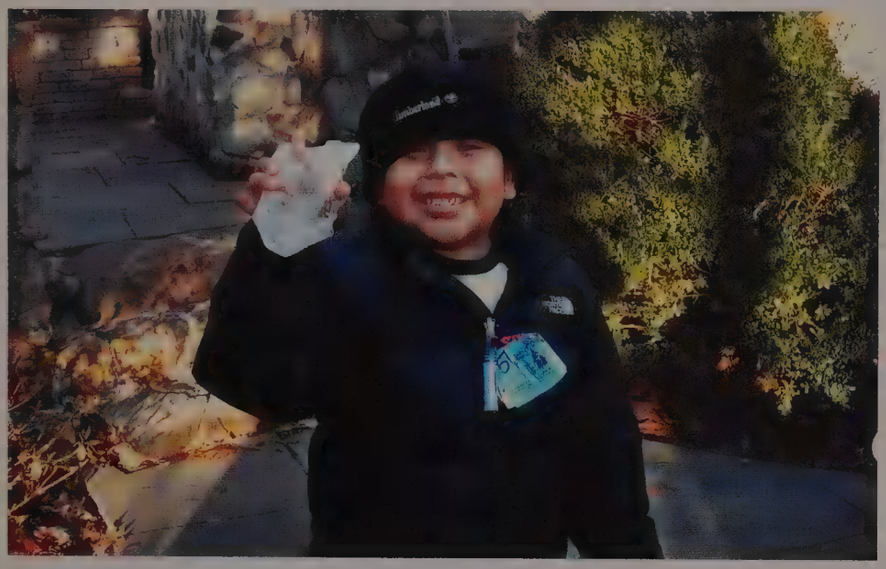
Weather is also a good context to explore how units affect the meanings we assign to numbers. For example, discuss with your child how fifty *degrees* feels mild or cool—so the number of fifty units does not seem like very much—compared to wind gusts that are fifty miles per hour—strong winds that make fifty units seem like a large number. Also consider number ranges: Storms that produce five to ten inches of rain at one time are considered severe, whereas winds that blow at five to ten miles per hour are calm, not severe.

Explore percentages and probabilities in conjunction with likelihoods. For example, consider the ideas that 100 percent chance (the "greatest" chance) means something will happen and that 0 percent chance (the "smallest" chance) means something will not happen. When children use percentages, they often ask questions that are fun for them to think about: "How sure are you that it will snow today?" on a 70-degree day or "What chance of rain do we have?" when it is already raining. (My son adopted the language of *percent* in response to its use on weather channels.) Let your child get comfortable with the limits before pursuing numbers inside the range.

Another fun activity is to consider how temperatures feel in relation to one another. That is, 80 degrees can feel warm after a 60-degree trend, but when the temperature drops to 80 degrees after a stretch of near-100-degree weather, 80 degrees can feel cool. All these activities strengthen your child's abilities to see numbers as more than counting devices, both inside and outside the classroom.

Figure 1

Going outside to feel the temperature (18 degrees) and finding ice



Photograph by Eileen Fernández; all rights reserved

Figure 2

Counting out $2.25 for lunch money

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Lunch Money

Children can have everyday interactions with money to support their mathematical development in and out of school. When they work with coins, the activity of counting includes considering units such as cents or dollars, denominations such as nickels and dimes, and opportunities for skip counting by fives and tens. In fact, skip counting becomes more sophisticated as children may be compelled to transition between counting by tens to counting by fives. My son uses his five fingers to help him when these transitions arise. Reasoning opportunities related to classifying and ordering occur as similar coins are grouped together and less valuable coins are ordered in relation to more valuable ones. Again, numbers take on meanings when you ask your child such questions as "[When lunch costs $2.25] can you buy your lunch if you only have $2?" Finally, considering conversions between different denominations introduces children to the important concept of equivalence in mathematics.

To provide opportunities in support of a child's money experiences in school, keep a container of pennies, nickels, dimes, quarters, and single dollars in an accessible place, such as on a kitchen counter. At home, my son and I gather $2.25 for lunch money nearly every day (see **fig. 2**). I have found that counting out lunch money is great way

for my son to learn the names of coins and explore relationships between different denominations. I try to vary the coin combinations we use on a daily basis to help him develop flexibility in recognizing and counting coins. One day, we might create $2.25 using all quarters; another day, we might have only four quarters to work with, thereby requiring the use of different coins. I sometimes hide certain coins to compel variety in the collections we create.

When you count coins with your child, anticipate questions about coin size in relation to their value. For example, "Why is a nickel worth less than a dime if the nickel is bigger than the dime?" Discuss how different numbers of things—for example, ten pennies and one dime—can be "the same" and "different." In this case, although ten pennies and one dime have the same value, the penny collection has more coins. Again, it is important to encourage children to create their own interpretations of the activities. My son and I once equated one hundred pennies to one dollar, introducing a meaningful discussion of the pros and cons of using coins and bills. For my six-year old, the pennies were "better" because you could "see how many you really have."

Concluding Remarks

Consider how your everyday activities can provide opportunities to practice mathematics every day. As you work with your child, keep in mind that "what some students learn in one year may take two or more years for others" (NCTM 2000, p. 84). On some days, my son counted out twenty-five cents as two dimes and one nickel, transitioning beautifully from counting by tens to counting by fives. On other days, this same challenge would stump him. Follow your child's lead, and you will be learning right alongside him. Respect for your child's individual mathematical development will translate into a love of learning mathematics and a natural appreciation for its role in our everyday lives.

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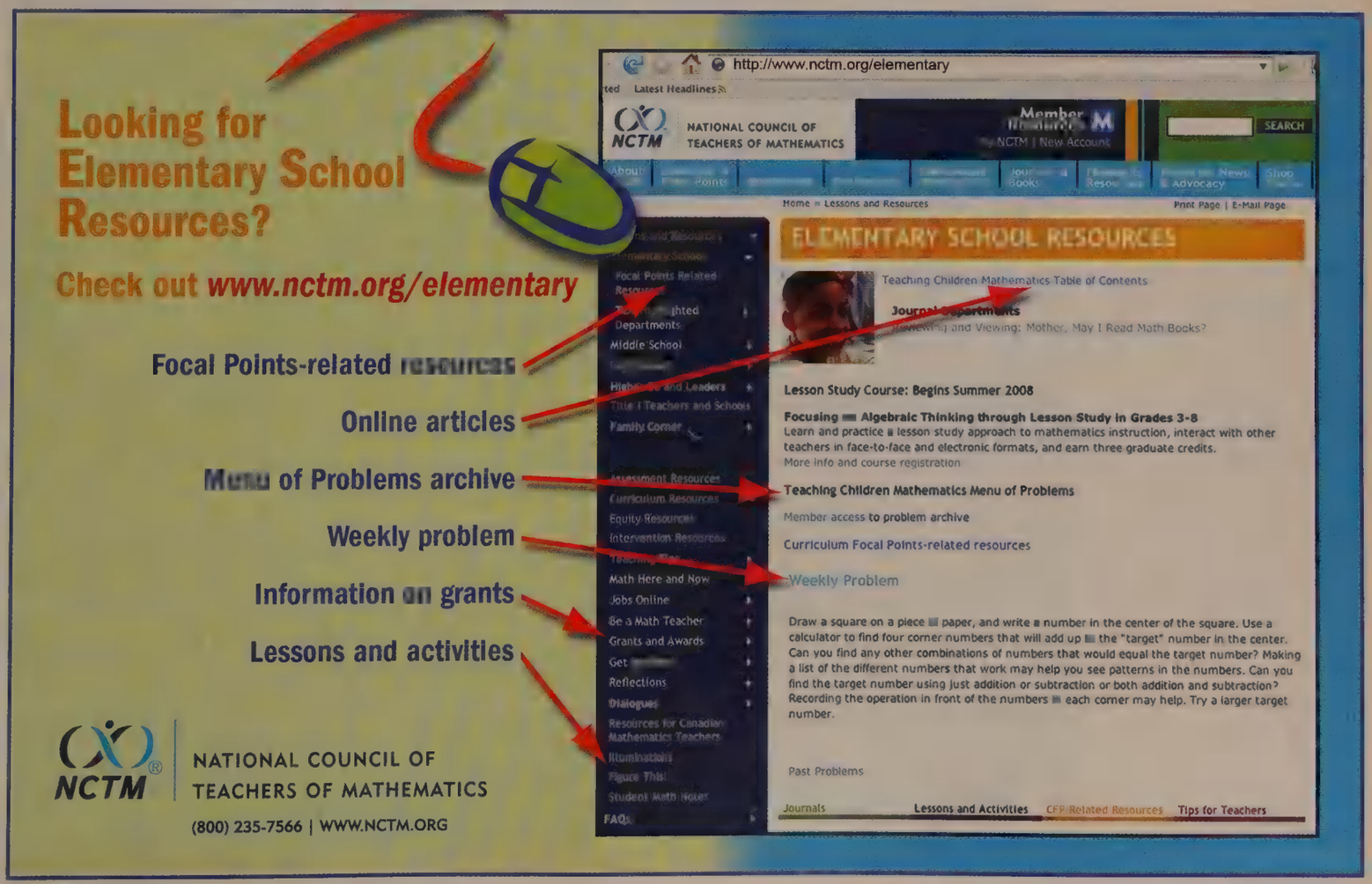
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Draw a square on a piece ▪ paper, and write ▪ number in the center of the square. Use a calculator to find four corner numbers that will add up ▪ the "target" number in the center. Can you find any other combinations of numbers that would equal the target number? Making a list of the different numbers that work may help you see patterns in the numbers. Can you find the target number using just addition or subtraction or both addition and subtraction? Recording the operation in front of the numbers ▪ each corner may help. Try a larger target number.
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The AquaDom—One Large Aquarium

Problem

This month's Problem Solvers road trip takes us to the AquaDom, the world's largest cylinder-shaped aquarium, located in the Radisson SAS Hotel in Berlin, Germany. Imagine being surrounded by the aquarium's inner walls as the enclosed elevator rises from the base to the top. The outer cylinder is 16 meters high, and the water level in the aquarium is about 14 meters high. The diameter is approximately 11 meters.

The hotel gift shop sells a scale model of the AquaDom, a cylinder 14 centimeters tall and 11 centimeters in diameter. Construct a paper replica of the scale model and explain how you built it. For this problem, disregard the elevator shaft; include only the aquarium's outer walls in your model.

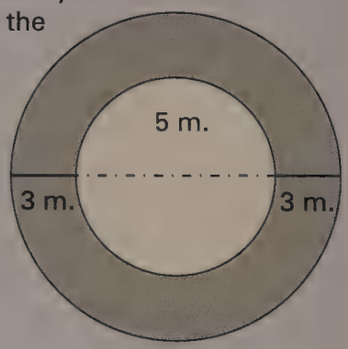
Would the model have the same volume if it were 11 centimeters tall with a diameter of 14 centimeters? Construct another model using the new dimensions. Describe how to determine the volume of each cylinder. Do the calculations and compare the results. Are they the same? If not, which cylinder has the greater volume? How do you know?

Variations

What different-sized aquarium models could be constructed with the same volume as the original gift shop model and a diameter that is larger than the height? Is there a pattern to your answers?

How does the volume of your original model (14 centimeters high and 11 centimeters in diameter) compare to the volume of the actual AquaDom (14 meters high and approximately 11 meters in diameter, including the aquarium and elevator shaft)? How do you know this? (Note: In this problem, we do not consider the volume of the internal elevator shaft.)

Students in upper grades can compare the AquaDom's construction to a roll of paper towels, where the paper represents the aquarium's water and the space inside the roll's core represents the elevator shaft. The AquaDom's elevator shaft has a diameter of 5 meters, and the base of the cylinder looks like the figure to the right. Use this information to determine the volume of the elevator shaft and then calculate the volume of the actual water in the aquarium. Remember that the height of both cylinders is 14 meters.



Edited by **Sarah Slater**, *sslater@gmail.chelsea.k12.mi.us, a third-grade teacher at Pierce Lake Elementary School in Chelsea, Michigan;* **Joseph Georgeson**, *jgeorgeson@usmk12.org, middle school mathematics department chair and teacher of eighth graders at the University School of Milwaukee in Wisconsin; and* **Brian Schad**, *schad@aaps.k12.mi.us, a fifth-grade teacher at Lawton Elementary School in Ann Arbor, Michigan.* *Each month this section of the department features a new problem for students. Readers are encouraged to submit problems to the editors to be considered for future "Problem Solvers" columns. Receipt of problems will not be acknowledged; however, problems selected for publication will be credited to the author.*

AquaDom Facts

- Elevation is 30 meters above the earth and is supported by pile construction 30 meters below ground
- Made of acrylic that varies in thickness throughout the structure: 22 centimeters on the bottom, 16 centimeters in the middle, and 8 centimeters on the top
- Height is 16 meters
- Water level is about 14 meters
- Diameter of entire cylinder is 11.56 meters
- Diameter of elevator shaft within the cylinder is approximately 5 meters
- Contains about one million liters of water

The goal of the "Problem Solvers" department is to foster improved communication among teachers by posing one problem each month for teachers of grades K–6 to try with their students. Every teacher can become an author: Pose the problem to your students, reflect on your students' work, analyze the classroom dialogue, and submit your resulting insights to this department. Through contributions to the journal, every teacher can help us all better understand children's capabilities and thinking about mathematics. Remember that even students' misconceptions provide valuable information.

Classroom Setup

Begin this activity by showing students a picture of the AquaDom (en.wikipedia.org/wiki/AquaDom) and discussing the AquaDom Facts from the problem. Explain that today they are going to construct a scale model of this unusual aquarium. Their models will have a height of 14 centimeters and a diameter of 11 centimeters.

Have construction paper and masking tape available for student use. As students complete this beginning task, notice the different strategies that they use to determine how to create a cylinder with a diameter of 11 centimeters from a flat sheet of paper. Allow students to develop their own strategies. Take pictures of them creating their models and of the finished products.

Also allow students to develop their own methods to determine the cylinders' volume. Do not suggest your own solution ideas (such as the use of pi). To determine the volume of the cylinder without using formulas, students may fill the container with sand, rice, cereal, or a similar substance. Containers labeled with liter and milliliter markings will help students complete this task. As students determine volume, observe and record their strategies and conversations and save student work samples.

For lower elementary grades, introduce the activity in the same manner and let students begin to problem solve. You may need to give guidance if students seem to be struggling. Brief introductory questions may help them begin the problem-solving process:

1. What shape is the bottom of a cylinder?
2. How big should it be? (Distance across is called *diameter*.)
3. How could we make a shape like that?

Lars Ziemann/theiStockphoto.com

Students then construct a second cylinder with a height of 11 centimeters and a diameter of 14 centimeters, repeating the process described above. Have them compare the results of the first cylinder to the second cylinder. They can physically pour the contents (like rice) from one container to the other, or they can compare calculated or measured results. When they determine and compare volumes of various cylinders, encourage your students to organize their information (diameter, height, volume) in a table to better display and analyze the results. Ideally, teachers would like to see which strategies students use to build their cylinders and then how they determine and compare the volumes. Ask the children to describe what they did to determine the volumes and to explain how they constructed and compared their models.

Is it possible to construct another cylinder with exactly the same volume as the original cylinder model but with a different height and diameter? This task could be completed by using the problem-solving strategy of guess and check. Would it also be possible to determine the size of this cylinder by using another problem-solving strategy, such as looking for patterns, making a table, or making a graph?

Remember that the problem-solving process for finding the answer is sometimes more important than the answer itself. Give students a later opportunity to review their problem-solving strategies. As they do, keep their focus on the different approaches they used rather than on their answers. Some students may be surprised to discover that the side of a cylinder is actually a rectangle. This revelation could lead to interesting discussions about two- and three- dimensional shapes.

Share Your Students' Work

Please try this problem in your classroom. We are interested in how your students responded to the problem, what problem-solving strategies they used, and how they explained or justified their reasoning. Include information about how you posed the problem, samples of student work, and photographs. E-mail your thoughts, reflections, scanned student work, and photographs to **Sarah Slater** at sslater@gmail.chelsea.k12.mi.us by **December 1, 2008**, or send them to her attention at Pierce Lake Elementary School, 275 N. Freer Rd., Chelsea, MI 48118. Include your name, grade level, and school name with your submission. Selected submissions will be published and acknowledged in a subsequent issue of *Teaching Children Mathematics*.

The editors wish to thank Susann Rataj, Sales and Marketing Coordinator at the Radisson SAS Hotel Berlin in Germany, for providing the AquaDom facts used in this problem.

(Solutions to a previous problem begin on the next page.)

Additional Resources

Did you know that NCTM has published a collection of some past "Problem Solvers" columns? Visit **nctm.org/catalog** for more on this and other NCTM resources, including professional development offerings, other publications, and online resources.

Sakshaug, Lynae E., Melfried Olson, and Judith Olson. *Children Are Mathematical Problem Solvers*. Reston, VA: National Council of Teachers of Mathematics, 2002. ▲

Where's the Math?

This month's problem is mathematically rich and can be approached on many levels. Topics include measurement, volume, circles, ratios, three-dimensional shapes, building models, and the number pi. Students develop an understanding of the concept of volume, how it changes, and how it is affected by the height and radius of a cylinder. Using formulas is not necessary, but the problem can lead to an understanding of the relationships among radius, diameter, area, circumference, and the number pi in circles.

Volume is a difficult concept for many students to grasp. In this problem, volume can be thought of as filling a three-dimensional space. What fills the space is less important than how the amount of fill changes as the dimensions of the space change. For example, when a sheet of paper is rolled to form a cylinder, is the volume the same regardless of which direction (horizontally or vertically) the paper is rolled? Assuming we use an 8.5-by-11-inch sheet of paper, which of the following has more volume: when the height is 8.5 inches or when the height is 11 inches? Keeping either the length or width the same could lead to patterns on the basis of which students could conjecture to answer the question and predict other volumes.

Ratios are also important in this problem. For example, how does the volume change when the scale model is one-tenth as large as the real cylinder? What happens when the length is twice the diameter? Students can make models to investigate these and other questions. Gathering data is one way to analyze these relationships; graphing and using tables are other options. The mathematical idea of change is intrinsic to this problem. Change is part of many problems in mathematics, and analyzing how changes in one dimension affect another is an important consideration.

Understanding volume is a more essential focus in this problem than focusing on any formal way to find that volume. Students must understand what volume represents before using formulas to find it. When they eventually learn a formula and use it, the results will make sense. Using this cylinder, I wonder how many scoops of water it would take to fill the average swimming pool. I guess that is a question for another month.

Solutions to the Height in Coins Problem

The Height in Coins problem involves concepts of measurement and number and operations, requiring students to apply problem-solving strategies and use mathematical reasoning. The original problem, in the October 2007 "Problem Solvers" section, was stated as follows:

Problem

Lorenzo and Maya are trying to solve a problem involving money and height. They are trying to decide which would be greater—the value of quarters arranged on a flat surface in rows or columns to represent their height (height value) or the value in nickels stacked vertically (on top of one another) to represent their height (height value).

Lorenzo: I think my height value in quarters would be greater because quarters are worth more than nickels.

Maya: I think my height value in nickels might be greater because the quarters are placed side by side and the nickels are stacked. Hmm, I wonder how we can figure this out?

Work with your partner or group to investigate the following questions. Be sure to record your strategies and findings.

1. Which would be greater—your height value in quarters when arranged on a flat surface in rows or columns or your height value in nickels when stacked?
2. What is your height value when using quarters? When using nickels?

Variation

For younger students, recast the problem in nickels and pennies. Have the students represent their height by arranging the nickels side by side and stacking the pennies and then determine their height value.

Extensions

Ask students to figure out who is taller—Maxine, whose height value in quarters is $12.75, or Daniel, whose height value in nickels is $25.40. Some students might begin to explore a scale factor that relates nickels to quarters with respect to height.

For an extension into data analysis, have students calculate the average height of their classmates and then investigate the average height value by using quarters and nickels.

Edited by **Mark Ellis**, *mellis@fullerton.edu, an assistant professor in the College of Education at California State University–Fullerton, where he prepares teachers of middle school mathematics and works with local classroom teachers in creating learning environments that support sense-making in mathematics; and* **Cathery Yeh**, *catyeh@aol.com, who teaches third grade at Arnold Elementary School in Cypress, California. Each month this section of the "Problem Solvers" department discusses the classroom results of using problems presented in previous issues of* Teaching Children Mathematics.

Mathematics educators Sueanne McKinney and Charlene Fleener from Old Dominion University in Virginia share what they observed when Anita DeSalvo's second graders at Lindenwood Elementary in Norfolk, Virginia, tackled the Height in Coins problem. DeSalvo began by having students work on a simpler problem:

> Supplied with only adding machine paper and coins, students were first challenged to determine

the "value" of the distance around their wrists in nickels and quarters. With little prompting, the students decided to first wrap adding machine paper around their wrists and then proceeded to "measure" their tape [with coins] by laying it flat and extending the quarters side by side on top of the paper. These second graders then realized that they would need to hold the tape vertically to determine its value when stacking nickels.... Students were given enough quarters and nickels to determine their wrists' value. Doing so allowed the students to gain experience with using coins as a nonstandard unit of measure, as well as practice with using their computational skills to determine total values when using different coin denominations.

The next day, DeSalvo asked students to think about how they could use their experience measuring their wrists to determine a classmate's height in coins. Several students offered ideas:

Divine: Maybe we could do it if he would lie down.
Shania: It would be easier with the quarters.
DeSalvo: Why do you think it would be easier?
Glalib: It's easier to place the quarters next to each other than stack the nickels. We would need a lot of nickels.
Mikey: If we stack all the nickels, they would fall down.

When it was clear that these young students were able to grasp the idea of measuring height with quarters and nickels, they were put into groups and asked to find the height of one student in quarters and in nickels. McKinney and Fleener describe what occurred as the groups began to work on this problem and encountered a perplexing issue:

> Each group worked on determining the height of an identified student, and began by having him or her lie down, while extending the [adding machine] tape from head to toe. As students proceeded to place the quarters side by side on top of the tape, many were a bit puzzled as to what to do once the quarters ran out.

Questions from their teacher, such as "How can you accurately estimate how many more coins are needed?" and "Can we do anything else with the paper we don't have enough quarters for?" led students to consider how to solve their dilemma of insufficient coins to cover the total height.

Such scaffolding helped students realize that they could fold the adding machine paper in sections equal to an already-determined number of coins and then use repeated addition to discover how many coins they would need altogether. McKinney and Fleener further explain:

> One group folded their adding machine paper over and over, representing the initial stack of nickels' height. They then counted the nickels in one stack (39), and continued to place the number 39 in each space designated by the folds. When asked what the 39 represented, Michael stated, "The 39 nickels." When they finished marking the paper, Michael then led his group to realize that the nine markings of 39 needed to be added. [To figure this out,] Shania added three groups of 10 at a time and counted by 30, which gave her 270. Then she persisted to add nine 9s, giving her 81. Adding those totals gave her 351.

Students were building the conceptual groundwork for later understanding of the concept of multiplication. This exercise also provided them with an excellent opportunity to strengthen their number sense. Shania's method of decomposing 39 into 30 and 9 and taking nine groups of each is a powerful example of repeated subtraction or "undoing" multiplication.

Once students had determined the number of quarters or nickels equal to the selected student's height, they were challenged to determine the value. With quarters, they knew that four coins make one dollar. They took the number of quarters in the student's height (52) and found that they could make thirteen groups of four—a value of thirteen dollars.

When it came to calculating the value of 351 nickels, students were asked by Diana Batliner, the school's mathematics specialist, who had come to help with the lesson, to represent 351 with base-ten blocks, a manipulative familiar to them. The interaction below shows how two second graders, Shania and Michael, wrestled with important mathematical ideas of number.

Batliner: Let's look at your base-ten blocks. Let's pretend that each unit piece represents one nickel. What would each tens stick be worth?
Shania [*after counting by fives*]: Fifty cents.
Batliner: Can we trade the hundreds flats into tens sticks and then figure out what the total value of the nickels is?

Figure 1

Students placed strips next to a classmate and realized he is exactly five strips long.

Photograph by Janice Wright; all rights reserved

Figure 2

Five strips equal fifty-five quarters.

Photograph by Janice Wright; all rights reserved

At this point, Shania and Michael traded the three hundreds flats for thirty tens sticks and showed a total of thirty-five tens sticks and one units piece.

Batliner: If we know [that] one tens stick is worth fifty cents, what does that tell us about the value of two tens sticks?
Shania and Michael: They would equal one dollar.

With Batliner's assistance, Shania and Michael grouped the thirty-five tens sticks by twos and then counted each group as one dollar. They announced that the thirty-five tens sticks would be worth seventeen dollars, with one tens stick and one units piece left over. Batliner asked the students the value of those two extra pieces.

Michael: Fifty-five cents. One stick is fifty cents and this [unit piece] is a nickel, so that's fifty-five cents.
Batliner: So, what is the total value?
Michael: Seventeen dollars and fifty-five cents.

This exchange shows evidence of students developing insights into the ideas of multiplication and division, something that will certainly help them in their future learning of mathematics.

Janice Wright had her third-grade students at A. E. Arnold Elementary in Cypress, California, explore the coin problem. As an introduction, she asked them to measure the length of an eraser with quarters and with nickels. When her students found that they did not have enough nickels, Wright reports, "one student suggested we measure half way, count the nickels, and double the answer." The students found that it took thirty-six nickels to go halfway and seventy-two for the entire length of the eraser. When figuring out the value of seventy-two nickels, her students had further insights:

> One student suggested multiplying by five; another suggested we could count by fives. A third student said that instead of finding the value of seventy-two nickels, we could just multiply thirty-six by ten. [This student] was able to explain to everyone's satisfaction why this would give the same result as seventy-two times five.

The students' insights in solving the problem in various forms display their conceptual understanding of multiplication. Because their responses to this initial question showed they were ready to start on the problem, students were put into cooperative groups of two or three and given nickels, quarters, and paper strips of a fixed length (similar to receiving a ruler with no markings). As the groups proceeded to solve the problem, they noticed that they

did not have enough coins to measure even half of one strip. They brainstormed to find different ways to work around this complication.

One group first determined the "worth" of each strip in quarters and realized that it was eleven and one-half quarters and that it would be very hard to calculate that half quarter. So, they cut down the strip to exactly eleven quarters in length and cut six more strips to match. In effect, they created their own standard unit of measure. They then laid the strips next to the student being measured and realized that he was exactly five strips long, which equaled fifty-five quarters (**see figs. 1** and **2**).

A second group of students noticed that one quarter was equal to one inch. Yet, when asked how they could use this information to solve the problem, they were unsure. With Wright's guided questioning, this group was able to see that because one quarter is about equal to one inch, a person fifty-two inches tall is also fifty-two quarters tall. Informally, these students were learning ideas of substitution and equivalence.

A third group realized they did not have enough nickels and used trial and error to fold one strip into smaller, equal-sized units. They measured the smaller unit with the stacked nickels and multiplied by the number of units in the entire paper strip. Each one of their units was equal to twenty nickels, which they knew had a value of one dollar because, as they stated to Wright, ten dimes was equal to one dollar. Because each paper strip was made of eight such units, the value of each strip was eight dollars. They measured all the group members with the eight-dollar "nickel strips" and, afterward, used a similar method to measure all the students with quarters.

Finally, Marjorie Ellis, a mathematics coach at P. S. 3 Elementary School in Brooklyn, New York, had teachers from across grade levels tackle the Height in Coins problem with modifications to match the conceptual needs of their students. Students from prekindergarten to fifth grade used coins as a nonstandard form of measurement. For instance, Ms. Forde's prekindergartners used pennies to measure not only their height but also the length of their hand and foot (see **figs. 3** and **4**).

The responses above show clearly that this problem got students involved in thinking about concepts of measurement and numbers and operations as well as processes of problem solving and mathematical reasoning. Many students started with the problem-solving strategy of taking a larger problem and making it simpler, evidenced by their finding the lengths of various smaller objects in nickels and quarters before finding their heights. Additionally, students developed a stronger understanding of the concept of measurement by working on this problem, explicitly by working with quantifying comparisons of a specific unit (in this case, nickels and quarters) with a given object. Giving students opportunities to work with simple, informal units for measurement, such as coins, helps them develop an initial understanding of the concept of length because it is easier for them to count iterations of a physical object than of a more abstract unit such as inches or centimeters. Judging from the range of student work submitted, with slight modifications, this problem offers meaningful learning for younger and older students alike. ▲

Figure 3

Prekindergartners use pennies to measure their height, hand length, and foot length.

Photograph by Janice Wright; all rights reserved

Figure 4

John's foot is eleven pennies long.

Photograph by Janice Wright; all rights reserved

Equity for English Language Learners in Mathematics Classrooms

By Gina M. Borgioli

Gina M. Borgioli teaches elementary mathematics methods courses at Indiana University–Purdue University Indianapolis. She is interested in educational equity issues and enjoys working with both pre- and in-service mathematics teachers.

Although English Language Learners (ELLs) quickly acquire basic interpersonal communication skills, most struggle for several years with reading and writing academic content in English (Cummins 1981). In particular, in English-only mathematics classes, children are likely to have difficulty reading and comprehending text,

reading word problems, and giving written or oral justifications of their problem-solving strategies in English (Clarkson and Galbrath 1992; Cuevas 1984; Cummins 1984; Moschkovich 2000; Ron 1999). A language barrier should not be mistaken for a learning problem or a deficit. English-speaking peers of ELLs should also avoid assigning them lower social and academic status because of nonnative language skills and perceived inability to perform well on social and academic tasks (Cohen et al. 2004).

Figure 1

Linking Theory to Practice—Tasks

Are the tasks linked to the students' backgrounds and experiences?
Are the tasks problematic, investigative in nature, and of high-level cognitive demand?
Do the tasks have multiple entry points and more than one solution, and do they validate multiple intelligences?
Are the tasks rich in their opportunities for mathematics as well as language development? How will these opportunities be explored?
Do the tasks require students to make their own sense of the situation or apply only externally prescribed procedures?

Figure 2

Linking Theory to Practice—Tools

Are students able to choose the tools or representations?
Are there opportunities for students to use any tool and language mode available to explain their thinking to others?
Is there an effort to bridge students' representations with conventional notation and vocabulary?

Figure 3

Linking Theory to Practice—Classroom Norms

Are students making and justifying mathematical conjectures?
Are class members negotiating mathematical meaning by listening to one another's thinking?
Is there a respectful atmosphere with a low affective filter?
Is collaboration emphasized over competition?
Is there broad student participation?
Is there a conscious effort to equalize participation within groups?
Is the communication of ideas in any language mode supported and valued?
Is a visual trail of important concepts created? Is it permanent?
Does the lesson include language objectives as well as mathematics objectives?

Teaching and Learning Mathematics with Understanding

In classrooms that promote teaching and learning mathematics with understanding (instead of memorizing abstract algorithms and mimicking teachers' solution paths), students collaboratively engage in making sense of mathematical tasks in their own ways and communicate their thinking with one another (NCTM 2000). Carpenter and Lehrer describe three critical instructional dimensions in such classrooms:

- *Tasks* are complex and problematic for the learner and involve meaningful problem solving as opposed to simple drill-and-practice exercises.
- *Tools* used to represent mathematical ideas and problem situations are meaningful to students, and teachers encourage students to reflect explicitly on the characteristics of these representations.
- *Classroom norms* encourage students to engage in discourse, justify their mathematical conjectures, and share their different ways of viewing important mathematical ideas (1999, pp. 26–27).

Secada and Berman (1999) noted that students' social and demographic backgrounds often position them to profit more or less from such mathematics instruction. With regard to diverse learners, such as ELLs, in mathematics classrooms devoted to teaching mathematics with understanding, these authors posed the question, "Does an emphasis on developing student understanding result in greater or less fairness?" (p. 23). They claimed an equity perspective could "alert educators to how this [teaching] innovation could exacerbate socio-demographic-group-based achievement differences" (p. 34). I contend that greater fairness or equity is possible, but *only if* the teacher *purposefully* attends to it as a goal. Without that explicit attention, fairness and equity are elusive. In fact, even a lesson well grounded in conceptual understanding can be virtually incomprehensible (Krashen 1985) to those for whom English is not the native language. By consciously pursuing equity as an objective, mathematics teachers can create nurturing environments in which ELLs can be successful at both learning mathematics with understanding *and* at becoming more proficient in English without sacrificing their home language or culture.

This article provides mathematics teachers with a framework of guiding questions (see **figs. 1**, **2**, and **3**) to enable purposeful practice toward class-

room equity, especially for ELLs. The questions are grounded in the mathematics and second language education literatures. They supplement Carpenter and Lehrer's (1999) three critical dimensions of mathematics instruction that promote understanding (tasks, tools, and classroom norms) with practical suggestions for attending to issues of equity for ELLs.

Tasks

Mathematics educators differentiate between tasks of lower- and higher-level cognitive demand. In low-demand tasks, students passively, repeatedly "perform a memorized procedure in a routine manner" (Stein and Smith 1998, p. 169), whereas higher-demand tasks ask students to "actively explore relationships, make connections, and attach meaning to their work by referring to various ways of representation" (p. 170). To create higher-level cognitive demand tasks that are comprehensible to ELLs (Krashen 1985), many researchers urge teachers to bridge students' backgrounds and prior knowledge to the new learning rather than by diluting the curriculum to make it understandable (De Jong and Harper 2005; Gutstein et al. 1997; Khisty 1997).

To create problematic and meaningful tasks that acknowledge and validate students' experiences (González et al. 2001), teachers must get to know their students' backgrounds and current or past life experiences. Through consistent questioning and close listening, teachers attending to equity can informally assess students' conceptual development and then create open-ended tasks with the potential of challenging students' current thinking and encouraging knowledge construction (Fosnot and Dolk 2001; Steffe, Cobb, and von Glasersfeld 1988). Such tasks have multiple entry points and solution paths, often have more than one correct answer, and require a wide range of intelligences (Gardner 1982), including visual and spatial acuity, nonverbal representational skills, logical reasoning, and written and oral communication skills (Secada, Fennema, and Adajian 1995). These tasks should involve students in problem-solving situations that are potentially rich in learning opportunities not only in mathematics but also in language (Garrison and Mora 1999).

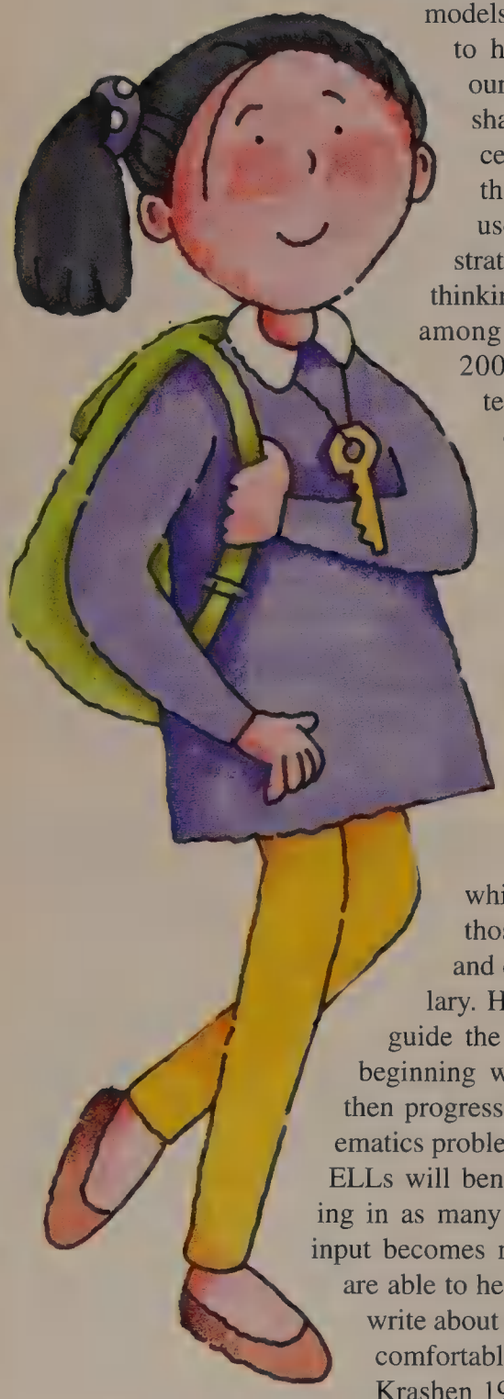
In Children's Math Worlds, a six-year project funded by the National Science Foundation, researchers involved inner-city, elementary-age Latino children in a "mathematizing co-construction process" (Lo Cicero, Fuson, and Allexsaht-Snider 1999, p. 62) that involved exploring the mathematics and language prospects inherent in real-life stories told by the children themselves. A child's photograph or story would spark an inquiry into problem solving that involved the entire class. Key aspects of the project were participants listening to one another; working together to "mathematize" one another's stories (a process whereby the children and the teacher use mathematics to make sense of the story or situation and then create an investigable situation grounded in conceptual mathematics); discussing, comparing, and understanding the significance of students' drawings and representations; focusing on students' solution processes rather than their answers; and involving their families via a "math helper" in the home. The researchers concluded that "high expectations combined with a teaching and learning environment in which meanings are co-constructed by teachers, students, and parents will greatly increase the success of inner-city Latino children in mathematics" (p. 69).

Tools

I define *tools* broadly to include concrete manipulatives (such as coins, blocks, tangrams, number cubes, spinners, calculators, protractors, string, etc.) or pictorial or graphical representations (such as students' representations, tree diagrams, area models, ratio tables, bar graphs, pie charts, hundreds charts or grids, graphing calculators, etc). Teachers should engage students in collaborative mathematical problem-solving situations in which students manifest their understanding of these situations using any tool that *students* find meaningful and useful (NCTM 2000). Students should have the opportunity to share and explain their solution

processes orally in any language (with sentences, phrases, or single words); through nonverbal communication (such as pointing or manipulating concrete objects); or in written form using pictures, diagrams, graphic organizers, graphs and charts, or symbols (Garrison and Mora 1999).

Students are the decision makers for tool usage and representation methods. Rather than using tools as answer-getting devices only, students must come to see them as a means of communicating and extending *their* mathematical thinking (Carpenter et al. 1997). Fosnot and Dolk encourage teachers and students to view these as tools for creating models, which serve as "powerful ways to help us understand and represent our world" (2001, p. 95). As students share and explain *their* solution processes, ways of understanding, and the tools and representations they used, students develop a repertoire of strategies and models to assist them in thinking about and making connections among mathematics concepts (NCTM 2000). Second-grade mathematics teacher Mrs. Schweitzer remarked on her changed view of tool usage: "I have used manipulatives and other useful aids to show [the children] my way of understanding. My work now is to let them find their own way of understanding" (Schifter 1996, p. 5).



Carpenter and Lehrer (1999) stress the need for teachers to guide their students in explicitly reflecting on the characteristics of each of their invented representations and the manner in which tools were used and to *bridge* those to abstract mathematical ideas and conventional notation and vocabulary. However, students' thinking should guide the instruction. Ron (1999) suggests beginning with students' everyday language, then progressing toward the language of mathematics problem solving and symbolic language. ELLs will benefit from experiencing this bridging in as many language modes as possible; the input becomes more comprehensible when ELLs are able to hear, do, see in writing, discuss, and write about the input in any language they feel comfortable with (Flores 1997; Khisty 1997; Krashen 1981). To accomplish this bridging, as we will discuss in the next section, teachers and students together must develop and embrace classroom communication and participation norms that support *everyone's* opportunity to share and justify their ideas while others respectfully listen, try to understand, and ask each other critical questions.

Classroom communication and participation norms

Classroom communication and participation norms are the ways in which students and teachers are expected to act, interact, and respond during instructional situations. Carpenter and Lehrer claim that these communication and participation norms also "govern the nature of the arguments that students and teachers use to justify mathematical conjectures and conclusions" (1999, p. 26). A respectful atmosphere with low affective filter is vital for ELLs (Krashen 1981). The equity-conscious classroom must be a safe, nonthreatening place where all thinking and reasoning are valued (Carpenter et al. 1997; Silver, Smith, and Nelson 1995) and every student feels like a contributing class member. Classrooms guided by teachers who purposefully attend to issues of equity for ELLs are characterized by an emphasis on collaboration over competition (Nieto 2000), communication in all four language modes (to encourage second language development and conceptual meaning making), and broad student participation (Wiest 2004). These teachers recognize that students may need to converse in their native language in order to assist one another in understanding the problem at hand and communicating their ideas (Cummins 1981; Flores 1997; Wiest 2004).

It is important to take time to listen to and understand the speaker (Carpenter et al. 1997), discuss and compare different students' solution methods, and allow the correctness of students' solutions to come from the logic of the mathematical thinking rather than from the word of the teacher or some other authority (Fosnot and Dolk 2001). Cohen and others have been studying student status relationships and issues of equity in cooperative learning groups for over twenty years: "In equitable classrooms, students use each other as resources in the context of equal access to a challenging curriculum and equal-status interaction in small groups" (Cohen et al. 2004, p. 143). They describe such a curriculum as being "intrinsically interesting, open ended, discovery oriented, investigative," and involving multiple intellectual abilities, such as "innovative problem solving, original and creative

thinking, mechanical ingenuity, artistic refinement, dramatic poise, or musical delivery" (p. 145).

Fosnot and Dolk (2001) describe a "math congress" where, after solving an open-ended mathematics problem in pairs or small groups, students share their thinking with the entire class and compare and contrast one another's strategies and models. In this sense, the classroom becomes a mathematical discourse community, providing an essential bridge for students' sense and meaning making. Khisty exhorts the use of inclusive discourse:

> Without participatory dialogue, learning remains outside the person; it is something that is removed from personal experiences and mental connections, and it is elusive and difficult to hold onto. Talk, therefore is the critical vehicle by which an individual internalizes meaning, becomes enculturated, and develops a sense of personal power in mathematics.... A simple concept such as talk becomes a political tool that either empowers students or disenfranchises them. (1995, p. 290)

It is vital that teachers facilitate this participatory dialogue in all four language modes—listening, speaking, reading, and writing (De Jong and Harper 2005) and by supporting ELLs' use of their native languages. Khisty recommends

> emphasizing the meanings of words by using variations in voice tone and volume, pausing either before or after a word in order to draw students' attention to it, writing or pointing to written words (from a previously prepared list) as the word is spoken, and frequently "recasting" mathematical ideas and terms with synonyms or other ways of saying them. (1995, p. 284)

Whenever possible, a visual, written, or pictorial record of key ideas (presented by students and teachers) should accompany any oral presentation. Some ideas for accomplishing such a visual record include the creation of a word and picture wall or a museum walk of artfully displayed samples of student work labeled with key words (in English and the native languages).

Implications and Conclusion

To summarize, mathematics teachers who purposefully attend to equity issues for ELLs will consciously embed language objectives and literacy activities within mathematics content objectives in order to assist ELLs in learning language as they learn mathematics (Cloud, Genesee, and Hamayan 2000; Echevarria, Vogt, and Short 2004). Most mathematics teachers are likely to consider this a radical statement. After all, as a teacher I might have responded, "Isn't that the ESL teacher's job?" In reality, language is the primary medium through which any discipline is negotiated, constructed, and learned. Therefore, mathematics teachers must expand their conceptions of their role to include and embrace that of language teacher and cultural facilitator (Brisk 1998; Moschkovich 2000).

Necessary paradigm shifts in teacher roles

Although most mathematics teachers have second language learners in their classrooms, the teachers receive little or no professional development specifically related to teaching language or sheltering instruction (De Jong and Harper 2005; Nieto 2002). This article offers a framework of "linking theory to practice" with guiding questions that support mathematics teachers' reflections and discussion on this important matter of "enabling teacher practice" (Gutiérrez 2002) toward equity for ELLs. Echevarria, Vogt, and Short (2004) provide a useful model that purposely integrates both content and language objectives into lesson plans.

Necessary paradigm shifts in dispositions

Although the paradigm shift requiring mathematics teachers to expand their conceptions of their roles is necessary, it is not sufficient. Dispositions toward ELLs must also change (Nieto 2000). The belief systems, held consciously or otherwise, of policy makers, administrators, teachers, students, and parents play a key role in educational equity. Unfortunately, "a deficit model permeates educational thinking and policy" regarding language minority students, where lack of academic achievement is attributed "to flaws ... within the students, their families, or their culture" (Khisty and Viego 1999, p. 71).

Sonia Nieto (2002) urges teachers to recognize and build on the strengths and diversity ELLs bring to classrooms and society. She offers a compelling reason *why* changes in teachers' dispositions are necessary: "Teaching language minority students successfully means above all challenging one's own attitudes towards the students, their

languages and cultures, and their communities. Anything short of this will result in repeating the pattern of failure that currently exists." She goes on to quote Cummins' insights into *how* changes in dispositions transpire: "Fortunately, good teaching does not require us to internalize an endless list of instructional techniques. Much more fundamental is the recognition that human relationships are central to effective instruction" (Cummins 1996, cited by Nieto 2000, p. 196). Khisty and Viego offer a promising example of a teacher who prioritizes personal relationships in her teaching. Through high expectations, challenging and equitable classroom discourse, and a focus on meaning making, this teacher "completely shatters the conventional deficit model thinking about poor, inner-city Latino students" (1999, p. 73).

"Enabling teacher practice" (Gutiérrez 2002) toward equity for all students requires continual, honest self-reflection and a willingness to confront and change our belief systems (Gutstein et al. 1997; Nieto 2000; Weissglass 1994). It requires teaching mathematics from a different paradigm. The goal is for *every* student *to know* that he or she has important contributions to make—in the mathematics classroom and in the larger classroom of life.

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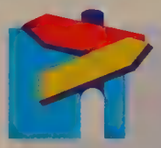

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Books

Elementary Mathematics Pedagogical Content Knowledge: Powerful Ideas for Teachers, *James E. Schwartz, 2008. 336 pp., $38.99 paper. ISBN 978-0-205-49375-3. Allyn and Bacon/Longman Publishers; www.ablongman.com.*

In elementary mathematics, teachers are beginning to realize that *how* they teach is as important as *what* they teach. Referring to five powerful ideas—Problem Solving, Reasoning and Proof, Communication, Connections, and Representation (the NCTM Process Standards)—Schwartz offers many examples of these strands throughout content areas. Each chapter provides a preview and learning objectives for the chapter, a description of the content area, the importance of the content area and common misconceptions related to it, the ways the Process strands support the content area, activities and exercises, online activities, a self-check, and selected references. Problem solutions are located at the end of the book. Throughout each chapter, margin icons indicate the text focus. These icons and the structure of each chapter make this an accessible text for preservice and practicing teachers alike. Although teachers could use the book as an individual resource, I think it is best suited for a collegial study group or an elementary mathematics methods class. It is as important for teachers to discuss mathematical ideas as it is for students to do so.—*Karen Soanes, Perry Central School District, Perry, NY 14530.*

The GIANT Encyclopedia of Math Activities for Children 3 to 6, *Kathy Charner, Maureen Murphy, and Charlie Clark, eds., 2007. 528 pp., $29.95 paper. ISBN 0-87659-044-X. Gryphon House; (800) 638-0928; www.ghbooks.com.*

The GIANT Encyclopedia of Math Activities for Children 3 to 6 was written for teachers by teachers. The editors have collected favorite activities that integrate mathematics from preschool and kindergarten teachers around the country. These teachers know what preschoolers are made of, and they share activities that teachers, parents, and day care providers can use to maximize young children's natural curiosities. The eighteen chapters use art, blocks, dramatic play, songs, games, literature, movement, science, cooking, and play to engage students while developing their mathematical understanding. Some of the activities were specifically developed for classroom mathematics centers, including sand-and-water tables; others are to be used as classroom routines, such as circle time. Each activity clearly identifies the mathematics concepts, materials needed, and steps involved to implement the lesson. Teachers will find an extensive table of contents as well as a very useful index of skills to assist them in creating learning units with a mathematical focus. An index of children's books further supports the mathematical concepts that are explored in many of the activities. Teachers looking for an abundance of tried and true early childhood mathematics activities in one book will want to add this book to their bookshelves.—*Barbara L. Blanke, California Polytechnic State University, San Luis Obispo, CA 93407.*

What Successful Math Teachers Do, Grades PreK–5, *Edward S. Wall and Alfred S. Posamentier, 2007. 168 pp., $27.95 paper. ISBN 1-4129-1503-1. Corwin Press; (800) 818-7243; www.CorwinPress.com.*

With this book, all prekindergarten–grade 5 teachers can be successful when teaching mathematics concepts. Based on NCTM's Principles and Standards, each chapter highlights effective teaching methods. The authors use vignettes to illustrate and explain best practices for teaching mathematics. Within each chapter, they divide the vignettes into grade levels for ease of finding lessons that pertain to your grade. Chapters also include an introduction to the concept, the strategy to teach it, exact wording of the NCTM Standard that applies to the concept, classroom applications, and possible pitfalls. Specific, consistent chapter layouts and accompanying diagrams make the book easy to read and understand. The chapter topics follow the ten NCTM mathematics strands, which include Number and Operations, Algebra, Geometry, Measurement, Data Analysis, Problem Solving, Reasoning, Communication, Connections, and Representation. What more do you need to bring your teaching skills to a higher level and raise student understanding of mathematics? This resource is one that all elementary teachers should include in their professional library.—*Judith Casey, Milton Avenue School, Chatham, NJ 07928.* ▲

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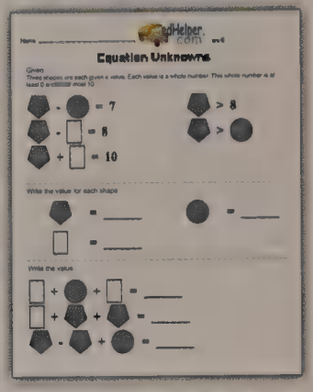

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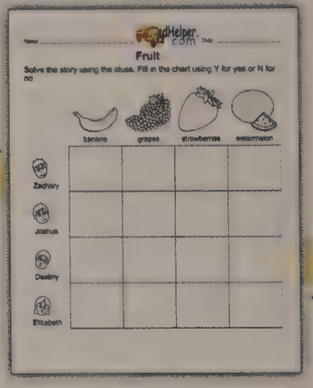

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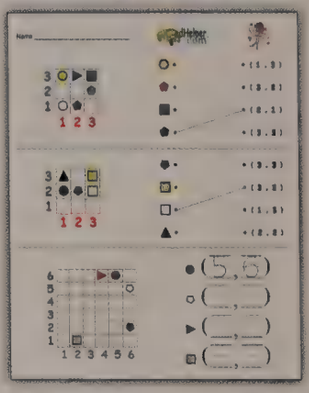

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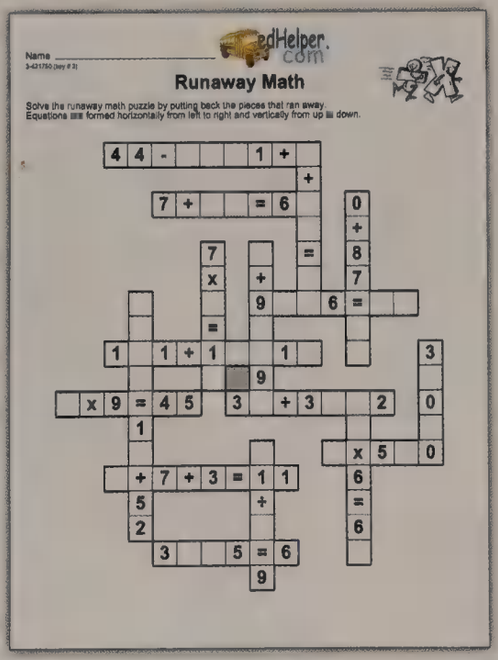
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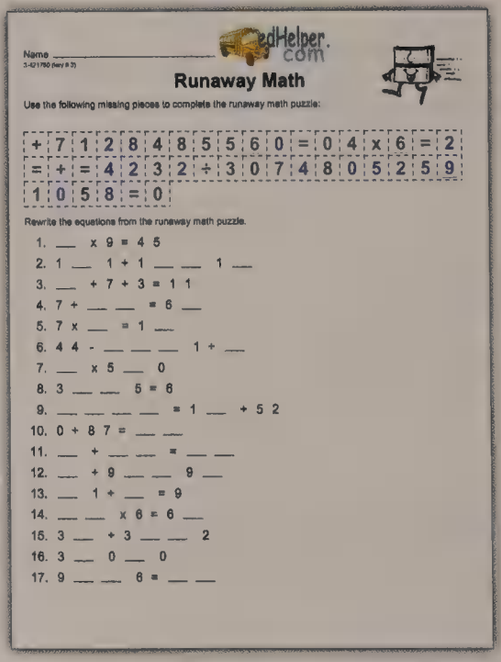
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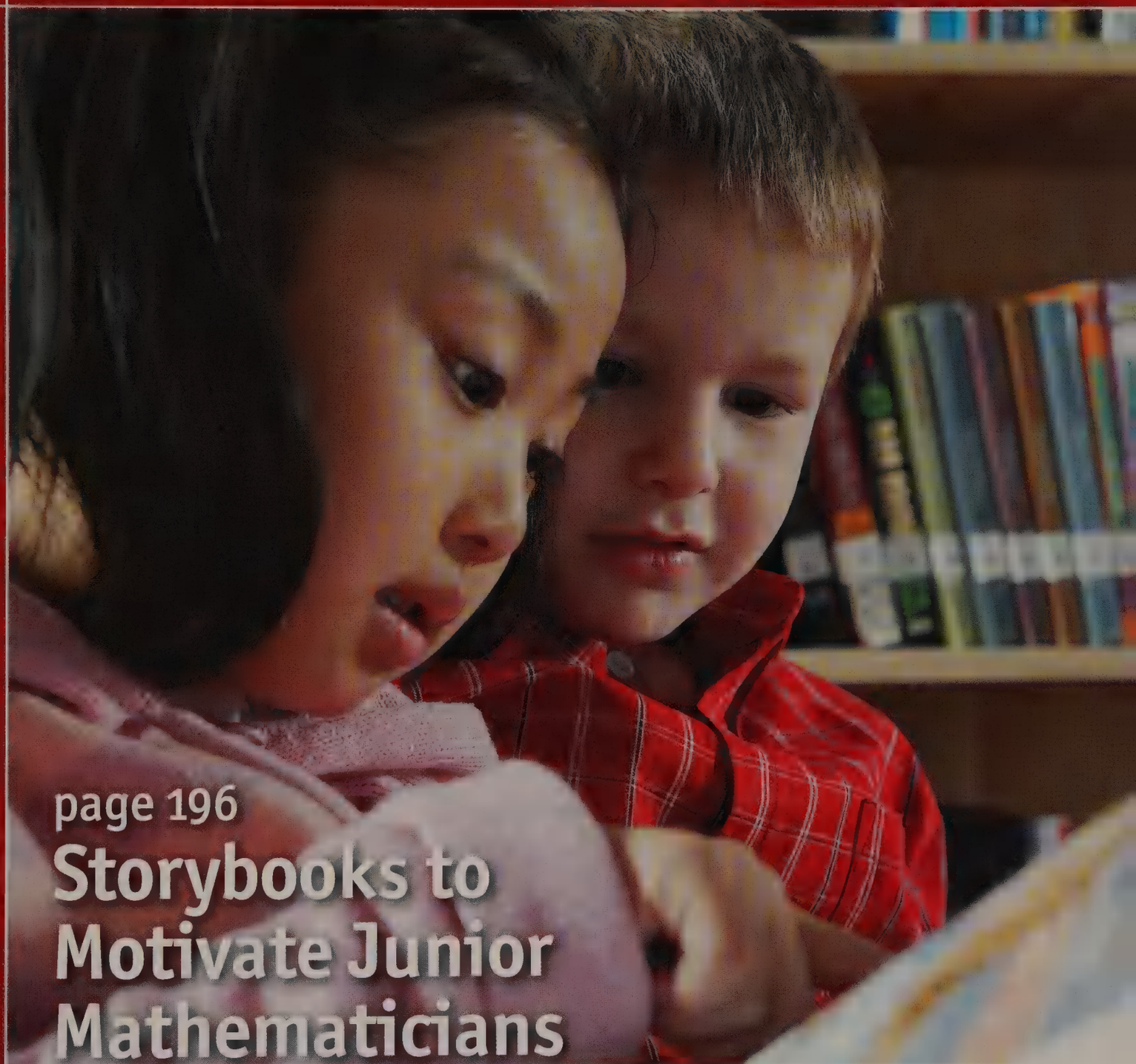

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TEACHING CHILDREN MATHEMATICS

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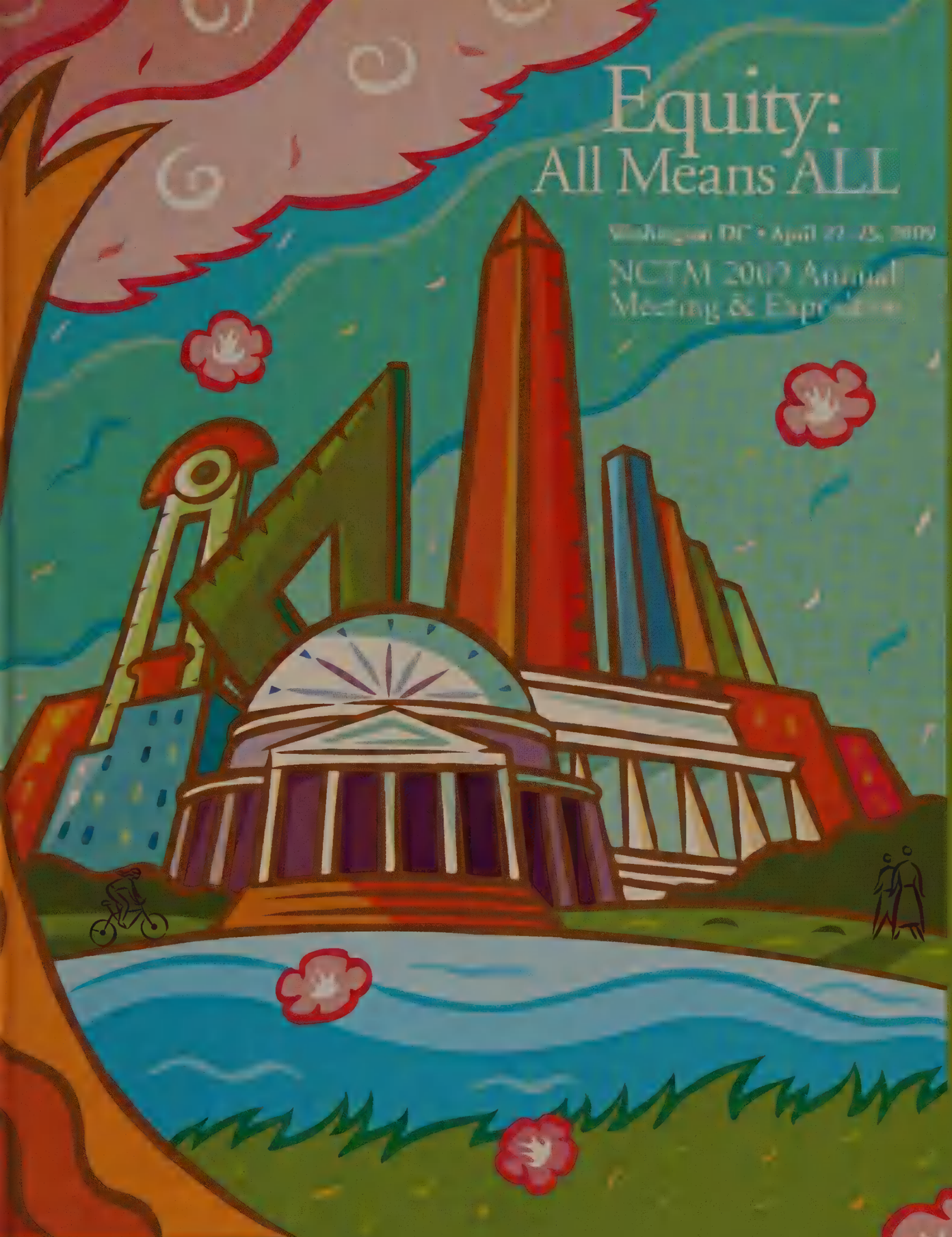

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MIM: Math in Motion for K–2

By Cheryl Chevalier, Mary H. Pippen, and Dorothy Stevens

Cheryl Chevalier, chevalierc@woodridge68.org, teaches second grade at Goodrich Elementary School in Woodridge, Illinois, with her teaching partner, Mary H. Pippen, pippenm@woodridge68.org. Dorothy Stevens, dw-stevens@att.net, retired from teaching kindergarten in Woodridge District no. 68. She is a substitute teacher in the district who also spoils her grandchildren.

In spring 2002, the authors attended an algebra workshop sponsored by the NCTM Academy for Professional Development. The workshop presented algebraic standards and activities exploring each aspect of the Learning Principle from *Principles and Standards for School Mathematics* (NCTM 2000). The hands-on activities were developmentally appropriate for students at grade levels K–2 and could easily be transferred into our classrooms to enhance the curriculum.

We were so inspired that we began developing a Math in Motion (MIM) program for our primary students that would make a connection between mathematics skills and literature. The program's premise is to use literature to inspire mathematics activities because literature grabs the attention of young learners. We chose a program name to convey that the activities would engage students in active learning through interactive lessons. At the beginning of each activity, we read a book to establish the story line and then reread it using problem-solving skills and manipulatives to act out the plot.

Why Do MIM?

The Learning Principle states that "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM 2000, p. 11). As early childhood educators, we know that children learn best with hands-on developmental activities:

> A page of abstract symbols, no matter how carefully designed or simplified, because of its very nature, cannot involve the child's senses the way real materials can. Symbols are not the concept; they are only the representation of a concept, and as such are abstractions describing something which is not visible to the child. Real materials, on the other hand, can be manipulated to illustrate the concept concretely, and can be experienced visually by the child. (Baratta-Lorten 1976, p. xiv)

Therefore, our objectives were threefold; we wanted to—

- use literature that engages all language learners,
- provide a strong mathematical foundation with manipulative activities, and
- provide activities to which all children can relate.

Our school has a culturally, racially, and economically mixed population; many students speak English as a second language. Providing mathematical activities that are "rich in language, where thinking is encouraged, uniqueness is valued, and exploration is supported" (NCTM 2000, p. 74) better prepares children for advanced mathematics concepts regardless of their background.

The MIM activities have proven to be engaging, exciting, educational, and within the children's own experiences. Another plus is that we have seen positive character building during our MIM time. Cooperating together, the children decide which parts of the story to act out. They share their problem-solving strategies to help one another, and they encourage one another. We often hear, "Let me show you," "Can I translate today?" or "Let's be partners and work together." It is particularly helpful to the English Language Learner (ELL) students when native English-speaking students share, demonstrate, and translate directions.

Getting Started

In planning our program, we applied for and received a grant of $500 from our local education association to cover the estimated cost of books, materials, and storage bins. We purchased books with mathematical story lines that applied to specific concepts in the mathematics curriculum, developed lesson plans to teach the mathematics concepts, and procured materials to go with the stories.

Three series—Hello Reader Math, Math Matters, and Math Start—proved to be excellent resources. These series are written at a reading and mathematics concept level appropriate to the primary grades. "Representing numbers with various physical materials should be a major part of mathematics instruction in the elementary grades" (NCTM 2000, p. 33). With these books, materials, and the many activities we received from the algebra and geometry workshops offered by the NCTM Academy for Professional Development, we are giving our students the hands-on experiences necessary for learning mathematics.

Management of MIM

The MIM program was specifically designed for small-group activities in kindergarten through second grade. Each classroom teacher uses three activities that are rotated over a three-week period. The activities are ranked as high, medium, and low to accommodate student ability and help facilitators select appropriate experiences.

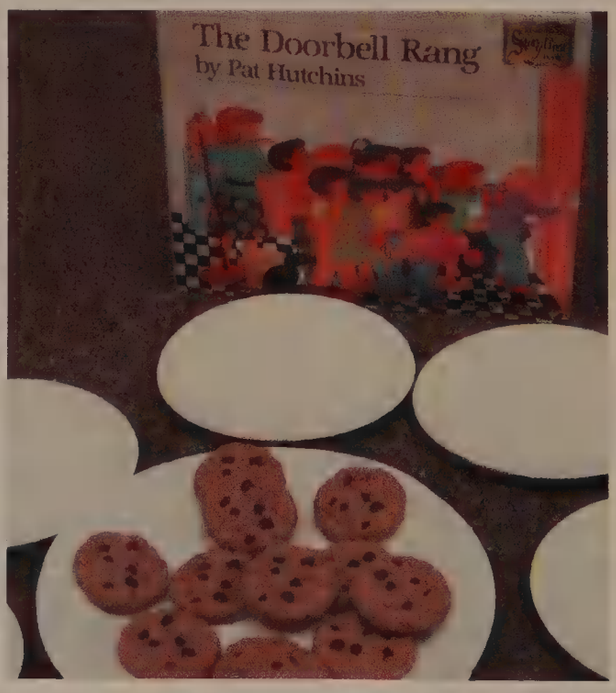
Each activity is stored in a clear plastic tub, complete and ready for facilitators to use. Everything they need is included—the book, the manipulative

materials, and a description of the activity. We house the boxes in a special MIM cabinet; a chart lists all the activities. Teachers check off each activity they use. We anticipated that when the younger children first used the manipulatives, they would be more focused on the concrete materials rather than the process and understanding. However, when we repeat activities, we find that the students are more focused on the mathematical concepts.

Books and Activities

The Doorbell Rang (Hutchins 1986) was one of the books we selected because it demonstrates the Number and Operations Standard. Specifically, the children learn division readiness through sharing equally. The MIM tub we assembled included the book, an instruction sheet, plastic cookies, a tray (or large paper plate) for the cookies, napkins, and real cookies. In the story, a mother bakes twelve cookies for her two children to share. Before they sit down to eat their cookies, the doorbell rings and more children arrive. This happens several times until there are twelve children and twelve cookies. The children realize "that would be one for each of them!" Just as the children are about to eat their cookies, the doorbell rings again, causing all the children to stop.

As we begin to read the story the second time, we place the twelve plastic cookies on the table. We then read and discuss what is happening in the story. Each time the doorbell rings, the children pause, contemplate, predict how the cookies can be shared equally, and rearrange them. These are some of the comments we hear: "Now we'll have to regroup and share again." "How many do you have?" "Let's put them all together and pass them out once more." Teachers observe the students problem solving and redistributing the cookies.



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As the story progresses, Grandma saves the day, bringing more cookies to enjoy. One student speculates, "I wonder how many she has brought." Our students continue to discuss how to share more cookies in various combinations. We end story time with a cookie treat.

Other storybooks that represent the NCTM Number and Operations Standard of "understanding the meanings of operations and how they relate to one another" (NCTM 2000, p. 392) are *Benny's Pennies* (Brisson 1993); *Alexander, Who Used to Be Rich Last Sunday* (Viorst 1978); and *Bunny Money* (Wells 1997). In the first book, Benny wonders what he should buy with his five new pennies. As the story unfolds, our students receive five pennies to spend along with Benny. Although Benny's family members make purchase suggestions, Benny shops in his own creative way. For example, he first looks for something beautiful and buys a red rose. In the end, he brings all his purchases to show his family. The MIM tub for *Benny's Pennies* contains everything Benny chooses to buy.

The other books also explore money-related problem solving. In the stories, children have a certain amount of money to spend, and readers see how the children make choices while spending it. The activities are excellent for young children, who may have had similar real-life experiences and can identify with the characters. The children often remark, "Oh, yes, that happened to me." When children identify with a character, their learning is more meaningful.

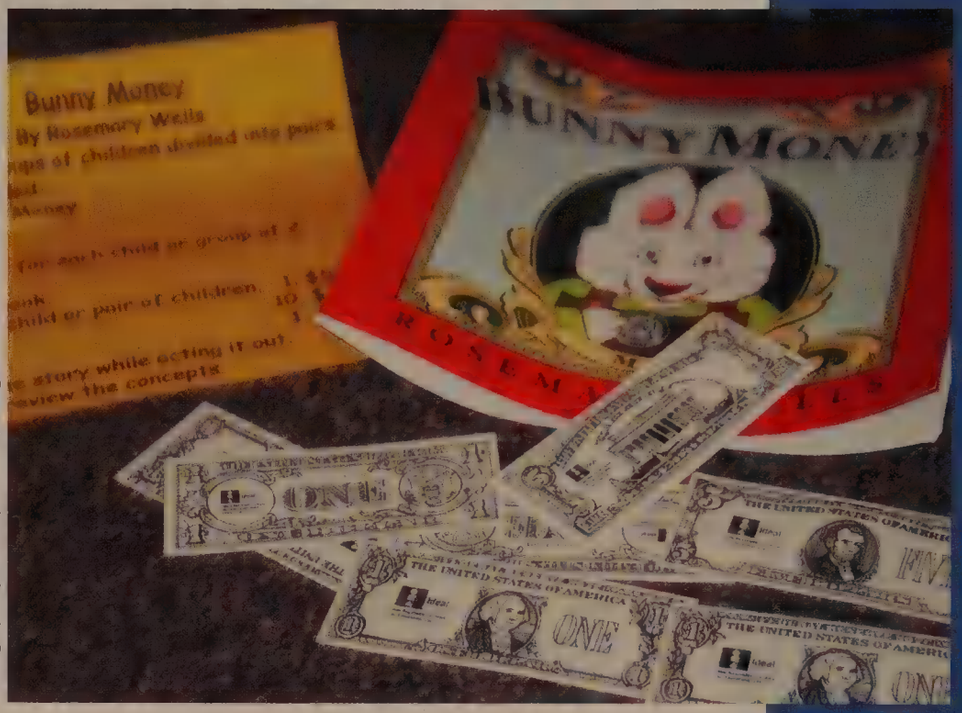
Mouse Count (Walsh 1991) explores counting in many different ways and reflects the Numbers and Operations Standard, too. In this delightful tale, ten mice play in the meadow until they get sleepy and decide to take a nap. Meanwhile, a snake comes by looking for something to fill his dinner jar. He first finds three mice, then four more, and, finally, the last three mice are added. In this story, the children practice the skill of counting on. Happily, the mice manage to "uncount" themselves, giving the children practice in counting backward.

The Crayon Counting Book (Ryan 1996) and *The Cheerio Counting Book* (McGrath 1998) explore numbers from zero to one hundred and reinforce children's understanding of the magnitude of one

hundred. Counting around a circle to one hundred is an excellent addition to this MIM activity. All the groups seem to enjoy this game.

"Early experiences with classifying and ordering objects are natural and interesting for young children" (NCTM 2000, p. 38). We also use nonliterature activities that we experienced at the NCTM algebra and geometry workshops, such as the Touch Me game. This game is popular with the children and a useful assessment tool. In kindergarten, we use a mat marked with a grid of 3 × 3-inch squares, gradually increasing the number of squares and ways of using them as the children develop better number sense. Younger children place the numerals in sequential order. In the activity we saw demonstrated at the algebra workshop, cards numbered one to one hundred are evenly distributed among the players. The first card is placed in its correct grid location. The children then take turns placing their cards, touching those already played (top, bottom, left, or right). In this game, children learn concepts of "before and after," "ten more," "ten less," and so forth. The cards we provide have two designs, giving the children the opportunity to form patterns and recognize odd and even numbers.

The Measurement Standard states that prekindergarten through grade 12 instructional programs should enable students to "understand measurable attributes of objects and units, systems, and processes of measurement" (NCTM 2000, p. 102). One related expectation for prekindergarten through second grade is that "all students should understand how to measure using nonstandard and standard units." Periodically, we record all our students' heights and weights to familiarize them with the use of standard measurement. One of our favorite books is *Much Bigger than Martin* (Kellogg 1996). Students readily identify with Martin, who wishes he were not the smallest child. For nonstandard measurement experience, the children enjoy *Twelve Snails to One Lizard* (Hightower 1997). In this story, each snail is one inch long; twelve snails are used to measure a foot-long lizard; and three lizards are used to measure a yard-long snake.



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Conclusion

Our MIM program is a work in progress. We continually review the curriculum to ensure that it runs smoothly and challenges all students. We also plan to track student growth through national and state testing. With Math in Motion, we know how much fun learning mathematics can be.

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Additional Children's Resources Recommended for MIM

"The Ants Go Marching One by One." Traditional song.

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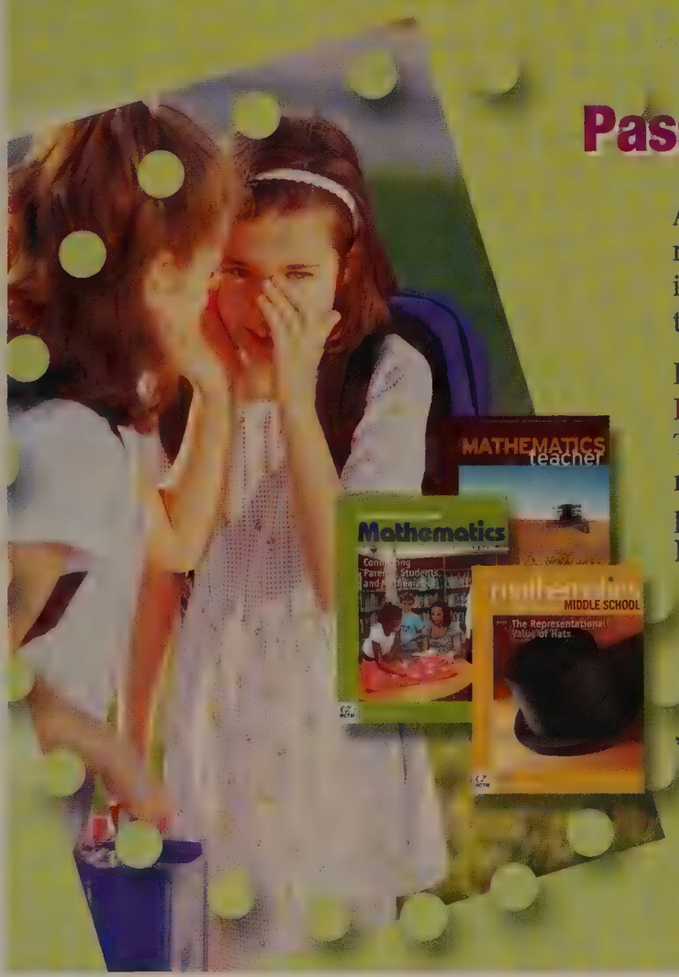

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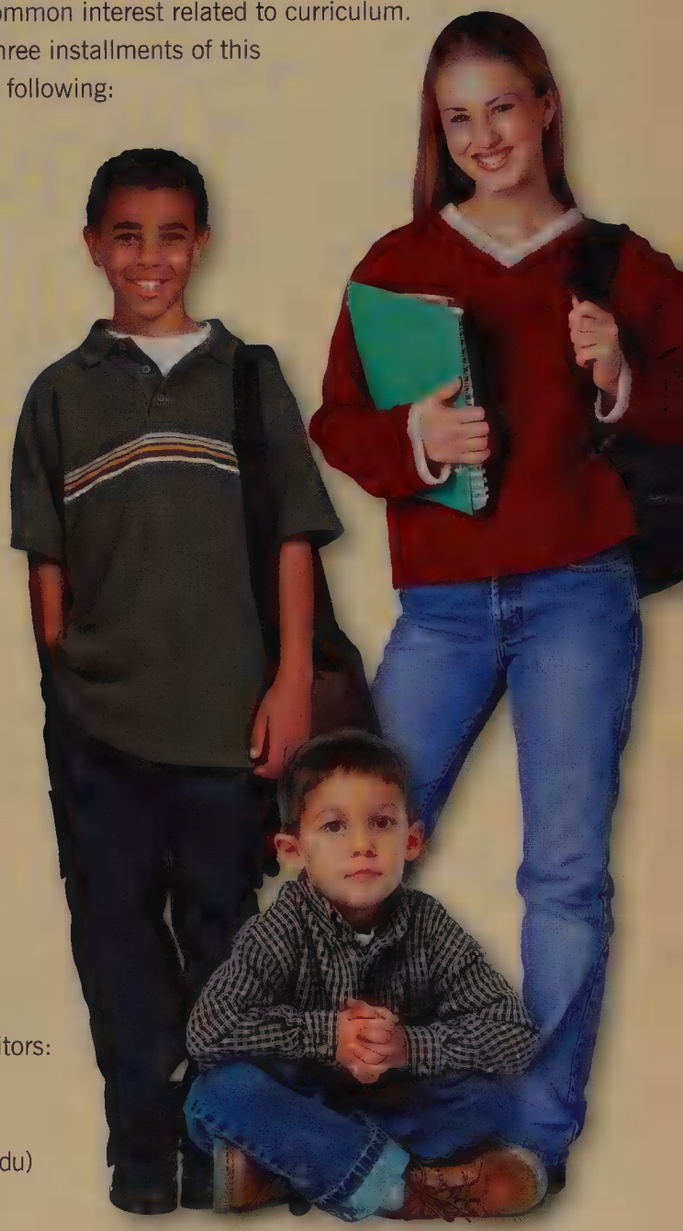
Contemporary Curriculum Issues

NCTM has provided strong leadership with regard to mathematics curriculum, most notably with the publication of *Curriculum and Evaluation Standards for School Mathematics* (1989); *Principles and Standards for School Mathematics* (2000); and, most recently, *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (2006). Curriculum goals and standards (the *intended curriculum*) and curriculum materials (the *written curriculum*) are powerful agents influencing teachers' instructional decisions and actions (the *implemented curriculum*).

The Editorial Panels of *Teaching Children Mathematics, Mathematics Teaching in the Middle School,* and *Mathematics Teacher* seek manuscripts that promote discussion and dialogue about important issues related to mathematics curriculum. A new department in each journal will provide a forum for stimulating discussion across a K–12 audience. The intent is to simultaneously publish three articles, one in each journal, all focused on a single important curriculum issue. Each article will discuss the issue from the perspective of the audience of the particular journal (elementary, middle, or high school teachers). Collectively, the set of articles appearing in all three journals is intended to increase communication and dialogue on issues of common interest related to curriculum. Possible topics for a curriculum-focused K–12 series could expand on the initial three installments of this department, which dealt with statistics, standards, and technology, or include the following:



- How is curriculum coherence achieved at the elementary, middle, and high school levels?
- How can mathematical habits of mind be instilled in students, and what role does the curriculum play in achieving this goal?
- How can curriculum articulation, across K–12 and from secondary to postsecondary institutions, be promoted and achieved?
- What are the central ideas of algebra, and how do these ideas develop across the K–12 curriculum?
- How do teachers perceive, interact with, and use curriculum materials, and what impact do these materials have on teacher practice?
- What role do curriculum materials play in supporting student learning?

This department will include three articles each year per journal. Manuscripts on any of the topics listed above or on other contemporary curriculum issues are welcome. The manuscript should be no more than 2500 words, not including figures. Submissions can be for one article for one particular journal, or they can be for a series of three articles, one for each journal. Manuscripts should be submitted at the appropriate Web site:

For *Teaching Children Mathematics,* go to **tcm.msubmit.net**
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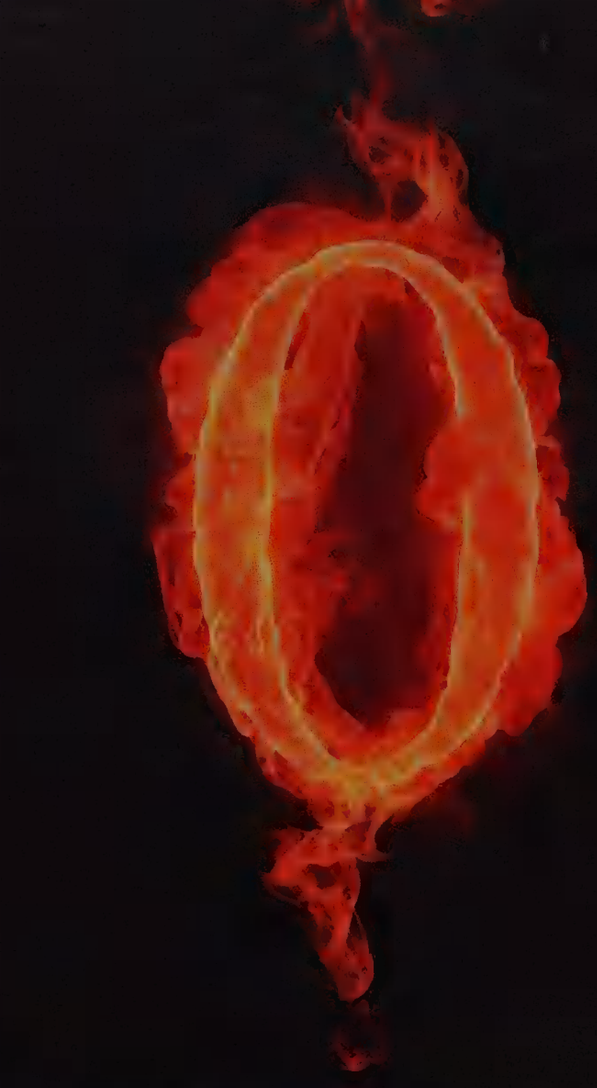
If you have questions or comments, please feel free to contact the department editors:

- *Teaching Children Mathematics:* Barbara Reys (reysb@missouri.edu)
- *Mathematics Teaching in the Middle School*: Glenda Lappan (glappan@math.msu.edu)
- *Mathematics Teacher:* Chris Hirsch (christian.hirsch@wmich.edu)



Questioning Zero and Negative Numbers

By Virginia B. Wilcox

Virginia B. Wilcox, wilcoxvb@yahoo.com, is a mathematics instructional coach in Puyallup, Washington. She was a classroom teacher in grades three through six for twelve years. She has taught mathematics methods at the college level and enjoys creating and presenting professional development around standards-based mathematics.

Last year, I was fortunate enough to take the Developing Mathematical Ideas (DMI) course on algebraic reasoning taught by Ruth Balf from the University of Washington, Seattle. DMI deepens participants' mathematics knowledge while also modeling how to listen to students and ask the right questions at the right time. As Willis and others note, "When students are learning mathematics, they use their existing knowledge and understandings to make sense of new ideas" (2007, p. 22). DMI allows teachers to uncover that existing knowledge. One of the primary messages is that children can and do construct their own meaning behind many complicated mathematical procedures and concepts. Taking a DMI course allows participants to return to their own classrooms and students with a fresh perspective.

The most fascinating piece of the course was examining how some higher-level algebraic concepts were viewed by very young children. I thought it impossible that children as young as first grade could construct advanced algebraic concepts. These are the very ideas that I struggled with myself in eighth and ninth grades. How could six- or seven-year-olds construct these notions without the same level of drill, instruction, and practice I had experienced?

A Young Student's Concept of Zero

After a few sessions of the DMI course, I found myself growing more curious about how my own first-grade daughter would respond when presented with the notion of the value of zero and the concept of numbers less than zero. To set up our discussion, I used the game described in *Number and Operations, Part 3: Reasoning Algebraically about Operations* casebook (Schifter, Russell, and Bastable 2007, p. 86). The game requires students to move along a horizontal number line that has unlabeled marks to the left of zero and numbered marks to the right. The game requires the child to draw action cards labeled with a plus (+) sign or a minus (–) sign followed by a numeral. Players move a game

piece forward or backward on the number line to coincide with the action card they have chosen. Once we began playing the game, it did not take long for the concept of a value less than zero to arise. My daughter landed on zero and then drew a –3 card. She seemed stumped for a brief moment and then moved her finger three spaces to the left of zero. The following conversation between mother (*M*) and daughter (*D*) ensued:

D: [*moving her token three spaces to the left of zero and counting aloud*] Zero, zero, zero.
M: Wait a minute … zero, zero, zero? What does that mean? How can you have three zeroes?
D: I'm right here. I'm at nothing [*pointing to the spot she landed on when she counted three spaces to the left of zero*].
M: What do you mean, you're at nothing? Your finger is somewhere. Are you telling me your finger is nowhere?
D: No, it's right here [*tapping the paper at the point three spots behind the zero*].
M: So, you are somewhere. What would you call that place?
D: Nothing. It's nothing. I can't call it anything.
M: But you just said you were here [*pointing to daughter's spot*]. That is something, isn't it?
D: Yeah … but … no … but … I don't know. This is confusing for me.

Zero is a hero

I felt that there was more beneath the surface, so I made the decision to push her just a bit. I took a chance and actually gave her an idea of how to clear up her confusion. I asked her to give the spot she was referring to a name. She called it the *zero cousin spot*. She quickly and enthusiastically changed this name to the *minus three spot*.

M: Okay, let's keep playing. Draw your next card [*daughter draws a +1 card*].
D: [*moving her token one jump to the right*] Now I'm on the minus two spot!
M: Why did you call it that?
D: 'Cause this [*as she pulls out a pencil and begins to write numbers to the left of the zero on the game board*] would be the zero cousin minus one spot [*pointing to the negative one position*]; this would be the zero cousin minus two spot [*pointing to the negative two position*]; and this is the zero cousin minus three spot [*pointing to the negative three position*]. I added one to the minus three spot so I'm here at the minus two spot."

"Wait a minute … zero, zero, zero? What does that mean?"

A first grader accurately constructs her own ideas about numbers to the right and to the left of the zero.

Photographs by Howard A. Wilcox III; all rights reserved

It was clear from our discussion that my daughter was constructing her own ideas (pretty accurately) regarding numbers not only to the right but also to the left of the zero. Still, the whole notion of zero had me curious. In the DMI class, we had watched videos of children struggling with the contradiction of zero representing nothing but still standing for something. Because my daughter was in such good spirits, I decided to keep the discussion going

and delve into her thinking on the concept of zero. We turned the game board over and used the same plus and minus action cards to display some basic mathematics problems. I put down the +4 card, and she pulled out four blocks. I then laid down the –4 card.

D: Hey … you can't do that!
M: Why not?
D: Well, you'd just have nothing, that's why.
M: Show me.
D: [*picking up all the blocks and hiding them*] See? Nothing!
M: I see! So, zero is nothing. You have nothing; it means nothing?
D: No, it means something. It means you don't have anything.
M: So, is zero important, then?
D: Yeah! Zero is a hero! He takes the place of other numbers.
M: What do you mean by that?

At this point, she spent a significant amount of time explaining in detail how numbers such as 10, 20, 30, and 100 could not be made without zero. She proclaimed zero to be "the most important number ever."

Understanding numbers less than zero

Having just touched on the concept of number less than zero, I pressed a bit further still:

M: Could you have something that's less than zero?
D: No, you just stop at zero. It just ends and there is no more.
M: Hmmm … so, what if you had this?

I searched the card stack to find numbers to fit what I had in mind. I pulled out the +3 card and showed it to her. She pulled out three blocks. I then pulled the –4 card and showed it to her.

D: [*after looking at the card for some time, takes all three blocks away*] See, nothing. I can't take anymore away, so it's nothing. There is nothing less than zero.
M: Yeah, but over here [*flipping the paper back to the previous game's number line*], you went behind the zero. Why couldn't you do that here?
D: 'Cause over here I could still count behind the zero because you had these lines. I don't have any more blocks or those lines to take away over here, so I can't show you this number [*pointing to the –1 position on the number line*].
M: So, three minus four would put me here [*pointing to the –1 position*] on the number line?
D: Yep, that was the zero cousin minus one spot.
M: There is a spot here on the number line for that, but you can't show it to me with the blocks?
D: Nope. I don't have any left.
M: Would you ever want to have zero of anything?
D: Yeah! Zero school!
M: Would you ever want to have less than zero of anything?
D: I don't think you can do that. Well … maybe if I gave you all of something I had but was still supposed to give you more, I'd have less than nothing. I don't know how to do that.

It was wonderful to see her struggle with the notion of how to represent and name the number she counted as three less than zero. I enjoyed watching her develop, revise, and explain her numbering system to me. I was also impressed with how definitively she could explain and prove her answers. Throughout the activity and conversation, she showed a much higher understanding of zero and its purpose than I expected for her age. She even managed to show some beginning conceptual understanding of what it could mean to have something less than zero. Although her knowledge is by no means complete, she is beginning to realize that she may actually have to "owe" someone something to represent numbers less than zero.

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Showing a much higher understanding of zero and its purpose than expected for her age, the author's daughter even managed to show some beginning conceptual understanding of what it could mean to have something less than zero.

Good Questions Show Us What Students Understand

How fascinating for me as both a parent and mathematics educator to watch these concepts as they were articulated for the first time. I was immediately reminded of one of the major talking points from the *First Steps in Mathematics: Number* course materials: "Good questions and tasks provoke students to show us what they know and understand" (Willis et al. 2007, p. 24). It is too bad that every mathematics teacher cannot implement this level of one-on-one questioning to really get to the heart of what each child they teach truly understands. As Tripp stated, "Students do misunderstand, but it is seldom because they cannot understand; most often it is because they understand something else" (1993, p. 88). I have a feeling that students who are written off as incapable of learning mathematics often fall into this category. They understand something; we educators have just not asked the right questions to figure out *what* they understand. Willis and others noted, "Understanding the mathematics helps educators make better professional judgments" (2007, p. 24). Reflecting on my conversation with my daughter, I would revise that idea to state, "Understanding the mathematics that children understand helps us make better professional judgments and, as a result, makes us better mathematics educators."

What implications does this scenario hold for current classroom teachers or anyone developing and applying new mathematical ideas and working with students? Young children have logical thoughts and ideas about mathematical concepts to which they have never been exposed. For educators to go into any lesson assuming that they will have to provide or convey the bulk of the ideas to the students is a fallacy. As educators, we know that students come to the classroom with preconceptions about how the world works. It is our job to engage those ideas and uncover the logic and meaning behind what the students already know. If students' initial understanding is not engaged, "they may fail to grasp the new concept or information, or they may learn what is taught for the purposes of passing a test but still cling to their initial preconceptions outside of the classroom arena" (Bransford and Donovan 2005, p. 1). Traditional mathematics instruction often attempts to override students' reasoning processes and replace them with rules and procedures that disconnect the problem solving from meaning making. Bransford and Donovan point out that "informal strategy development and mathematical reasoning can serve as a foundation for learning more abstract mathematics ... this link is not automatic" (pp. 218–19). The technique of building on existing knowledge and the need to engage students' preconceptions encompass the difference between true learning and simple memorization.

Engaging and building on student preconceptions, then, poses an instructional challenge. Although there is surely no single best instructional approach, Bransford and Donovan (p. 223) suggest identifying certain features of instruction that support the following goals:

- "Allowing students to use their own informal problem-solving strategies, at least initially, and then guiding their mathematical thinking toward more effective strategies and advanced understandings." Most of the standards-based curricula being used in many schools today allow for this to occur.
- "Encouraging math talk so that students can clarify their strategies to themselves and others, and compare the benefits and limitations of alternate approaches." The Developing Mathematical Ideas (DMI) courses foster and encourage this level of mathematical discourse.
- "Designing instructional activities that can effectively bridge commonly held conceptions and targeted mathematical understandings." The *First Steps in Mathematics* program (Willis et al. 2007) not only focuses on these conceptions but also allows teachers to identify and address the misconceptions constructed by students with high levels of competence.

I hope that my daughter's teachers help her build and consolidate her previously constructed concepts. In order to understand any new concepts in depth and organize them in a way that makes sense to her and can be useful as she confronts increasingly difficult ideas and theories, she must have opportunities to build on concepts she has already constructed. The challenge for all teachers is to "provide sustained and then increasingly spaced opportunities to consolidate new understandings and procedures" (Bransford and Donovan 2005, p. 232).

An instructionally valid and appropriate lesson goes beyond the actual plan written on paper. It transcends the prepared blackline masters. It is more than the pages one plans to cover, the

books that will be shared, or the activities in which students will be engaged. A lesson will be instructionally appropriate and sound if the only things the teacher has written down are the high-level, probing questions she plans to ask her students about their understanding of the mathematical concepts during the lesson. Teachers may not be capable of asking every student each individual question (the way I was fortunate enough to do with my daughter), but the end product is far superior because the lesson is driven by students, their responses, and their innate understandings—not by the pages that must be covered or completed before a certain date. Making lessons learner-centered sends the message that the previously constructed knowledge not only is valued but also can be used to teach others in the class and move all participants (even the classroom teacher) forward in deepening their mathematical knowledge. Teachers must know their content well, understand the misconceptions students may bring to the concept, and be able to "build effectively on what learners bring to the classroom. They must pay close attention to individual students' starting points and their progress on learning tasks. They must present students with 'just-manageable difficulties'—challenging enough to maintain engagement and yet not so challenging as to lead to discouragement. They must find the strengths that will help students connect with the information being taught" (Bransford and Donovan 2005, p. 14).

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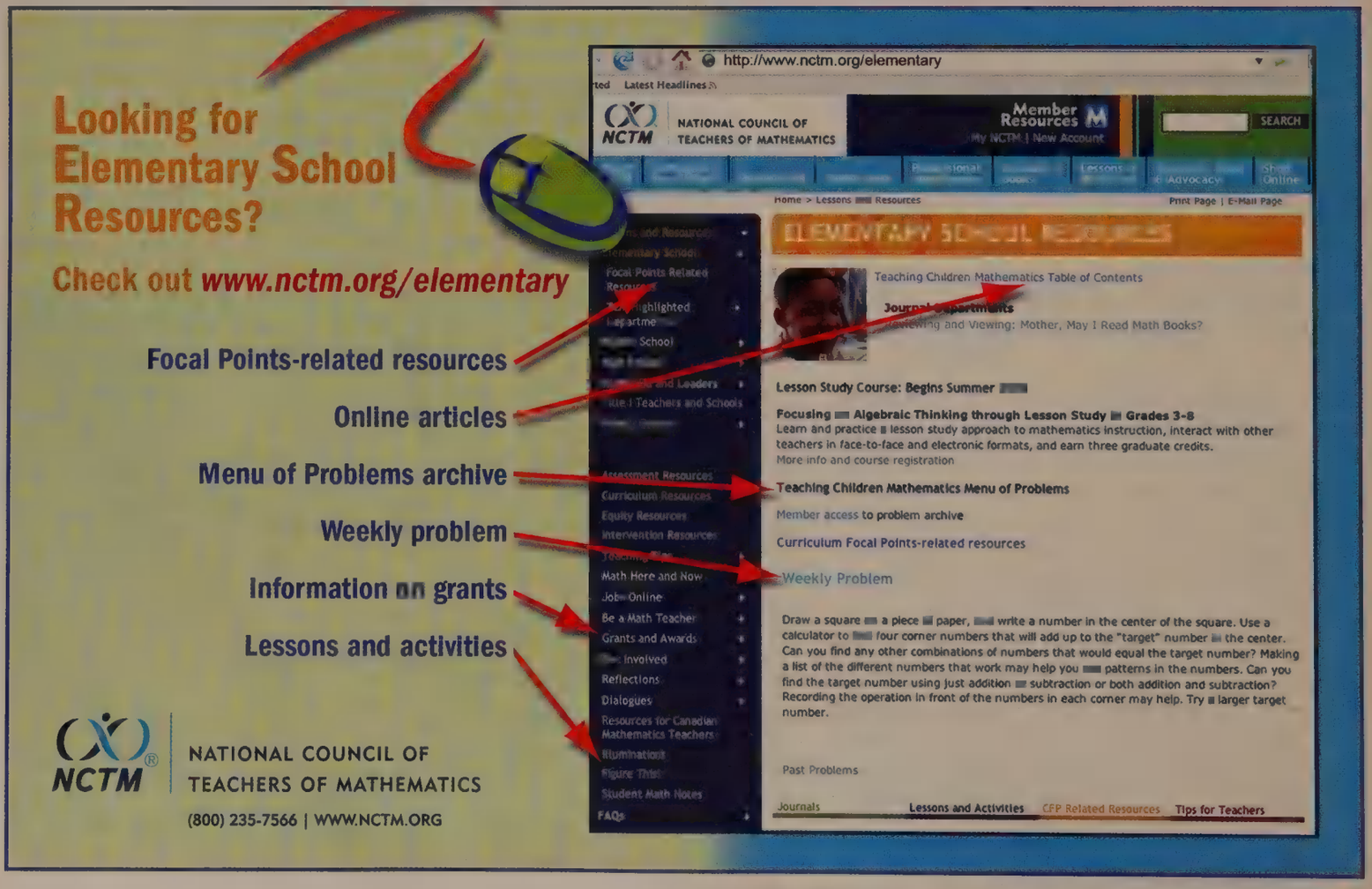
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The *Teaching Children Mathematics* (*TCM*) Editorial Panel is currently seeking manuscripts for the "In My Opinion" department that address issues related to algebra readiness and the teaching of mathematical algorithms. Some possible topics include—but are not limited to—the following:

About Algorithms

- What is an algorithm?
- Must all students learn the same algorithms? If so, what are those algorithms? What makes them the most appropriate? Are traditional algorithms the best for all students and all situations? How do these algorithms support understanding and successful learning in higher mathematics and in life?
- Are there dangers to learning algorithms in a strictly procedural manner? If so, how do those dangers play out in our students' future?
- Does understanding an algorithm support or hinder computational fluency? How?
- Does understanding an algorithm support learning in higher mathematics courses? How?
- Do number sense and place value understanding have a role in learning algorithms? Why or why not? If so, what is their role?
- Are the algorithms most typically taught in U.S. elementary schools the best? Why or why not? What are their advantages and disadvantages?
- What is computational fluency? What level of computational fluency is necessary for success regardless of students' educational aspirations?
- Will technology ever replace the need for algorithms?

Algebra Readiness

- What is algebra readiness?
- What are the underpinnings of algebra readiness, and where do they begin?
- What is the best way to develop algebra readiness in elementary school?
- What are the indicators that a student is algebra ready?
- What role do NCTM *Curriculum Focal Points* suggestions play in algebra readiness?

The most effective manuscripts are provocative, forceful statements that will spur *TCM* readers to consider a specific viewpoint on an issue or some aspect of mathematics teaching. Authors should be aware that *TCM* also encourages readers to submit their reactions for later publication in "Readers' Exchange."

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Using Sudoku Bulletin Boards to Teach Mathematical Reasoning

By Carol A. Buckley

Carol A. Buckley, cbuckley@gettysburg.k12.pa.us, is an extension teacher at Eisenhower Elementary School in Gettysburg, Pennsylvania. She is interested in developing critical thinking and mathematical problem-solving skills in young students.

"Maddening!" "Addicting!" "Can't do it." "Do it every day." These are phrases my peers have used in reference to Sudoku puzzles. I enjoy completing Sudoku puzzles but had not previously considered that incorporating them into my instruction would motivate my primary-grades students. After reading "Exploring the Sudoku Challenge" (Wilburne 2006), I decided to offer Sudoku as a student option for the logical reasoning category of a tic-tac-toe instructional strategy. In a tic-tac-toe strategy, three tasks in three different categories are organized to resemble a tic-tac-toe board. Students choose three tasks in a straight line: across, up and down, or diagonally. My students' enthusiasm and curiosity about completing Sudoku were immediately evident.

K–3 Classrooms

As a teacher of mathematics for young students, I put careful thought into how to teach students to think and reason. With encouragement from colleagues, I have developed a teaching style that is largely constructivist. The vast majority of my lessons are inquiry based, so I am familiar and comfortable with facilitating class discussions that require students to "think." My approach works well in one particular class of gifted and talented students. Second and third graders who repeatedly demonstrate their advanced reasoning and usually enjoy an academic challenge, they responsed to this activity no differently.

The Pennsylvania Academic Standards for grade 3 clearly include drawing inductive and deductive conclusions within mathematical contexts (Pennsylvania Department of Education 2006). NCTM's Focal Points encourage teachers to have students "apply logical reasoning to justify procedures and solutions" (2006). Encouraging students to develop and apply problem-solving strategies has always been a priority for me. All year, both identified and nonidentified gifted students had been stopping by my classroom with copies of completed Sudoku puzzles. The apparent fascination with Sudoku gave me the idea to incorporate it into my mathematics classroom.

The Lesson

To ensure success, I started students on basic puzzles. I created puzzles constructed of 2 × 2 sections on a grid and four basic colors as a teaching tool to define the concept of Sudoku, as I thought this would be easier than the numbers (see **fig. 1**). I carefully planned how to explain the process of never duplicating any of the colors in a 2 × 2 square (section), in any row, or in any column. I anticipated how to support my students' thinking in true constructivist style without telling them how to solve a puzzle and allowing them to develop their own strategies for solving the problem.

While I was still carefully weighing my words to protect the integrity of an inquiry-based lesson, the students who knew how to do Sudoku completed their puzzles. I had barely distributed all of the materials, and the students who were new to the puzzle craze were asking for assistance from their peers. Students already familiar with Sudoku informed the others about typical puzzles with numbers and defined the "rules" about using each of the colors (or numbers) only once in any given row or column. I clarified any details that were missing in the student explanations. Beyond using words to explain, students also modeled behaviors they wanted their peers to try when solving puzzles: They ran their fingers along columns and rows, checking for specific colors. Some held a finger or elbow on a particular space to mark their place. In no time at all, most of the students had exhausted my supply of "easier" homemade puzzles with colors. I thought I was being clever to have Plan B ready to go with a 4 × 4 puzzle made of letters (why not use M-A-T-H?). These, again, were too easy. The students were eager for more difficult and engaging material, so the challenge was on. Hence, the idea of an interactive Sudoku bulletin board was born (see **fig. 2**).

Figure 1

Initially using puzzles with four basic colors (instead of numbers) was a teaching tool to define the concept of Sudoku.

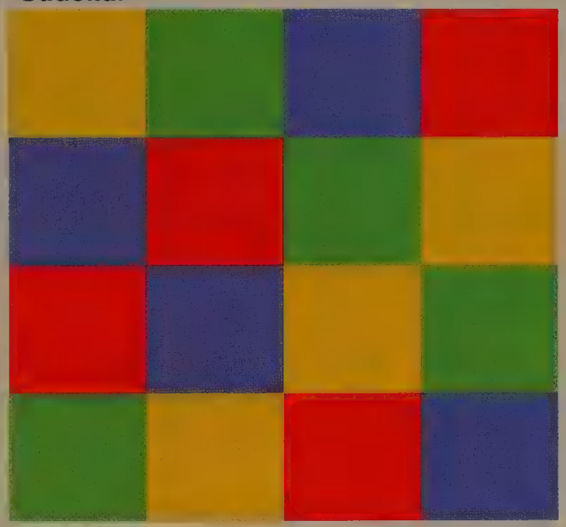
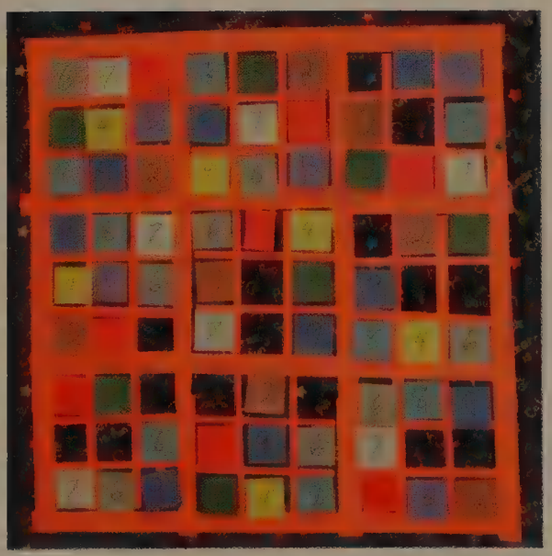
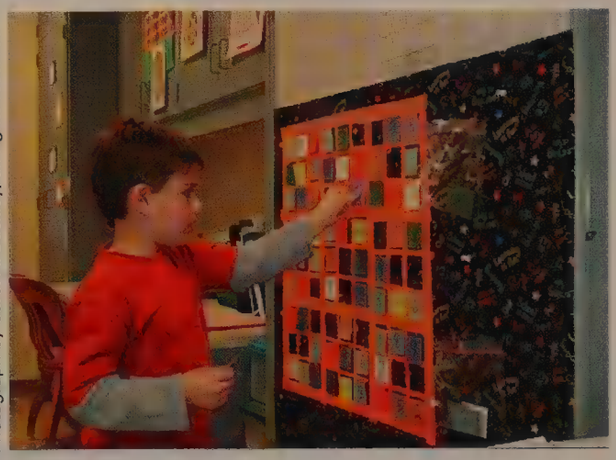


Figure 2

When students became eager for more difficult and engaging material, the idea of an interactive Sudoku bulletin board emerged.



During the initial introduction to the puzzles, some of the students appeared confused and unable to grasp the concept. For those who struggled, I detected no apparent pattern regarding age or gender. I pulled them into a small group to re-teach with more modeling and more of a direct-instruction approach, gradually releasing to the students the responsibility for their own learning. Whereas I had avoided suggesting specific



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The Sudoku bulletin board challenged and excited most students.

strategies to the whole group, I did guide the thinking in this small re-teaching group toward specific strategies, including looking for a few numbers that could be filled in immediately. I asked questions, such as "How can I be absolutely certain a seven goes here?" or "Why can't I use a three here?" I reminded two students to look at the numbers in other boxes—but within the same column or row—to rule out possible answers. I left it up to them to discover these strategies.

Differentiation

The bulletin board employs both colors and numerals, so students can choose which method is easier to use in solving the puzzle. For instance, all ones are orange, all twos are tan, and so forth. Students can make sure to have either one of each number or one of each color in each row, column, and each 3×3 section divided by a heavier line. The numbers that are part of the original puzzle have been stapled to the bulletin board, so students know that those do not move. I place Velcro strips in the other spaces that need to be filled and Velcro tabs on the back of the number cards that can be used to complete the puzzle.

This activity lends itself to differentiation within the classroom. For students who are experienced in solving Sudoku puzzles, the teacher may offer a puzzle with a higher level of difficulty. For beginners, a less intense puzzle is an option.

I typically build options into my assignments to accommodate students' different interests and learning styles. I did not require them to complete the bulletin board puzzle. It was one of three student choices under the logical reasoning category in the tic-tac-toe strategy, and about one-third of the class chose a different assignment. Students who chose the Sudoku puzzle had the option of solving it alone or with a buddy. If a student chose this assignment but another student was working on it at that time, the second student could work on another assignment until the bulletin board was available. Once the frame was in place, it was easy for me to change the numbers to display a new puzzle for another child.

The Sudoku bulletin board was a perfect fit for most students. It was academically challenging and allowed them to be out of their seats and physically manipulating objects. Student reaction was priceless when they came into class to see the bulletin board on the first day. Many of them were excited and wanted to be the first to attempt the puzzle.

Student metacognition

When I work on a puzzle, one strategy I use is to write little numbers in corners; consequently, the page ends up heavily marked. I was not at all sure that students (or I, for that matter) would be able to complete the puzzle on the bulletin board without using that strategy. I was unsure what strategies they *would* use. Some students haphazardly chose a guess-and-check method, which was not very effective. One student used fingers, elbows, and other body parts as place holders while wiggling around to look at the rest of the puzzle. This student succeeded in completing the first bulletin board puzzle.

Having previously solved many nonroutine problems in my class and having discovered several standard problem-solving strategies, another student was sure he would find a pattern to the missing colors. Students made comments such as, "I can't use a seven here, because there is already one in this row." One student tried to first place all the cards with the same digit (i.e., all the threes) and was disappointed to find that I had made more cards of each digit than what was needed to complete the puzzle. A student unsuccessfully tried to start at the first 3×3 square section on the top left and work his way across the top of the puzzle, doing individual 3×3 sections first without regard to the entire column and row. Another student tore little pieces of paper and numbered them one through nine. He then placed them on an imaginary grid on the floor between him and the puzzle. This was his representation of the "numbers in the corners" strategy that he uses with paper puzzles. I was fascinated to watch and hear their problem-solving strategies unfold.

And, of course, one student seemed to intuitively know which numbers were needed where and solved the puzzle with little effort. Unfortunately, this student was not initially able to articulate his strategy other than to say, "I just knew where they went." On further prodding, he rolled his eyes into his head at different angles (presumably looking at the empty holes in the puzzle that he had etched in his mind).

He started telling me that "either a nine or a three has to go here, and even though there isn't a four in this box, this one can't be a four because there is already a four in that row." He was apparently using the same method that I like to use on paper, but he was miraculously able to hold that information in his head.

Most students seemed to rely on the numbers on the cards and seemed to make little or no connection to the card colors. Either way, I heard "Aha!" expressed a fair number of times, which is always delightful to a teacher's ears. The awareness ranged from finally understanding the Sudoku puzzle concept to correctly identifying a number that went into one of the open spaces, ruling out a number previously considered for a spot, and even successfully completing the puzzle.

Conclusion

The interactive Sudoku bulletin board has allowed students to work cooperatively, an unanticipated bonus. Success in completing the puzzle has been mixed. I have had both second- and third-grade students complete it successfully. I am so amazed at the persistence and reasoning skills of these youngsters that my next project is to introduce Sudoku to even younger students in my school with some of the simpler puzzles that were "too easy" for my second and third graders. I hope that some of the older students (grades 2 and 3) will create puzzles of their own for the younger students to try as well.

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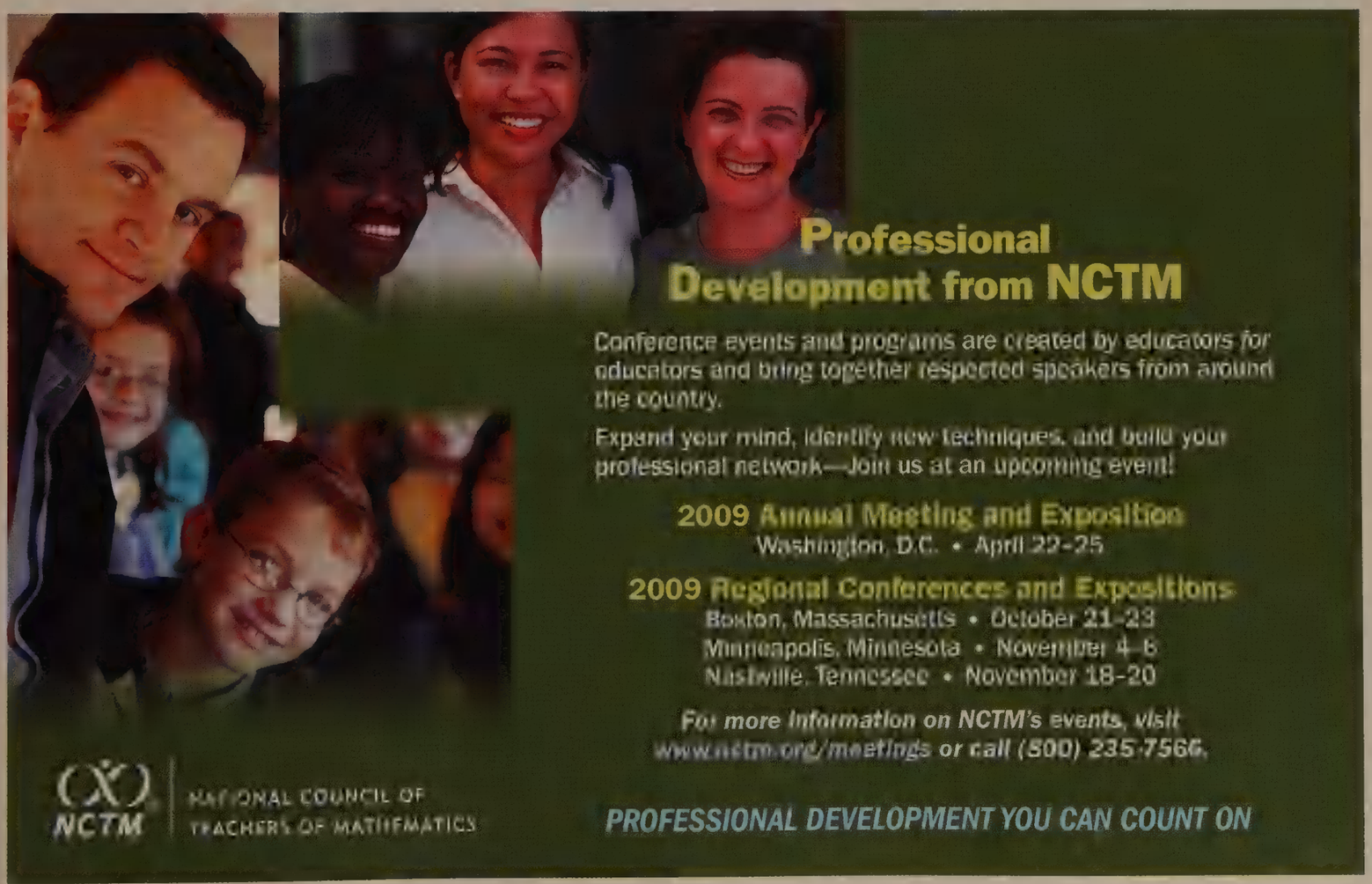
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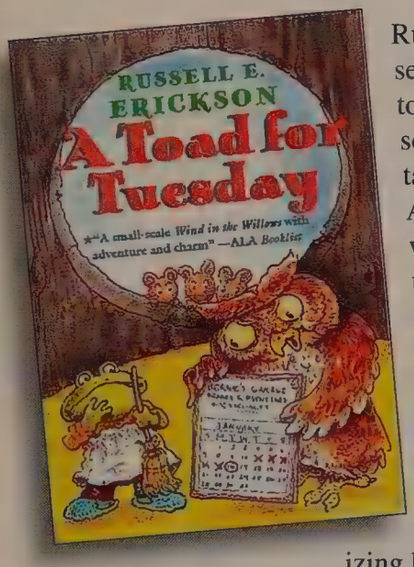
Gilda O. Shultz

Making Sense of Time with *A Toad for Tuesday*

"If you had a whole day to do whatever you wanted, what would you do?" I ask my second-grade students. "What if you had five days?"



I use the book *A Toad for Tuesday* by Russell E. Erickson to develop students' sense of time. The story begins with two toad brothers, Morton and Warton, enjoying some Beetle Brittle. Warton decides he must take some of the remaining brittle to their Aunt Toolia. But Morton warns that it is winter and will be most difficult for a toad to travel in such conditions. Warton insists and prepares some skis to make the journey easier. Warton leaves on a Wednesday, thinking it will take him just three or four days to reach his aunt. Along the way, he is captured by Owl, who threatens to save Warton for a special meal the following Tuesday, Owl's birthday. Warton, realizing he has just five days until Tuesday, tries to plan his escape …

Day One

I introduce the book, show the cover to the children, and ask them what the owl is holding. This approach provides me the opportunity to assess the students' existing knowledge of a calendar.

"He is holding a calendar!" responds Nick.

Next I say, "Let's look at our calendar for the month. Our class calendar shows the month, weeks, and days. People use calendars to help them remember important dates and activities and keep track of time. What day of the week is the sixteenth? What clues can help us figure this out?" I ask the children to think about this first and then share with their partners.

"First, you find the sixteenth, and then you can look up at the top to see what day it is," Mackenzie states proudly.

"You're right, Mackenzie. That was a good strategy to figure out what day the sixteenth is. Does anyone else have another strategy?" I continue.

"Well, I know the fifteenth was on a Monday, so the sixteenth has to be on a Tuesday," Bailey points out.

"What month is Owl's calendar showing? Why do you think he has circled the sixteenth, and why do you think he has crossed out the eleventh through the fifteenth?"

"Maybe he is doing something special on the sixteenth," Tia answers.

"I bet his birthday is on the sixteenth, and the other days are things he has to do to get ready for his birthday party," Chase offers.

"I think he is going on vacation from the eleventh through the sixteenth," Zoe suggests.

I want the children to think about the importance of the number of days crossed out and what types of activities might happen over a five-day period.

The first twenty-one pages of the story tell us that Warton packs, makes skis, and prepares lunches for his three- to four-day travel to Aunt Toolia's home. I talk with the children about how long these activities might take. On the board, I write labels: "About 5 Minutes," "About 5 Hours," and "About 5 Days." I name some activities and ask the children to think first and then share which category each activity should be placed in.

"How long would it take to make your bed?" I question. As I indicate the end of think time, the class answers in unison, "About five minutes!"

"What about going to a museum or an amusement park?" I ask. Again, in unison, the class responds, "About five hours!"

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"Now, what about going on a vacation?" They are catching on quickly and are easily able to discern approximate times for the activities. "About five days," they cheerfully answer.

I redirect their focus to the book's cover. "How many days had Toad crossed out?" After thinking and sharing, the class announces together, "Five days!"

"I was right!" Zoe exclaimed. "Toad is going on vacation, and maybe he is bringing Warton with him!"

The children are adept at determining the length of the activities I am asking about, so I narrow the field to make the next step more challenging. I encourage them to think of some activities that might take about a minute, about an hour, and about a day. Before they share responses, I question, "Which is longer, a minute or an hour?"

I pause a few moments for children to think and then ask them to share with a partner. After they share, I call on José. "It's an hour because there are sixty minutes inside of one hour," he proclaims.

Next, I give each pair of students three 3×5 cards. They take turns recording headings on the cards: "A Minute," "An Hour," and "A Day." I allow them five minutes or so to record activities on the cards for each of the categories. Next, I ask each pair to discuss with another pair the activities that they brainstormed. Once the pair-sharing comes to a close, students offer their ideas for each of the categories as I record them on chart paper (see **table 1**). As each suggestion is made, the class gives thumbs up or thumbs down to indicate their agreement.

The children come up with ideas for activities that take about a minute and about an hour. A brief discussion ensues about activities that take about a minute or an hour, centered on how long it takes to watch a movie. They agree that most movies are more than an hour; however, they reach a consensus that watching a movie best fits into the section labeled "About an Hour."

The real discussion comes when we address the suggestions that take about a day. Most of their ideas fall better into the category of "About 5 Hours" that we discussed previously. They decide as a group that it is difficult to think of an activity that actually take an entire day. In fact, the only suggestion they feel really fits into the category is the time from when you wake up until you go to bed. The questions that arise as a result of this activity really help the children think deeply about approximate lengths of activities, which provides the necessary understanding for the next day's lesson.

Table 1

Second Graders' Activity Chart

About a Minute	About an Hour	About a Day
Sharing with a partner	Silent reading	Going to school
Brushing your teeth	Doing PE	Skiing
Writing your name and date on your paper	Playing a soccer game	Going on a field trip
Walking to the cafeteria for lunch	Watching a movie/TV	Flying in a plane to visit grandparents
Taking the trash cans in from outside	Dance class	The time from when you wake up until you go to bed
	Doing homework	

Day Two

I begin the lesson by first reviewing the chart the students created the previous day and asking if they have any additional thoughts or questions.

James offers, "I had another idea for the 'About a Day' column. It's the time from when the sun comes up to when it goes down." I ask the children to think about James's idea. Does it fit into the category? Why, or why not?

After some pair sharing, Isabella adds, "It's better than some of the things, like 'going to school' or 'going on a field trip,' but it's still not a whole day."

Curious, I ask Isabella to explain further. She continues, "A whole day is twenty-four hours, and most of those things on the list don't even take half a day. If a whole day is twenty-four hours, then half a day is twelve hours."

At this point, I ask the children to think about what Isabella has shared. Sanjay says, "She's right because twelve plus twelve equals twenty-four, so half a day is twelve hours. Going to school or going on a field trip would only take about six hours or less." Many thumbs went up in agreement with this comment.

Jamie contributes, "The time from the sun coming up to going down is more than a school day, but it's definitely not a whole day." I am thrilled with the direction the discussion is taking, as this understanding of time will help them with the following lesson. As a result, we add a question mark to the "About a Day" column to indicate that we have many questions about activities that fit in this category.

I now direct my students back to the book. Before I continue to read, I briefly revisit what has happened up to page twenty-one. On page twenty-three, Owl explains why the calendar is so important to the story. He says that in five days it

will be his birthday, on a Tuesday. "Hmmmm," I think out loud, "I wonder what Warton will do in the five days leading up to Owl's birthday?" After Warton and the Owl chat and have tea (pp. 25–31), I ask the class if they think this chat took minutes, hours, or days.

At first the children think that it probably took minutes, but a few children, such as Mallika and Connor, say that it may have taken hours because they both have had experiences with their parents where "chats" lasted from dinner well into the evening. At this, more of the children start nodding heads, and thumbs are going up once again.

"It looks like many of you are having additional ideas, so let's share with a partner first and then we'll discuss this together."

After sharing, Chloe directs our attention back to the story: "It said that they had tea … two cups, I think. Warton also got his slippers on, and then it also said something about the 'latest hours of the night.'"

I follow her comments with, "What do you think 'latest hours of the night' means?"

Jesus answers, "It means that they talked until late in the night." Again, thumbs are up everywhere. Now the children are in agreement that the chat probably took hours.

During Warton's five days, he does many activities, such as clean the house, eat breakfast, unravel his sweater to use the yarn to make a ladder, and eat his lunch. Again, discussion ensues about how long these activities would take.

"Maybe we can make a new chart that just says 'Minutes,' 'Hours,' and 'Days' instead of our old one that says 'About a Minute', 'About an Hour', and 'About a Day.' Then we can put Warton's activities on it," Matthew suggests.

I had not anticipated doing this, but I quickly realize the value the chart will have when it is time for the children to work on the problem I have prepared. I draw the chart on the board, and within minutes the children have placed Warton's activities on it (see **table 2**).

Table 2

Wharton's Activities

Minutes	Hours	Days
Ate breakfast	Cleaned the house	Made a ladder
Ate lunch	Unraveled his sweater	Time Warton spent at Owl's house

I continue to read. On page 40, it is Sunday. I ask questions about how many days have gone by since Warton was captured and how many days remain until he will be eaten, reminding the children about the activities Warton accomplished during his time at Owl's house. At the end of the story, I ask whether or not Warton was able to travel to Aunt Toolia's home in the three to four days that he predicted at the beginning of the book. Students quickly explain that his prediction was incorrect; he has already spent five days at Owl's house. I ask if they think Owl can fly Warton to Aunt Toolia's in fewer than three days. (Sophia had stated after partner discussion that because they were flying, Owl would probably get there in hours, and the class had agreed with overwhelming thumbs up.)

The questions that I want the children to think about now are how many times they can participate in an activity in a given number of days or, given the time it takes to do particular activities, what combination of activities they can accomplish in one or more days. To answer these questions, they need to understand two things: how long certain activities take (in hours) and how many hours are in a day. I decide to stick with hours and days to start off and extend to the minutes in hours and the minutes in days when I am confident the students are secure with the former.

Before I challenge them with the activity, though, I ask them to reflect back to the two charts we made. If we could add anything to the first chart now that we have spent some time exploring activities and approximate times, what would it be? Nick asks, "Can we change the titles?" I agree that this would make the tasks easier to categorize (see **table 3**).

The upper portion of the chart shows the items we originally brainstormed; the lower portion shows the new activities. During the brainstorming session, the children decide they want to move some of the items into more appropriate categories with the new headings. I also notice that in their independent work time, some of them added additional activities that had not been previously brainstormed.

I ask the children to focus on the middle column and estimate the approximate hours each of the activities would take. They work with a partner to determine what is reasonable. Then, each pair works with another partner pair to share ideas and confirm the reasonableness of their approximate times. I do not want to write the hours next to each item because I think the times might vary and I want the children to use their own ideas as long as they have some validation from their peers that the times are appropriate.

I set the stage then for their next activity: "Let's take a look at 'going to a museum.' Think about what you and your partner estimated would be a reasonable time to spend there." Ideas ranged from one to four hours. "I agree that all of those times would be a reasonable amount of time to spend at a museum. I'm going to think about what I would spend.... I think it would be about three hours." I write "3" on the board. "Now here is my question: If I spent three hours at the museum, about how many times could I go to the museum in one day?" I allow a few moments of think time. "Before you share with a partner, let's go back to the discussion we had yesterday. How many hours are in a day?" With this question, hands start flying. "Even though you wouldn't really go to the museum in the middle of the night, let's think about how many times we *could* go if we stayed up all night long. It also might seem unrealistic to go to the same museum over and over in one day. But, if I were in Washington, D.C., where there are well over 150 museums, how many different museums could I visit in one day?" I can see their heads spinning as they think through this problem.

After thinking and partner sharing, Shyann speculates, "Well, you could add threes until you get to twenty-four."

I follow with, "Okay. Let's try that. How many threes should I put?"

Alek jumps in, "Well, three plus three equals six, and another three plus three equals six; then, six plus six equals twelve. There are four threes in a half day, so there must be eight threes in a whole day. So, that means you could go to eight museums in one day."

Astonished at Alek's thinking, I ask him to come up to the board and write his thinking. He writes, "3 + 3 + 3 + 3 = 12" two times and then "12 + 12 = 24," explaining that twelve hours (four threes) is a half day, so twenty-four hours (eight threes) is a whole day.

Mason exclaims, "So, you just need to add up all the hours until you get to twenty-four!"

And Victor follows up: "But you wouldn't really go to a museum in the middle of the night, would you?" The children nod their heads in agreement with Victor.

I had contemplated bringing up this issue earlier in the lesson, so the question prompts me to move forward with it. "That's a really good question. It isn't very realistic to be doing much of any activity during the night other than maybe sleeping, so what do you think would make better sense?"

Table 3

Revised Activity Chart

Minutes	Hours	Days
Original Brainstormed Ideas		
Sharing with a partner Brushing your teeth Writing your name and date on your paper Walking to the cafeteria for lunch Taking the trash cans in from outside	Silent reading Doing PE Playing a soccer game Watching a movie/TV Dance class Doing homework	Going to school Skiing Going on a field trip Flying in a plane to visit grandparents The time from when you wake up until you go to bed The time from when the sun comes up until it goes down
New Activities		
Eating breakfast, lunch, or dinner Doing PE Doing homework Silent reading Checking out a book from the library Talking on the phone Running around the track	Going to school Going on a field trip Going on a plane Going to an amusement park Playing in a baseball game Going to a birthday party Going to a museum Playing at a friend's Going to the beach Going to Knott's Berry Farm Playing outside	Taking a vacation Working on a big report Being off from school for winter break, spring break, or summer

Following their think-pair-share time, many students have ideas. Edith suggests, "Maybe we could just add up the hours in twelve hours since that is about the time the sun is up and you could do activities." Heads nod, and thumbs are up.

Angel chimes in, "Or you could add up fifteen hours because the sun is up for twelve hours, but then there's probably a few more hours before you go to sleep, like maybe three, and twelve plus three equals fifteen."

Yainid is eagerly waving her hand. "Maybe you could add up fourteen hours, and then ten hours could be for sleep or getting ready to sleep, and fourteen plus ten equals twenty-four. You could put sleep on the chart as an activity under 'Hours.'"

I am so pleased with the direction our class discussion is taking. It provides a perfect transition for

Figure 1

Examples of student activities for a day

(a) José
going to school = 6 hours
playing in a baseball game = 2 hours
going to a birthday party = 4 hours
playing at a friend's house = 2 hours
sleeping = 10 hours

6 + 2 + 4 + 2 = 14
10 4
10 + 4 = 14
14 + 10 = 24

(b) Tia
going to school = 6 hours
going to a museum = 4 hours
going on a plane to visit grandparents = 3 hours
sleeping = 10 hours

6 + 4 = 10
3 + 10 = 13
10 + 13 = 23

When I asked Tia if she had a whole day's worth of activities, she realized she needed one more hour and added the following:
going to the library = 1 hour
23 + 1 = 24

(c) Zoe
going on a field trip = 5 hours
going to school = 6 hours
going to a movie = 2 hours
eating breakfast, lunch, and dinner = 2 hours
playing in a baseball game = 2 hours
going to an amusement park = 8 hours

5 + 6 = 11
2 + 2 + 2 = 6
11 + 6 = 17
17 + 8 =
7 + 8 = 15 and 15 + 10 = 25

Again, I directed Zoe's attention to the number of hours in a day as I had done with Tia. Zoe quickly realized she had one additional hour, so she went back and wrote the following:
going on a field trip = 5 hours
going to school = 6 hours
going to a movie = 2 hours
eating breakfast, lunch, and dinner = 2 hours
going to the library = 1 hour
going to an amusement park = 8 hours

5 + 6 = 11
2 + 2 + 1 = 5
11 + 5 = 16
16 + 8 =
6 + 8 = 14 and 14 + 10 = 24

the next task I want them to ponder: "What if I ask you to think about all the activities you could do in one day, including sleeping? Can you figure out the activities that you could do in twenty-four hours?"

Isabella asks, "Do you mean we can use any activity, including sleep?"

"Yes, Isabella, you can include any activity and the amount of time you would take to sleep," I respond. I can tell the children are ready for this challenge. They ask no more questions and get right to work figuring out what activities they would do in one day. See **figure 1** for a few examples that the children generated. Note that several of them had to revise their tallies.

Before finishing for the day, I ask my second-grade students what they have learned over the two days we spent making sense of time. Mallika tells us, "I learned that the time the sun is up is about twelve hours."

Tuheen smiles. "I learned you can't go to Disneyland for twenty-four hours unless you're sleeping there!"

Mackenzie says, "I learned that the time you are awake is about fourteen or fifteen hours."

And Connor exclaims, "I learned that Washington, D.C., has over 150 museums!"

After spending two days on this lesson, I am extremely pleased with how my students are now thinking about time and their ability to relate their thinking and their number sense to the context of a day's activities.

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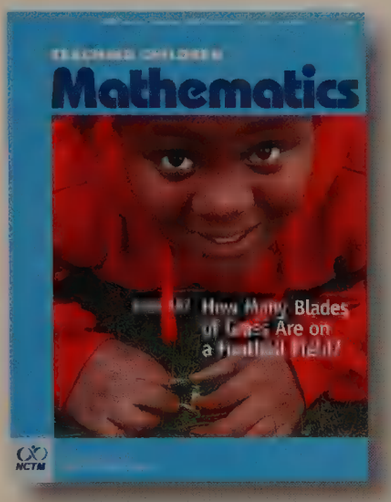

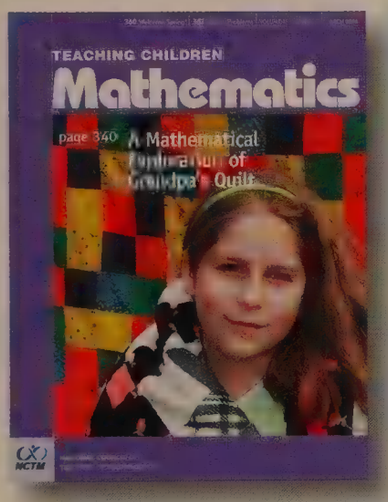

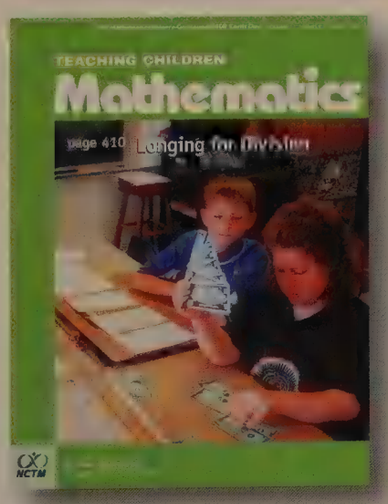

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Flags by the Numbers

Problem

The design of the presidential flag of Italy (see **fig. 1**) is similar to the number flag in this month's problem. Both flags are composed of nested squares. In our number flag, however, each outer square also has four corner numbers, which are used to determine the corner numbers of the next inside square.

Create a number flag by entering a number in each of the four corner circles of the original square (see the example in **fig. 2**). The numbers may all be different, or you may choose to use the same number more than once.

Next, place four additional circles at the midpoint between each of the original circles. Within each of these "inside" circles, record the difference between the numbers in its outside corner circles (always subtract the smaller number from the larger number). The inside circles now become the outside corners of a new square that is rotated 45 degrees from the original (see **fig. 3**).

On the newest square, between each pair of corner circles, place four additional "inside" circles. Repeat the step of recording the difference between the corner numbers in these new circles. Continue this process until one set of four corner numbers all become equal and then place a zero in the center. **Figure 4** shows a completed example from our original square. We call this completed example a *size-four* flag because it has four squares total, including the original outside square and the final inside square that contains threes on each corner circle.

Figure 1

Figure 2

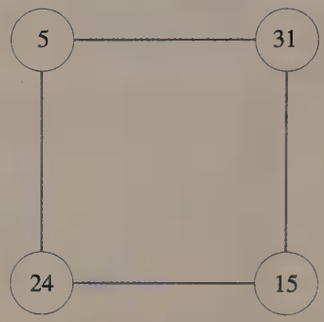


Figure 3

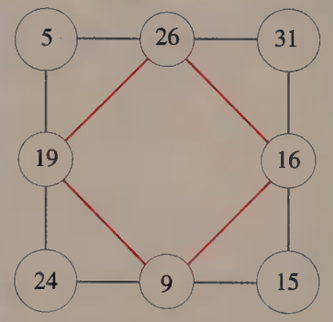
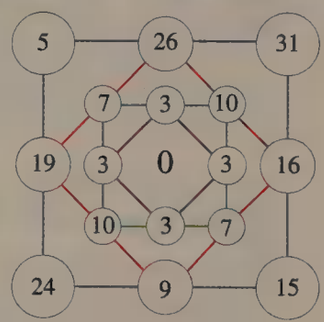


Figure 4



What other flag sizes can you make? Explore this subtraction puzzle to answer the following questions. Remember, you can begin your flags with any numbers you wish.

1. To create a size-one flag (meaning that there is just one square with no internal squares), what would have to be true about your starting numbers?
2. Is it possible to begin with four numbers that will result in a size-two flag or a size-three flag? If so, provide at least one example of each.
3. If you are able to use only numbers one to ten, what is the greatest flag size you can make? Explain how you know this.
4. What four numbers should you start with, and where should they be placed, so that you achieve the largest flag size possible? Make sure to explain how you know these numbers will result in the most squares.

Variations

1. How would the use of fractions or decimals affect this problem? Is it possible to use different fractions and decimals and still end with four equal numbers on the last inside square?
2. How would changing the operation between the corner numbers from subtraction to addition, multiplication, or division change the nature of this problem or the possible sizes of the resulting flags?

The goal of the "Problem Solvers" department is to foster improved communication among teachers by posing one problem each month for teachers of grades K–6 to try with their students. Every teacher can become an author: Pose the problem to your students, reflect on your students' work, analyze the classroom dialogue, and submit your resulting insights to this department. Through contributions to the journal, every teacher can help us all better understand children's capabilities and thinking about mathematics. Remember that even students' misconceptions provide valuable information.

Classroom Setup

Introduce the problem by using the example (or a similar problem) to demonstrate how number flags are created. You may choose to complete several examples with your students. You might also choose to begin with an example that uses smaller numbers (less than 20) in each of the four original corner circles, enabling students (especially younger ones) to understand the process without getting confused by the subtraction. If your students are familiar with the terms *vertices* and *midpoints* or if you would like to use this problem to introduce those terms, replace the problem terms *corners* and *inside circles,* respectively.

When you have completed your explanation, make sure students understand how to determine each new set of numbers and how the number of squares relates to the size of the resulting flag. After it is apparent that students understand how to complete the problem, allow them time to explore the posed questions by creating flags with different original numbers and different-sized flags as outcomes. You may have them draw their own squares and flags, or you may use the blank template on the **activity sheet** (page 220) to help them get started. If the students are exploring a flag larger than size five, they can either add more squares to the center of the current diagram or take the four numbers that are on the innermost square and begin a new larger square (and diagram) with those same numbers—thus, the flag would be continued over multiple diagrams.

The main challenge is to have students determine which four numbers should be used and where they should be placed so that the greatest flag size (or total number of squares) is achieved. Even if students are not able to determine or prove this result, they should be able to construct flags of different sizes and discuss the properties of the numbers used to construct such flags. Some students will also enjoy exploring the variations. You or your students may develop your own interesting extensions to this problem.

As students begin this problem, many will use the problem-solving strategy of guess and check, which will provide opportunities to experiment with various numbers and check their results after completing each of the subtraction squares. Other students may employ alternative strategies, such as working backward. After students have created several flags, encourage them to begin looking for and describing patterns. They might discover the

Where's the Math?

Many students have one model for subtraction: take-away. When one number is subtracted from another, something must have been "taken away." This model is useful but limited. The problem this month requires subtraction, but the context is comparison rather than take-away. When one number (for example, people's ages) is compared to another, the comparison is done with subtraction, although nothing is actually "taken away." Finding how far something is from a particular place also requires comparison and normally results in the positive distance. Similarly, in this month's problem, the subtraction results in the absolute distance, or positive difference, between two given corner values. Thus, this problem involves more than just subtraction, because the positive difference or "distance" between number pairs could lead to the idea of absolute value and distance on a number line.

Students working on this problem may also use algebraic thinking and reasoning when determining numbers that produce flags of particular sizes and when comparing their results to those of their classmates. Shape, dimension, and recursive pattern concepts could evolve from looking at the tilted squares that continually rotate and get smaller. This problem also provides a context for introducing or reviewing geometry vocabulary, such as *vertices* and *midpoints*.

The Flags by the Numbers problem is visually appealing and contains challenging numeric patterns. Students can practice the subtraction operation multiple times while also developing the idea of a model other than take-away for the subtraction concept. The problem also challenges students to explore numeric relationships, analyze strategies and solutions, and explain and justify their work and reasoning.

Edited by **Brian Schad**, *schad@aaps.k12.mi.us, a fifth-grade teacher at Lawton Elementary School in Ann Arbor, Michigan;* **Joseph Georgeson**, *jgeorgeson@usmk12.org, middle school mathematics department chair and teacher of eighth-grade students at the University School of Milwaukee in Wisconsin; and* **Sarah Bunten**, *sbunten@gmail.chelsea.k12.mi.us, a third-grade teacher at Pierce Lake Elementary School in Chelsea, Michigan. Each month this section of the "Problem Solvers" department features a new problem for students. Readers are encouraged to submit problems to the editors to be considered for future "Problem Solvers" columns. Receipt of problems will not be acknowledged; however, problems selected for publication will be credited to the author.*

relationship between the original numbers and the final size of the flag or the connection between the placement of numbers and the final results.

Make sure that students record their strategies and explain their solutions and approaches. Later, during your in-class discussion, have students investigate the different strategies their classmates used and allow them to compare and analyze the numbers used to construct different flags sizes. Challenge them to look for patterns or similarities between the various solutions.

Share Your Students' Work

Please try this problem in your classroom. We are interested in how your students responded to the problem, what problem-solving strategies they used, and how they explained or justified their reasoning. Include information about how you posed the problem, samples of student work, and photographs. E-mail your thoughts, reflections, scanned student work, and photographs to **Brian Schad** at schad@aaps.k12.mi.us by **January 15, 2009**, or send them to his attention at Lawton Elementary School, 2250 S. Seventh St., Ann Arbor, MI 48103. Include your name, grade level, and school name with your submission.

The editors wish to thank Dr. John Ginther, a retired Eastern Michigan University professor, for the idea and use of this intriguing problem.

(Solutions to a previous problem begin on the next page.)

Additional Resources

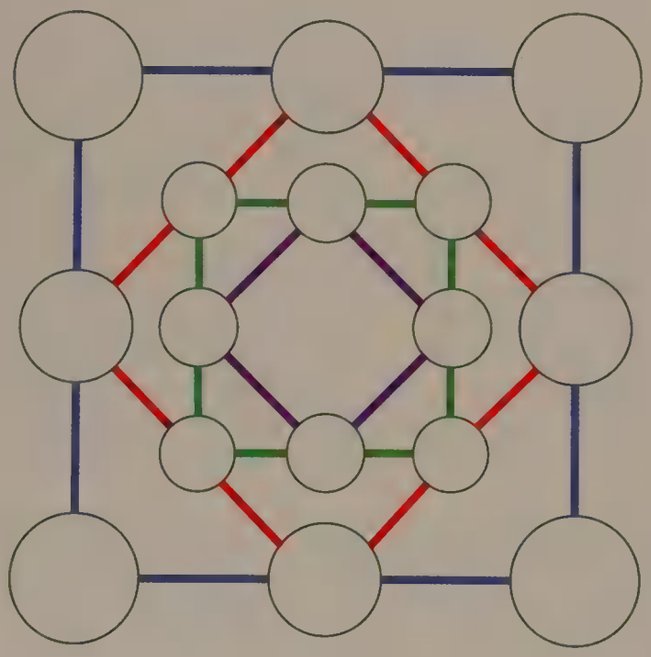
Did you know that NCTM has published a collection of some past "Problem Solvers" columns? Visit **nctm.org/catalog** for more on this and other NCTM resources, including professional development offerings, other publications, and online resources.

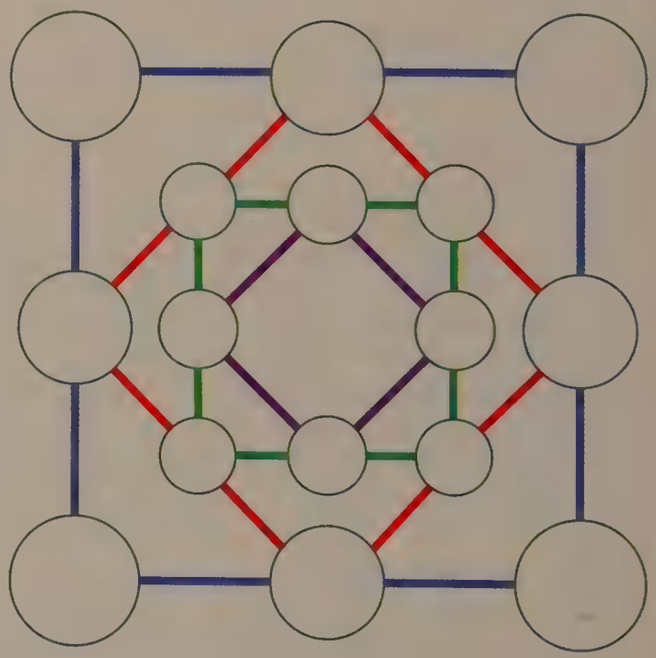
Sakshaug, Lynae E., Melfried Olson, and Judith Olson. *Children Are Mathematical Problem Solvers*. Reston, VA: National Council of Teachers of Mathematics, 2002. ▲

Activity Sheet. Flags by the Numbers

CREATING SIZE-FIVE FLAGS

Name _____





From the November 2008 issue of TEACHING CHILDREN Mathematics

Solution to the How Many Triangles? Problem

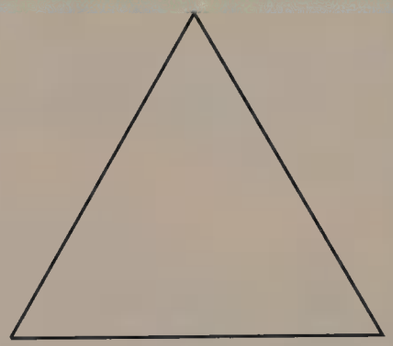
The How Many Triangles? problem combines geometric and algebraic reasoning. It helps students make sense of mathematical vocabulary in the context of building successive figures by folding and connecting midpoints. The original problem, which appeared in the November 2007 "Problem Solvers" section, was stated as follows:

Problem

Starting with a large equilateral triangle (see **fig. 1**), connect the *midpoint* of each side to the midpoints of the other two sides. Then count and record how many smaller triangles resulted (see **fig. 2**). Repeat this procedure for each of the smaller triangles and count and record how many new triangles you have. Continue this process until the triangles get too small to work with. You have created an approximation of a famous fractal known as the Sierpinski triangle.

1. Describe the patterns—numerical and geometrical—that you find as you continue to connect the midpoints of each triangle's sides.
2. Explain how you can determine the number of small triangles in the eighth figure in this pattern. (Note: The original large triangle counts as the first figure). Use your method to find how many small triangles are in the eighth figure.

Figure 1



Note: Re-create or enlarge this graphic so that the triangle used with students measures 8 inches in length per side.

Figure 2

Variation: Ask younger students to explain how to determine the number of triangles in the fourth figure in the pattern.

Extensions: Create the Sierpinski triangle with your class. After connecting the midpoints of each side to the other sides of the first large equilateral triangle, color in the triangle in the middle. For any triangle that is not colored, connect the midpoints of each side to the other two sides. Again, color in the triangle in the middle. Continue this process until the triangles are too small to work with. Have the students investigate how the number of unshaded triangles grows each time. Challenge them to explain how to determine the number of unshaded triangles in the eighth iteration. Older students can explore the relationship between a linear measure, such as side length, and a two-dimensional measure, such as area, by examining the side length and the area of each new smaller triangle. What pattern do the changes in side length and area create?

Edited by **Mark Ellis**, *mellis@fullerton.edu, an assistant professor in the College of Education at California State University–Fullerton, where he prepares teachers of middle school mathematics and works with local classroom teachers in creating learning environments that support sense-making in mathematics; and* **Cathery Yeh**, *catyeh@aol.com, who teaches third grade at Arnold Elementary School in Cypress, California. Each month this section of the "Problem Solvers" department discusses the classroom results of using problems presented in previous issues of* Teaching Children Mathematics.

Photograph by Mary Kay Varley; all rights reserved

The first student to realize that we did not need to cut the original triangle apart continued folding on top of each folded triangle.

Mary Kay Varley at the Fort Worth Country Day School introduced this problem to her fourth-grade class by asking students to recall their previous work with number patterns. She wanted students to have in mind the idea that growth patterns can be represented visually as well as numerically. She drew examples of triangular numbers and square numbers, each of which students had explored previously (see **fig. 1**).

Varley then gave each student a large triangle to work with. A discussion about naming the triangle followed, with students quickly recognizing that it was an equilateral triangle because all sides were equal in length and all angles were equal. Her students had difficulty determining the meaning of *midpoint* but were okay once they understood that triangles have *three* midpoints, each located at the middle of one of the three sides.

Next, Varley asked students to fold each vertex to the midpoint of the opposite side. This required students to recall that *vertex* means the corners of the triangle where two sides meet. After folding each vertex so that it touched the midpoint of the opposite side, students counted the number of smaller triangles formed by the three folds and found four. This information was recorded (see **table 1**) and displayed on the overhead projector.

To continue exploring the pattern, students were instructed to cut off one of the small triangles and repeat the folding process. However, one student noticed that if he folded all three outer triangles onto the center triangle, he would have, in effect, just one small triangle to work with for the next set of folds. Students seemed to like this idea, as it did not require them to make any cuts. However, as the triangles got smaller and smaller, it was clear that the thickness of the paper (with so many triangles folded on top of one another) made further folding very difficult.

With subsequent folds, students became very interested in finding out how many smaller triangles would be formed. Varley describes the mathematical thinking that emerged as they continued working on the problem:

Figure 1

Varley sketched examples of triangular numbers and square numbers.

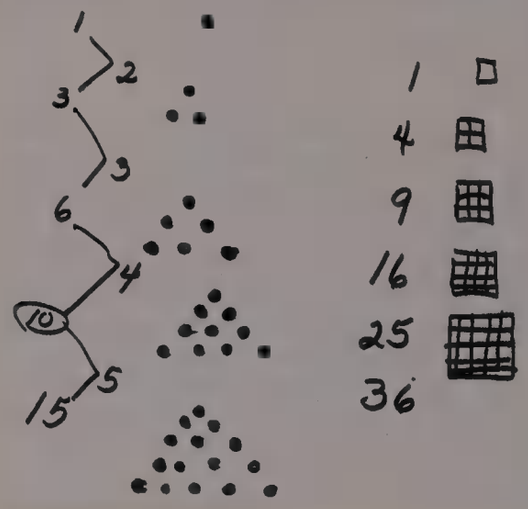


Table 1

Data from Varley's Class

Number of folds	0	1	2	3
Number of triangles	1	4	16	64

Photograph by Mary Kay Varley; all rights reserved

Students had fun discussing the best way to fold and count the number of triangles.

> Several students realized that the triangles were getting too small to count accurately. This led to a wonderful discussion of why mathematicians use patterns to develop a formula to answer their investigations. Students wondered how much longer they could keep folding, and we [discussed] the concept of infinity.... Several students couldn't help but repeat the line from the movie *Toy Story*, "To infinity and beyond!"

Students were doing mathematics and having fun at the same time.

Once the class had recorded data from three sets of folds (see **table 1**), it was time to think about a more efficient way to continue the process, because folding was no longer practical. Seeing the numbers in the table (1, 4, 16, and 64), many students thought about square numbers. But where were 9, 25, 36, and 49? Varley describes what happened next:

> A light bulb seemed to go off in several students' minds. Andrew decided that each of our answers would be a square number because we were always multiplying by four, which is a square number.

Students made the connection between previous knowledge and the new discovery of repeated multiplication by the same number. Varley took the opportunity to introduce the concept of *exponent* to them, defining it as representative of repeated multiplication with the same number, in this case, four.

Students were then challenged to think about how to find the number of triangles after any number of folds. More than one of them came up with the idea of looking at the number of folds and multiplying by four that many times. Varley writes:

> I was elated because it would now make sense for them to see what happens to numbers when we begin working with powers of ten. They began to realize the power of multiplication ... and saw the growth pattern as a multiplicative process.

This response demonstrates the power of giving students opportunities to work on what is, at first, a simple problem—but one that has the potential to lead them to explorations of big mathematical ideas. One of the most important concepts of the middle grades (and one with which students often struggle) is multiplicative reasoning. We encourage you to try this problem with your students and watch their interest grow in figuring out patterns through mathematics. ▲

Christine Gallego, Dora Saldamando, Grace Tapia-Beltrán, Karolyn Williams, and Thomas C. Hoopingarner

Presidential Parade

"Math by the Month" activities are designed to engage students to think as mathematicians do. Students may work on the activities individually or in small groups, or the whole class may use them as problems of the week. No solutions are suggested, so students look to themselves for mathematical justification and develop confidence to validate their work.

The following activities are designed to help you and your students explore our presidential election and history through mathematical investigations involving number sense and operations, data collection, patterns and algebra, measurement, communication, and representation. ▲

WEEKLY ACTIVITIES

PRESIDENTIAL PARADE K–2 — NOVEMBER 2008

3 **Compare coins.** Ask an adult for a penny, nickel, dime, quarter, and half dollar that you can closely compare. Identify the presidents on the coins. What is the same about the coins? What is different? Place the coins in order from smallest to largest in size. Make a rubbing of them in this order by placing a thin sheet of paper over them and rubbing the paper with the edge of a crayon. Now order the coins by their value and create a new rubbing.

10 **In an election** for class president, 20 students use two colors of beans to represent their votes. A red bean represents a vote for candidate 1, and a blue bean is a vote for candidate 2. One possible outcome of the election is 20 red and 0 blue. Another possible outcome is 19 red and 1 blue. What other outcomes are possible? Work with a partner to draw and complete a table. If you find any patterns that could help you find all possible combinations, write about them. How many possible combinations of votes exist? How do you know whether you have found them all? When you finish, compare your table with another pair's.

Candidate 1 (red)	Candidate 2 (blue)	Number Sentence
20	0	20 + 0 = 20
19	1	19 + 1 = 20

17 **Create a pattern** with colors: Make a cube "train" using 10 red, white, and blue cubes. Record your pattern on a hundreds chart. Color square 1 on the chart the color of the first cube in your train, square 2 the color of the second cube in your train, and so on until you have recorded the colors of all 10 cubes. If you continued the pattern you used to create your cube train, what would be the color of the 12th, 15th, and 20th cubes? How do you know? Ask an adult to help you roll a sheet of paper into a tube. The tube should fit over your train so that it hides it from view. Then show a friend only the first three cubes of your train, leaving the rest of the train hidden from view in the tube. Ask your friend to study the first three cubes sticking out from the tube and predict the color of the next cube. To see if your friend predicted correctly, slide the train out far enough to show the next cube. Did your friend predict the correct color? Continue until your friend can use your pattern to correctly predict the next color in your pattern.

24 **Compare votes** between your class and the nation. Which national presidential candidate would win in your class? Before your class voting day, ask your classmates to gather election information from newspapers, magazines, and the Internet and share it in class. List the presidential candidates on ballots that each student can complete privately. Once everyone has voted, count the votes and share the information with your class. How close was your class election? How many votes were cast? How many votes did each candidate receive? What is the difference between the number of votes the winner and loser received? Compare the outcome of your class's balloting with the national election results.

Thomas C. Hoopingarner is the mathematics curriculum and instructional specialist for Sunnyside Unified School District in Tucson, Arizona. The other team members teach at Pueblo Gardens Elementary School in Tucson Unified School District: Christine Gallego teaches first grade; Dora Saldamando teaches fourth grade; Grace Tapia-Beltrán teaches fifth grade; and Karolyn Williams teaches a third-fourth multiage class.

Edited by **Dana Islas**, *who teaches kindergarten at Pueblo Gardens Elementary School in Tucson, Arizona. Readers are encouraged to submit problems to be considered for future "Math by the Month" columns to the department editor at* **dana.islas@tusd1.org**. *Receipt of the problems will not be acknowledged; however, those selected for publication will be credited to the author. Readers are also invited to submit creative solutions and adapted problems to* **tcm@nctm.org** *for potential publication in "Reader's Exchange." Please include "Reader's Exchange" in the subject line. Because of space limitations, responses beyond the 250-word limit are subject to abridgment. Submissions are also edited for style and content.*

PRESIDENTIAL PARADE 3–4

NOVEMBER 2008

Create a timeline. Use the information about our most recent presidents and their dates in office to make a timeline. List them in order from the one with the shortest term to the one with the longest term. Which president served the shortest term? The longest term? Which presidents had terms of the same length?

Richard Nixon (37th)	January 20, 1969–August 9, 1974
Gerald Ford (38th)	August 9, 1974–January 20, 1977
Jimmy Carter (39th)	January 20, 1977–January 20, 1981
Ronald Reagan (40th)	January 20, 1981–January 20, 1989
George H. W. Bush (41st)	January 20, 1989–January 20, 1993
Bill Clinton (42nd)	January 20, 1993–January 20, 2001
George W. Bush (43rd)	January 20, 2001–January 20, 2009

The White House building site and the architect who designed the home for all the presidents were chosen by George Washington, but he was the only one who did not live in the White House. It was first occupied by President John Adams in 1800. How many years have U.S. presidents lived in the White House? Show how you know. Share your thinking with a classmate.

The longest presidency in U.S. history was Franklin D. Roosevelt's (from March 4, 1933, until April 12, 1945). He served three full terms and died early in his fourth term. How many days did he hold office? Record your thinking.

The Electoral College elects U.S. presidents. Based on population, each state gets a particular number of votes in the Electoral College (see the chart below for examples). Put the following 25 states in order from least to greatest according to the number of Electoral College votes they receive. What different combinations would a candidate need to win the election if winning requires one vote more than half?

Alabama	9	Indiana	11	Minnesota	10	New York	31	Rhode Island	4
Alaska	3	Kansas	6	Mississippi	6	North Carolina	15	South Carolina	8
California	55	Louisiana	9	Nebraska	5	Ohio	20	Texas	34
D.C.	3	Maryland	10	New Jersey	15	Oregon	7	Washington	11
Florida	27	Michigan	17	New Mexico	5	Pennsylvania	21	West Virginia	5

WEEKLY ACTIVITIES

PRESIDENTIAL PARADE: 5–6

NOVEMBER 2008

An annual salary of $25,000 is what George Washington made after he was voted into office in 1789. As of 2008, George W. Bush, our 43rd president, had an annual salary of $400,000 + $50,000 for expenses. For how many years has the United States been governed by a president? What is the difference between the annual salaries of the two Georges? Using your data, what do you notice happening? What is the percentage of increase of the presidential salary? If this trend continues, predict the annual salary for future presidents in 50 years, 100 years, and 200 years. Explain your thinking.

Presidential squares! A square number is produced when a number is multiplied by itself. A square number can be represented by a square array. When considering our presidents in order, which of our presidents were square presidents? Which presidents held office during a year that was a square number? What number will be our next square president? What is the earliest year that our next square president could hold office? What is the latest year that our next square president could hold office?

What's in a name? If only 42 people have been president, explain how it is that George W. Bush is our 43rd president.* George has been the name of 3 of our presidents. The most popular presidential name has been James (6 presidents). John and William tie (each had 4). Franklin was the name of 2 presidents. Using this information, find the fraction to represent all the presidents whose first name was James, Franklin, William, George, or John. How do you know? What percentage is this? Find the fraction of presidents not named George, James, John, William, or Franklin. (*President Cleveland served two nonconsecutive terms.)

Presidential faces appear on American coins and paper currency. Investigate and then make a list of the presidential images that appear on the different pieces of American currency from the penny to the fifty-dollar bill. Which coin and bill combinations could you use to make $158.93? How many different presidential faces did you use? Can you use all the presidents to create that amount? Now find the fewest number of coins and bills to make $158.93. Work with a friend to find other combinations.

Mathematics Thinkers and

Everyone Needs a Positive Story

All children—regardless of personal characteristics, background, or physical challenges—deserve an opportunity to learn mathematics to the best of their ability. This declaration is loud and clear, and NCTM's mission of providing sound mathematics instruction and holding high expectations for *all* children (NCTM 2000) is important in our mathematical world. Mathematics permeates our daily lives from the moment we awaken to the moment we fall asleep. In many ways, our mathematical successes and failures get woven into the life stories that make us who we are.

Unfortunately, for some students, mathematics is more a burden than a joy. A sizable number of students struggle in mathematics, and their stories often do not have happy endings. Second only to reading, mathematics is the domain in which many students encounter great difficulty (Mazzocco 2001), which may arise from a lack of prerequisite knowledge, poor instruction, or a learning disability. Children with learning disabilities in mathematics often have perceptual or memory problems, and they tend to use poor strategies (Hallahan and Kauffman 2003; Van de Walle 2007). If such problems are not addressed early with sound instruction, students may come to believe that they are not capable of succeeding in mathematics. Students' ideas as to why they succeed or fail are called *attributions*, and these fuel their motivation and drive.

By Debby Zambo

Debby Zambo, Debby.Zambo@asu.edu, is an assistant professor in the College of Teacher Education and Leadership at Arizona State University in Phoenix. Zambo teaches educational psychology courses and is interested in the emotions children feel in relation to learning in the content areas.

Attribution Theory

Attribution theory is a psychological theory that explains motivation in terms of attributions—explanations, justifications, and excuses—that individuals make (Weiner 1994, 2000). According to this theory, most attributions can be characterized along these three dimensions:

1. Locus—Is the cause of success or failure internal or external?
2. Stability—Is the cause of success or failure likely to remain the same or change in the near future?
3. Control—Can the cause of success or failure be controlled?

In many ways, attributions are the perfect example of knowledge construction in action. Children interpret new events in terms of their existing knowledge and beliefs about themselves and develop what seems like a reasonable explanation of what has happened to them. Because attributions are self constructed, they may or may not be true (Schunk 2004; Weiner 1994, 2000). False attributions can still provide a window into how children feel about themselves as learners in a domain. If teachers listen carefully to the stories children tell about themselves and mathematics, they will hear the attributions children make.

This insight can help teachers better understand students' motivation and behavior. For example, if a child succeeds in mathematics, her story will likely contain a happy tale and be full of positive attributions as to why she is able to succeed. Holding a happy story with positive attributions will encourage her to set higher goals for herself because she believes that she has the capacity to succeed (her ability is inside) and that her ability will remain

(it is stable). When obstacles to learning arise, this child likely will seek assistance, work hard, and try new strategies (she has control). Success helps students develop positive attributions that, in turn, help them become capable characters in their life stories.

In contrast, children who have not succeeded in mathematics often attribute their failure to themselves (believing they do not have what it takes to succeed). They believe that no matter what they do, mathematics will always be difficult for them (it is a stable trait) and that there is nothing they can do to improve (failure is uncontrollable). In their minds, doing mathematics leads to little joy and causes feelings of frustration, shame, and guilt. Children who feel this way set low goals for themselves, put forth little effort to practice, and fall further and further behind (Ames and Lau 1982). Just as positive attributions influence behavior and motivation, so do negative attributions, persuading children to perceive themselves as incapable characters in their life stories (Schank 1990).

As a teacher and researcher, I believe that teachers can learn much about children and their concepts about mathematics and themselves by listening to their stories and the attributions they contain. I also believe that just as teachers can learn from children's stories, children can learn from the stories of others. Listening to a story in a carefully chosen picture book, identifying with a character or characters, and hearing what they attribute their success and failures to can help children develop more positive ways of thinking, which, in turn, may influence their motivation and behavior in a classroom (Schiro 2004; Sipe 2001). Providing sound instruction and helping children develop positive attributions can assist their development of a positive story about themselves and mathematics, fueling their motivation and making NCTM's ideal of opportunity a reality for every child. I base my claims on my experience in working with children with mathematics disabilities. Each scenario that follows provides specific examples of how my ideas can be transformed into practice. See the **appendix** on page 234 for additional storybooks and applications.

Four Vignettes

Justin's story

Perceptual difficulties in mathematics manifest themselves in rotations, inversions, and distortions of symbols, signs, and words. Children with perceptual difficulties mix up operational signs such as + and × and confuse numbers such as 6 and 9. Many young children display these behaviors, but children with perceptual difficulties do not outgrow this

confusion. Even in the upper grades, they experience problems in detecting operations, placing numbers on a line, and putting numbers in columns and rows (Thornton et al. 1983). Because of problems with alignment, place value is difficult for them, and basic algorithms are challenging. To understand how perceptual factors influence attributions and how negative attributions may be built into a child's personal story, I introduce Justin (all names are pseudonyms), a third-grade student.

The school year has just begun, and the school psychologist recently finished testing Justin because of his lack of mathematics progress last year. Standardized testing reveals that Justin's intelligence is in the average range, but a significant discrepancy exists between his verbal reasoning and his spatial ability. Justin has tremendous difficulty writing numbers on paper and lining up numbers in columns and rows. Before testing confirmed this weakness, Justin's teachers considered him careless and inattentive. In the past, he received many reprimands and failing grades as a result of their misconceptions.

When Justin arrives in Ms. Sally's class, she wants to learn all she can about the struggles he has with mathematics. She reviews the testing results in his file and decides that if she is going to help Justin, she must listen to his attributions, help him set realistic goals, and show him strategies he can use to compensate for his perceptual difficulties. To begin to understand Justin, Ms. Sally listens carefully to his story and analyzes the attributions he makes. As Justin works on his mathematics assignments, she determines where he places blame for his struggles (inside himself or outside?), the stability of his beliefs (will things always be this way, or can they change?), and the controllability of his situation (can he control the cause of his struggles?). Ms. Sally hears him make comments such as, "I can't do this; math is hard. I'm just dumb and will always be this way." It is evident that Justin perceives himself to have a low aptitude in mathematics, believes there is nothing he can do to change this, and, as a result, sets low goals for himself. These negative beliefs are not good and are taking their toll. Although he is only in third grade, Justin is becoming more and more frustrated with mathematics, and he is not getting the practice he needs. Armed with insight, Ms. Sally begins her work.

Ms. Sally and Justin sit down and discuss the results of his tests. Ms. Sally explains how Justin's trouble perceiving operational signs and numbers leads to his difficulties. Ms. Sally tells Justin that she believes in him and conveys high expectations for him in mathematics this year. She explains that they will be working all year on solving problems and that she will help him develop strategies he can use to get unruly numbers under control. To make her ideas more concrete, Ms. Sally reads *One Hundred Hungry Ants* (Pinczes 1993) to Justin. She chose this book because it reveals the importance and necessity of lining up in an orderly fashion and being in control.

Justin likes the story about one hundred hungry ants marching single file to a picnic and discovering that moving in one long row is too slow to get to the picnic on time. To speed things up, one ant decides to organize the others. He gets them to march side by side and in rows of four, five, and then ten. On pages where the ants are rearranging themselves, utter chaos ensues; and Ms. Sally uses these pages to help Justin understand the importance of using ordered rows, columns, and lines: "Look at how paying attention to where everyone should go helps the ants move more smoothly. Think of numbers on a page as the ants in this story."

To help Justin gain a sense of control over numbers and operations, Ms. Sally tells him to "think of yourself as the ant in the story who orders the others to help them get to their goal." To aid Justin in his battle, she has him solve addition problems by building one-, two-, and three-digit numbers with base-ten blocks. She gives him a place-value mat to keep the blocks aligned and provides him with a template when he transfers numbers onto paper. Ms. Sally also writes the operation signs in color to help Justin focus on the operation he needs to perform.

Ms. Sally helps Justin work on positive affirmations to say to himself as he works, especially when the going gets tough:

- I can do mathematics with hard work and a strategy.
- I am like the little ant who keeps numbers in a line.
- The more I practice subtracting, the better I get.

Thinking more clearly about himself and mathematics has led Justin to set some realistic goals that focus on self-improvement instead of comparison to others. To determine if his goals are being met, Justin now monitors his own progress with a simple bar graph. In four short months, Justin's grades have

Table 1

Mathematical Difficulties and Strategies

Mathematical Difficulties	Strategies to Use
Perceptual	
Visual Problems with column alignment, place value, operand adherence Auditory Problems with picking up auditory information in noisy rooms and difficulty with tuning out noises	Visual Provide worksheets with centimeter grids and templates for traditional algorithms. Use real models when possible (i.e., base-ten blocks). Assign a buddy to check work. Keep workspace free of clutter. Auditory Seat child in front of room. Keep room quiet when concentration is needed. Clearly provide information and repeat key facts and ideas.
Memory	
Short-Term Trouble recalling what was just learned Can recall new information for a period of time but loses it rapidly Long-Term No immediate difficulty but later difficulty retrieving information Poor automaticity of basic facts (recall is slow and inaccurate) Poor quantitative knowledge, number concepts, counting, and arithmetic skills	Short-Term Present facts and instructions slowly and one at a time. Check for understanding: Ask child to restate directions and have child listen to peers restating directions and ideas. Provide a special quiet space when child must work alone. Use external memory aids. Long-Term When giving problems verbally, be sure child has access to a written version. Be patient for answers; allow ample wait time. Use games to exercise memory skills for basic facts. Provide external memory aids (e.g., computers, timers, calculators). Teach children how to use these tools; encourage and reinforce them when they do. Encourage children to use mathematics across domains and in the real world.
Integrative	
Difficulty with abstract ideas, concepts, and connections Makes many errors Relies on immature strategies Limited flexibility due to limited strategy knowledge	Provide explicit training in strategies. Encourage child to restate problems in own words. Have student express ideas in verbal and written form. Frequently require explanations and justifications to heighten awareness of new ideas and make connections. Allow time for repetition and practice of new skills. Provide opportunities for child to teach concepts to peers. Use multiple presentations (illustrations, words, drawings, concrete objects, actions).

gone from failing to passing, and his frustration has lessened tremendously! Justin's story is changing and becoming more positive. Instead of being in an uncontrollable situation, Justin now works hard to get unruly numbers under control. Other strategies Ms. Sally uses with Justin are listed in **table 1**, which also contains ideas for working with children with auditory perception problems.

Karen's story

In addition to perceptual difficulties, students who struggle with mathematics may also have short-

Figure 1

Attributions survey

Mark how much you agree or disagree with each of the following statements by circling one choice.

SA	A	D	SD
strongly agree	agree	disagree	strongly disagree

Children who fail at mathematics will always fail at mathematics.	SA	A	D	SD
Children who are good at mathematics are born that way.	SA	A	D	SD
I control my successes in mathematics.	SA	A	D	SD
Even if someone fails at mathematics, with hard work she can change things.	SA	A	D	SD
If I work hard, I will succeed in mathematics.	SA	A	D	SD
Children who are good at mathematics work hard to get that way.	SA	A	D	SD
When it comes to mathematics, I cannot control what happens.	SA	A	D	SD
I succeed at mathematics because of strategies I use.	SA	A	D	SD
My teacher is the reason I fail at mathematics.	SA	A	D	SD
When it comes to mathematics, I'm just unlucky.	SA	A	D	SD
If someone is bad at mathematics, they will always be this way.	SA	A	D	SD
When it comes to mathematics, I feel out of control.	SA	A	D	SD

term memory deficits. Short-term memory is the system that stores information for fifteen to twenty seconds, so children with this type of memory problem have difficulty recalling information even a few seconds after they see or hear it (Baddely 2001). When copying from the board, these children forget what they are doing and never complete the task. When working on mathematics problems, they lose track of what they need to remember from the beginning to the end (Thornton et al. 1983).

Karen is a fifth-grade student who shows classic symptoms of short-term memory deficit. She forgets what she is doing midway through mathematics problems, which is becoming an issue because Karen's classmates are laughing at her forgetfulness and choosing not to work with her in cooperative groups. Karen's forgetfulness is a problem outside of school, too. Her mother is an accountant, and Karen wants to grow up to be like her. Unfortunately, Karen's dream will not become a reality if her ability and attitude do not improve.

When asked to tell her story about mathematics and herself, Karen tells one of humiliation and withered dreams. Karen attributes her failure to her lack of ability and feels there is nothing she can do to make things better. Karen wants to be good at mathematics so badly that she is becoming increasingly restless and unhappy with herself. Recently, when it was mathematics time, she hit her fists against her desk and began crying. Needless to say, her teacher, Mr. Hector, is very concerned. He sees a girl in distress and knows that if he is going to help her, he must listen to her story to hear the attributions she is making. He must provide sound instruction to help Karen change her beliefs. To gain a clearer picture, Mr. Hector develops a brief survey of twelve questions (see **fig. 1**) and asks Karen to complete it during mathematics time.

After Karen completes the survey, Mr. Hector uses it as a springboard for conversation. Mr. Hector and Karen sit down and discuss how she feels about mathematics. Mr. Hector takes notes as she speaks and notices that several themes keep surfacing. Karen consistently blames her struggles in mathematics on her poor memory (an internal flaw) and believes there is nothing she can do to make things better (it is uncontrollable). Things will always be this way (it is stable). Karen dislikes who she is because she feels she will never achieve her future goals. Armed with this insight, Mr. Hector develops a plan that begins with reading *The Greedy Triangle* (Burns 1994). Mr. Hector chooses this book because he feels it contains an important message for Karen to learn.

In the beginning of the book, a triangle is happy just being itself. The triangle likes its shape because it can support roofs, make music, and become a slice of pie. But one day, the triangle feels dissatisfied with being a triangle and doing the same old things. It

wants excitement and more from life, so it goes to the shapeshifter to be transformed into a more interesting shape. The shapeshifter turns the triangle into a quadrilateral, and this new shape enables many new experiences. As a quadrilateral, the former triangle becomes a baseball diamond, computer screen, and picture frame. After a period of time, the triangle-turned-quadrilateral once again becomes unhappy. The same thing happens no matter what shape (heptagon, pentagon, decagon) the shapeshifter changes the triangle into. With each transformation, the triangle is happy for a while but eventually becomes restless, dissatisfied, and unhappy with whatever shape it becomes. After many shapeshifts, the triangle loses its angles and becomes circular. Not having a side to stand on, the triangle has a difficult time doing its favorite things. In the end, the triangle begs the shapeshifter to turn it back to its original shape. When the shapeshifter does, the triangle is happy again. It realizes it is happiest being itself.

Mr. Hector and Karen read the story several times so she will remember the plot and internalize its moral. Whenever Karen gets frustrated and wishes she were someone else, Mr. Hector reminds her of the greedy triangle. Mr. Hector also works with Karen to help her learn how to circumvent her limited memory space. Karen's mathematics instruction consists of a problem-based approach, so when Mr. Hector gives assignments, he makes a conscious effort to speak slowly and clearly so Karen can process what she must complete. In their classroom, Karen now has a quiet area to work free from distractions. She carries a mathematics notebook in which she draws pictures of concepts and writes down important information and facts she needs to recall. Whenever she works on problems, Karen uses manipulatives and checks her work with a calculator. Multiple strategies and cues are being used to help Karen. Some of these are listed in **table 1** (short-term memory).

These strategies are helping lessen Karen's anxiety and frustrations. Karen is learning to be happy with herself despite her limited memory. Her grade has risen from a D to a C. Karen has even been heard joking with her classmates that she is a triangle in a very round world. But that's okay with Karen because she knows that the world needs children of all shapes.

José's story

Another type of memory problem that children who struggle in mathematics often have involves their long-term memory. Children with this type of difficulty initially show no immediate problem with learning new concepts. These students have problems when they try to recall information after time has passed. Mastery of basic facts would be a hallmark for these children because they just can never find the facts in their long-term memory (Van de Walle 2007).

José is a sixth-grade student with a long-term memory deficit, and, as a result, his performance in mathematics is uneven. He can remember what he learned for short periods of time, but after extended periods, he simply forgets what he previously knew. Because of his forgetfulness, much of José's time in mathematics has been devoted to learning basic facts over and over again. As a result, José thinks mathematics is nothing more than remembering facts. He sees no connection to real life or to his own personal needs. José is getting pretty bored. Whenever time for mathematics class approaches, José begins to withdraw. He feigns illness and asks to go to the nurse or simply puts down his head. Seeing a child who is struggling, his teacher, Miss Susie, decides to intervene. Wanting to understand how José feels and knowing that he loves to draw, Miss Susie asks José to draw a picture of himself doing mathematics. José gladly complies and draws a small character in a dark cave full of large numbers and operational signs. When Miss Susie asks José about his drawing, he explains that it shows his mind as a dark cave in which facts and answers seem to get lost. As they talk, it becomes clear that José feels incapable of success in mathematics, and this perception is causing him to feel hopeless and out of control.

To help José develop a positive story and better attributions, Miss Susie has him read *Math Curse* (Scieszka and Smith 1995). She chooses this book because it tells the perils of a child who lives in a world full of mathematics problems. When the child wakes up in the morning, getting dressed becomes an issue of knowing how much time to allot. Eating breakfast involves measurement. Lunch includes a fraction problem. Each school activity demands mathematics skills, too: English class is full of word problems, and P. E. requires solving many distance and measurement problems. Trapped in a world full of mathematics, the child is willing in the end to accept the facts and take on the challenge of

mathematics throughout the day. Miss Susie wants José to understand that mathematics is part of all our lives and that, regardless of his challenges, he can overcome them like the child in the story does.

José's parents and Miss Susie work together to help José create a more positive view. Miss Susie is enthusiastic whenever she teaches a lesson, and she models her use of mathematics throughout the school day. Attendance becomes a subtraction problem, and lunch orders become problems of estimation. Instead of focusing on mere recall, Miss Susie's lessons are all based on problem solving. Students work on applied problems and real-world mathematics. To help José do this, Miss Susie has taught him numerous strategies, and José now carries a calculator wherever he goes. José's parents are helping him see the use of mathematics outside of school. José has taken a particular interest in how his father uses mathematics in his construction job. José's parents have told him they hold high expectations that some day he will become a construction boss. As a result of these interventions, José's story of himself as a learner and doer of mathematics is improving. José now attributes his success to a better understanding of the practicality of mathematics and the use of external sources to extend his limited long-term memory space. His esteem is rising, as are his grades. More strategies used with José are presented in **table 1** (long-term memory).

Devonne's story

The final student we will meet is Devonne, a sixth-grade girl who can solve problems that rely on basic facts but falls short when higher-level thinking involving abstract concepts are needed. Devonne is a concrete thinker whose ideas come in bits and pieces instead of integrated wholes. When she works on problems, Devonne relies on immature strategies. Unfortunately, these challenges are causing great difficulties for Devonne in mathematics. When asked to tell her story of mathematics and herself, Devonne conveys exasperation, sadness, and loss. In the lower grades, she was good at mathematics, but now that the subject is getting more complicated, she attributes her failures to a weakness in herself—a weakness unlikely to change. In Devonne's mind, she was once a starring character, but now that mathematics has become more demanding, she no longer plays a main role. Her poor self-concept is causing Devonne to give up when it comes to mathematics. At this point in her development, Devonne has learned helplessness and is completely closed to learning new ideas. To help Devonne regain pride, her teacher, Mr. Zavala, reads *The King's Chessboard* (Birch 1998).

Mr. Zavala chooses this book because it is about a wise old man who does a favor for a king just because he sees the need. Unfortunately, the king perceives the favor as recognition of his weakness. Because he is the ruler, the king does not want to be in debt to anyone and insists that the man accept a reward. So, the man asks that a grain of rice be placed on his chessboard and that every day the grain be double the amount of the day before. The first day, one grain of rice is placed on the chessboard, two the next day, four on the third day, eight on the fourth day, and so on. The story and illustrations show how rapidly the amount of rice increases. By the twenty-fourth day, four men carry a heavy sack of rice weighing over one hundred pounds. By the thirty-sixth day, sixteen wagons, each holding a ton of rice, are hauling away the man's reward. In the end, the royal mathematicians note that there is not enough rice in the whole world to carry out the man's request. Realizing this, the king goes to the man and admits that he cannot fulfill his promise. The man says it is not necessary because he is happy simply having done the favor for the king. He was satisfied without the rice and only took it at the king's insistence. He helps the king realize that even kings are human and must sometimes acknowledge their weakness. The king comes to understand that pride can make a fool of anyone, even a great king like himself.

After reading the book aloud, Mr. Zavala works to help Devonne understand that when it comes to mathematics, she sometimes needs help. Just as the man wanted to help the king, other people are available to provide support for Devonne. Mr. Zavala also provides instruction and strategies for Devonne by—

- encouraging her to restate problems in her own words;
- providing plenty of wait time to allow her to think;
- asking her to express her ideas in verbal, written, and pictorial form;
- requiring explanations and justifications of ideas;

- allowing time and opportunities for repetition and practice of new skills in unique ways (computers, performances, rhythm and rhyme);
- providing opportunities for Devonne to teach concepts to peers;
- using multiple presentations when he teaches a lesson (e.g., illustrations, words, drawings, concrete objects, action); and
- connecting mathematics to other classes, such as social studies and reading.

Used consistently, these strategies are helping Devonne's story change. She no longer sees herself as helpless but as capable. Devonne now volunteers in a second-grade mathematics class. She is very proud of being a peer tutor, and when she talks about herself and mathematics, she often refers to *The King's Chessboard*. Devonne claims that she is like the king in the story. She knows her limitations and is clever enough both to realize when she needs assistance and to ask for it.

Conclusion

NCTM's vision that every child can succeed in mathematics is profound. In this article, I have provided scenarios that reveal how this ideal can become a reality for students with specific disabilities in mathematics. When teachers listen to stories and the attributions they contain, they gain insight into the motivation and behavior of children. When teachers hear negative attributions, there is much they can do. They can model enthusiasm, hold high expectations for every child, teach children strategies, help children set realistic goals, and use assessment to show progress toward those goals. However, just as important is a classroom where a problem-solving focus is used. When students understand that mathematics is all around, makes sense, and is applicable to their lives, they form a new view of this important domain.

Children who struggle in mathematics face challenges just as important as children with any other type of disability (Hale and Fiorello 2004). Mathematics is such an important part of today's world that children, from an early age, recognize that mastering the subject is essential for success. When children feel they have no control over mathematics, become frustrated and sad about their lack of progress, and lose their future dreams, mathematics destroys self-worth and self-esteem. Fortunately, just as teachers learn from students' stories, students can also listen and learn from the stories others tell. Listening to a story in a well-chosen storybook and identifying with its characters can help children who struggle gain a better image of themselves. Children like Justin, Karen, José, and Devonne can change their attributions and become stars in the stories of their lives. All students need a positive story of themselves as doers and thinkers of mathematics.

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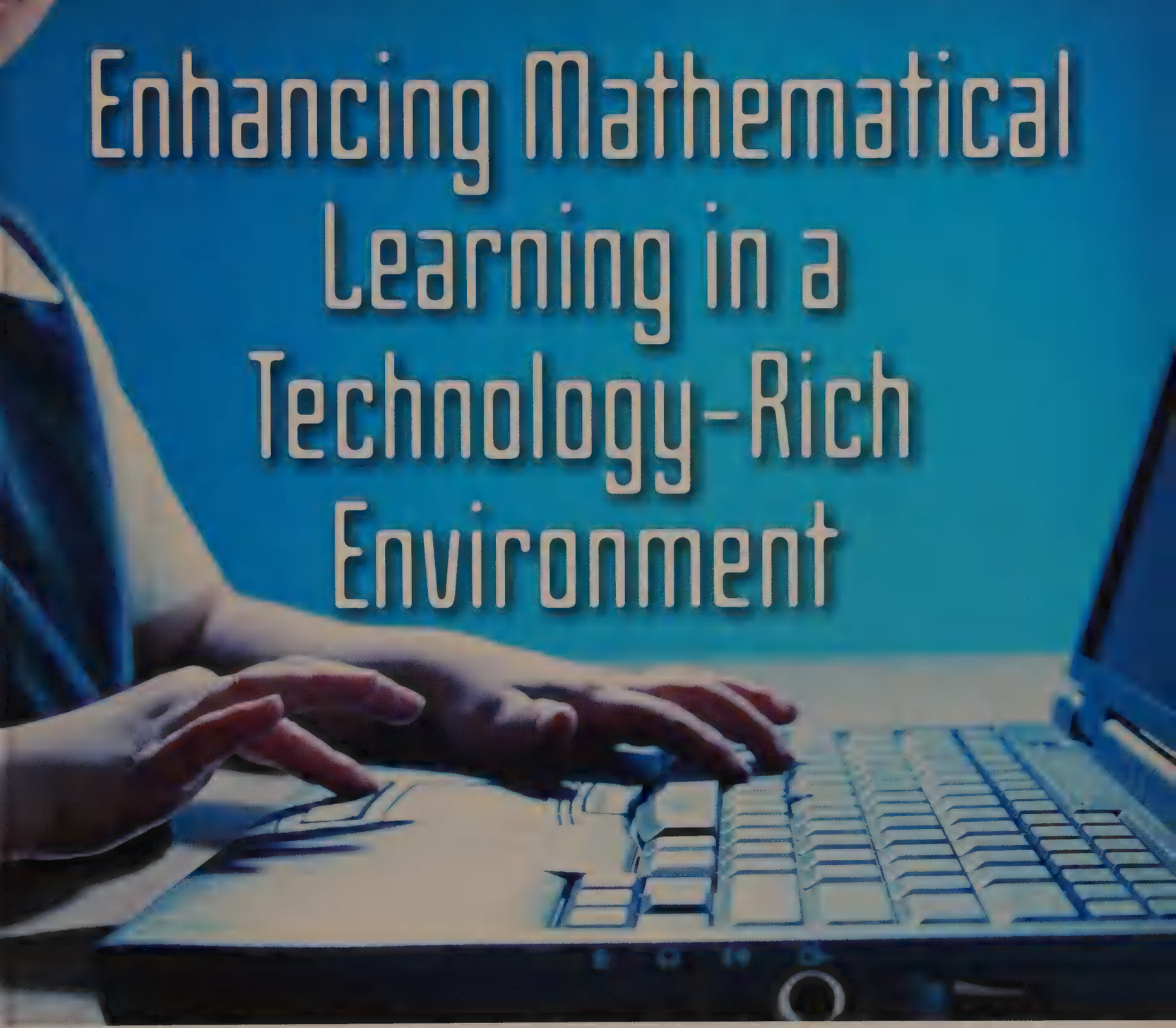
Appendix

Book Titles, Plots, and Mathematical and Affect Components

Book Titles	Book Plots and Mathematical and Affect Components
Grandfather Tang's Story: A Tale Told with Tangrams (Tompert 1990)	Grandfather Tang tells of two fox fairies that transform themselves into various animals. The story is about loyalty and bravery in the face of danger. As children listen, they can use tangram pieces to change the fox fairies into the various beasts. Working with tangram pieces is a good activity for children with visual-spatial difficulties.
Seventeen Kings and Forty-two Elephants (Mahy 1987)	Seventeen elephant-riding kings set out on a journey and encounter new creatures along the way. This story can be used as a basis for various math problems. It also can help children understand that mathematics is a journey full of challenges, fun, and new insight. Appropriate for children with all types of mathematics difficulties.
Count on Your Fingers African Style (Zaslavsky1980)	This wonderful ethnic tale poses different finger-counting strategies from various African communities like the Kamila, Taita, and Masai. A good book to help children think about different cultures and the mathematics they use. Appropriate for children with all types of mathemtics difficulties.
No Fair (Holtzman 1997)	Two children play games and experience the concept of being fair. This is an easy reader, written to help children gain a positive view of mathematics and learn a moral lesson as well. Appropriate for children with all types of mathematics difficulties.
The Shape of Things (Dodds 1996)	This book helps children understand shapes because it shows shapes in the world. Asking children to find and trace real-world shapes is a great hands-on experience for students with visual-spatial difficulties.
Sir Cumference and the First Round Table: A Math Adventure (Neuschwander 1997)	In the land of Camelot, people known as the Circumscribers are to be invaded. King Arthur calls upon his men to find a solution, but the table they sit at is rectangular, and this causes great communication difficulties. In the end, a circular table is built to help the knights discover that a peaceful solution is best. The book reveals problem solving at its best and can be used to encourage children to solve their own problems. Great for children with integrative difficulties.
Mind-Stretching Math Riddles: Math for All Seasons (Tang 2002)	Math riddles are all around, and this book poses many problems that can be seen every day. It also gives hints and strategies for solving the problems posed. A good book for rehearsing facts for students with memory difficulties, it also sharpens auditory skills for students with this type of math difficulty.
One Good Horse: A Cowpuncher's Counting Book (Scott 1990)	A tale introducing readers to life on a cattle ranch, the final, panoramic view of the countryside allows readers to retrace the journey of two buckaroos and the scenes along the way. It also helps children understand that mathematics can be like a journey full of the splendor and beauty of the wild, wild West. Appropriate for children with all types of math difficulties.

Enhancing Mathematical Learning in a Technology-Rich Environment



In *Principles and Standards for School Mathematics* (NCTM 2000), the Technology Principle asserts: "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p. 24). More specifically, a technology-rich environment for mathematical learning influences five critical features of the classroom (Hiebert et al. 1997): the nature of classroom tasks, the mathematical tool as learning support, the role of the teacher, the social culture of the classroom, and equity and accessibility. An essential question when working in a technology-rich mathematics environment is how technology can be used (appropriately) to enhance the teaching and learning of mathematics.

This article describes teachers working collaboratively in a technology-rich environment to plan mathematics lessons that address the needs of their diverse students, in particular, English Language Learners (ELLs) and students with special needs. Through classroom examples, we discuss how a technology-rich learning environment influences a classroom's critical features. Moreover, we define

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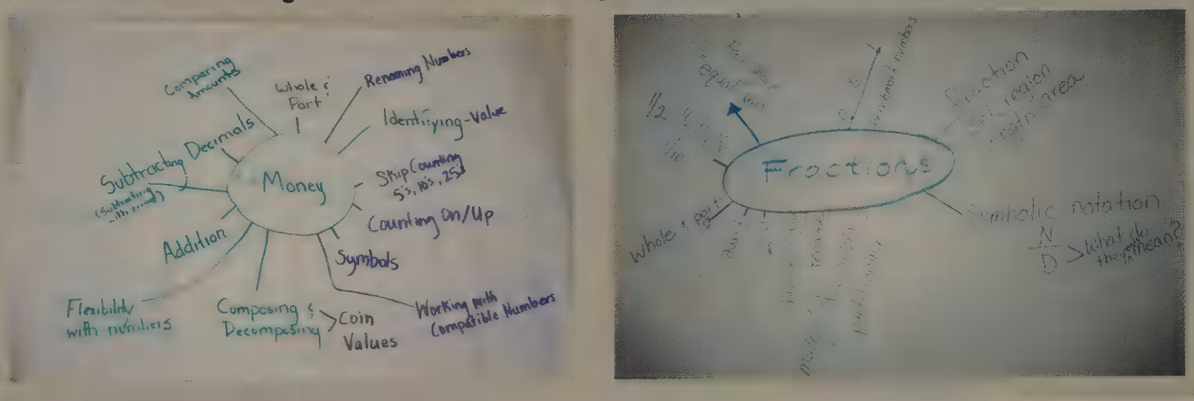
Figure 1

Student work showing mathematical knowledge mapping



unique technological properties that amplify opportunities for extending mathematical thinking.

The participating school is a Title I elementary school in a major metropolitan area with approximately 600 students: 51 percent Hispanics, 24 percent Asians, 16 percent Caucasians, 3 percent African Americans, and 6 percent others. More than 50 percent of the student population receive free or reduced lunch, 44 percent receive services for English for Speakers of Other Languages, and 49 percent are identified as limited in English proficiency. On the basis of student need at this school, an important school initiative sought to incorporate nonlinguistic representations into students' daily activities to help build their background knowledge. Representing knowledge (nonlinguistic representation) is one of nine categories of instructional strategies proven to advance student achievement (Marzano et al. 2001). An essential part of the initiative promotes the integration of technology in all content areas to provide the diverse student population with interactive, visual, and multimedia tools. To enact the initiative, teachers in each grade level collaborated with a university mathematics educator to design lessons incorporating technology tools and nonlinguistic representations to engage, motivate, and respond to the needs of a diverse group of learners.

To begin planning collaborative lessons, teachers identified a mathematics area at their grade level that presented a teaching and learning challenge. For one of the highlighted classroom examples, we will share a third-grade money lesson that was part of the measurement strand. The lesson objective was to count a collection of mixed coins and then find and record a variety of ways to show a given amount of money. The future building-block target was to make change for amounts up to five dollars.

The second featured lesson was a fourth-grade fractions lesson with the objective of renaming fractions and finding equivalent fractions. This lesson was a prerequisite to adding and subtracting with unlike denominators using models. Once teachers identified the lesson objectives, the lesson-planning team worked together to construct a mathematics knowledge map outlining the key components of both interrelated prerequisite and future knowledge mathematics concepts building blocks. Additionally, they identified effective representations or models to teach each concept (see **fig. 1**).

In these two lessons, the planning team included third- and fourth-grade classroom teachers, the special education teacher assigned to those grade levels, the mathematics specialists, and the university mathematics educator. Teachers ranged from novices to experienced teachers with varied strengths and weaknesses in the areas of technology integration, mathematical content knowledge, and teaching practice—a range that provided opportunities for all participants to develop deeper pedagogical content or technology knowledge.

Creating Technology-Rich Mathematics Learning Environments

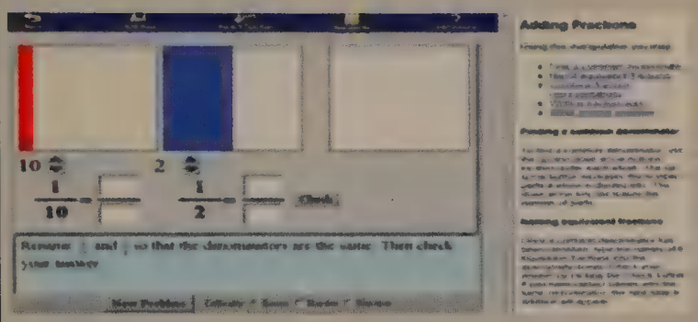
When creating a technology-rich mathematical learning environment, teachers must understand what using technology "appropriately" (Garofalo et al. 2000, p. 67) means when integrated into teaching mathematics:

1. Introducing technology in context
2. Addressing worthwhile mathematics with appropriate pedagogy
3. Taking advantage of technology

Figure 2

An Advancing Mathematical Thinking planning sheet

Advancing Mathematics Thinking with the Use of Technology

Web site	Addition of Fractions		
Math Strand	Number and Operations	Grade Level	Fourth Grade
Description of mathematical concept (NCTM)			

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Analysis of Mathematical Representations and Models	
X Concept tutorial/skill practice _____ Investigation/problem solving _____ Open exploration	
Representation ▪ Create and use representations to organize, record, and communicate mathematical ideas ▪ Select, apply, and translate among mathematical representations to solve patterns ▪ Use representations to model and interpret physical, social, and mathematical phenomena	Connected pictorial and numerical representations
Communication ▪ Organize and consolidate their mathematical thinking through communication ▪ Communicate their mathematical thinking coherently and clearly to peers, teachers, and others ▪ Analyze and evaluate the mathematical thinking and strategies of others ▪ Use the language of mathematics to express mathematical ideas precisely	Peer talk: Discuss applet's function and the mathematics process (step by step)
Connections ▪ Recognize and use connections among mathematical ideas • Understand how mathematical ideas interconnect and build on one another to produce a coherent whole • Recognize and apply mathematics in contexts outside of mathematics	Connecting renaming before combining; finding common multiples with arrows breaking pieces
Reasoning and Proof ▪ Recognize reasoning and proof as fundamental aspects of mathematics ▪ Make and investigate mathematical conjectures ▪ Develop and evaluate mathematical arguments and proofs ▪ Select and use various types of reasoning and methods of proof	Analyzing and making sense of the algorithmic process
Problem Solving ▪ Build new mathematical knowledge through problem solving ▪ Solve problems that arise in mathematics and in other contexts ▪ Apply and adapt a variety of appropriate strategies to solve problems ▪ Monitor and reflect on the process of mathematical problem solving	Discovering what happens when fraction pieces are renamed and combined

4. Connecting mathematics topics
5. Incorporating multiple representations

In planning the lessons, we used these guidelines to structure the learning environments with virtual manipulatives and applets.

In addition to knowing how to integrate technology appropriately, teachers must focus on worthwhile mathematics and effective pedagogy when using technology. An effective way to optimize the mathematical thinking opportunities presented by technology is to plan the mathematics task focused on the five Process Standards (NCTM 2000): Problem Solving, Reasoning and Proof, Communication, Connections, and Representation. We used a template during planning to guide the activity and the classroom discourse so that teachers were focused on advancing students' mathematical thinking processes (see **fig. 2**).

The remainder of the article describes two lessons in which technology was used as an instructional strategy. More specifically, we describe the task and the technological tool that supported the learning, the role of the teacher in capitalizing on learning in the technology-rich environment, and how the technology gave more access to learning opportunities and more equity to diverse learners.

Counting Change Makes Sense

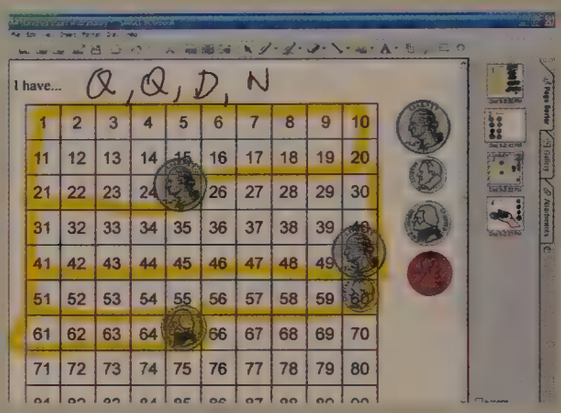
The third-grade lesson objectives were to teach students to count a collection of mixed coins and find and record a variety of ways to show a given amount of money. Making change for amounts to five dollars was a future knowledge building block. To address the objectives, we designed the task on the SMART Board with a hundreds chart and coins that had infinite clones to count change. Using the hundreds chart (see **fig. 3a**), students worked with benchmark numbers such as five, ten, and twenty-five, learned to skip-count when counting change, and practiced using numbers flexibly. The second activity, "Show Me the Money," embedded two tasks. First, students counted the money in the virtual hand (see **fig. 3b**) by dragging the coins and skip-counting. The following scenario offered the other task: "I have in my hand a total of thirty-three cents. Show me all the possible ways to make that amount" (see **fig. 3c**).

We used technology to provide students with multiple representations. The electronic hundreds chart helped students see the relationship between coins and their value. Using the SMART Board, students were able to touch the screen and drag the coin directly onto the hundreds chart to help them count by twenty-five, ten, five, or one. We used the highlighting pen to shade in money amounts and to show the value of each coin. For many children, counting money is especially challenging because its representation is nonproportional; that is, although a dime has more value than a nickel, the dime is

Figure 3

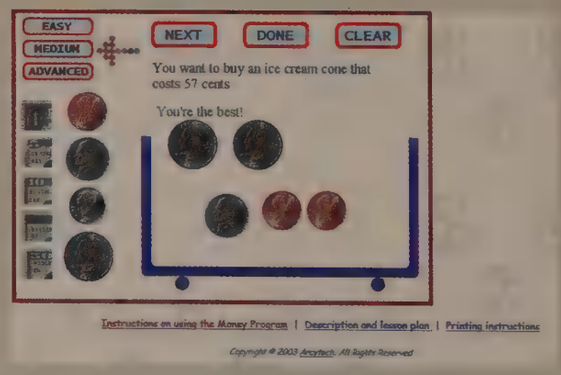
Student work using tech tools

(a) The SMART Board to count change



(b) A virtual hand to drag coins and skip-count

(c) Multiple coin combinations for thirty-three cents


EASY
MEDIUM
ADVANCED
NEXT
DONE
CLEAR
You want to buy an ice cream cone that costs 57 cents
You're the best!
Instructions on using the Money Program | Description and lesson plan | Printing instructions
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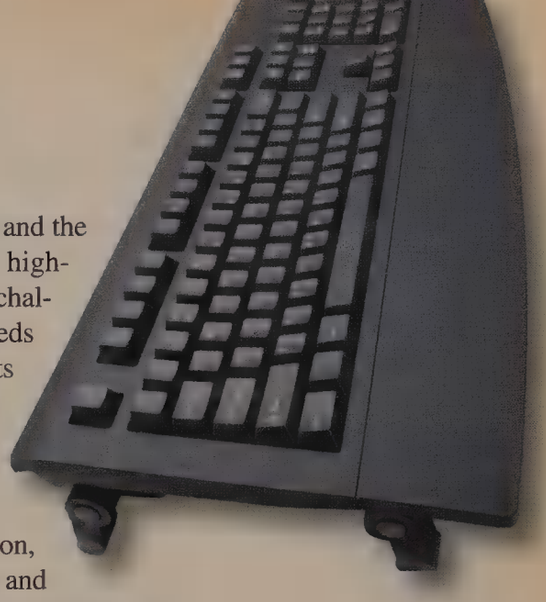
physically smaller. The task was designed to relate the proportional representation of the hundreds chart with the nonproportional representations of the coins so that students would gain better understanding of each coin's magnitude and worth. Demonstrating how to count up to thirty-three cents at the SMART Board, one student commented, "With twenty-five cents, I need to shade in a lot more: two rows for the two tens and five ones; and to get to thirty cents, I need to add a nickel and then three more pennies." Shading the hundreds chart was an instructive visual representation of the coin values.

The teacher's role

The SMART Board technology facilitated the teacher's ability to give students opportunities to show multiple ways to count change. Important teacher tactics included allowing students to display different solution paths on the SMART Board simultaneously, asking students to compare different thinking strategies for making compatible numbers, and initiating productive discussion on efficient change-counting strategies. Simultaneously displaying multiple student solutions allowed students to compare and make some important generalizations about counting coins. For example, when given the coins (a quarter, dime, dime, and nickel), one student shared, "I count the quarter first and then the nickel to get to thirty cents [and] then add the two dimes to get to fifty cents." Another student said, "It is easier for me to add the quarter, then the two dimes to go from twenty-five, thirty-five, forty-five cents, [and] then add the nickel to get to fifty cents." Many students began to adapt their thinking and model the strategies shared in class that made it easier to skip-count money. The task also allowed them to discover ways to compose and decompose numbers using different coin combinations.

Equity and access for diverse learners

Technology enhanced students' learning by allowing diverse learners to understand the concept through multiple representations. Students recorded the numeric value right next to the coins as they counted change on the hundreds chart, thereby allowing the visual representations to be closely tied to the numeric representations. For some English Language Learners, being able to write words such as *quarter*, *dime*, *nickel*, and *penny* next to the coin gave them better access to the lesson. The technology features allowed for better communication, problem solving, reasoning, and connections among concepts. In fact, the dual representations of the coins and the hundreds chart allowed for some high-ability students to engage in more challenging tasks. By using the hundreds chart and counting on, these students used the tools to determine how much change one should get back if one pays with a dollar bill. For example, the cost of a candy bar is sixty-eight cents. The child counts on, "Sixty-nine, seventy," using pennies and then counts on, "Eighty, ninety, one hundred," using three dimes; the total is thirty-two cents in change. Having multiple tasks embedded within each task also allowed for differentiation in instruction.



Exploring Equivalent Fractions

The lesson objectives for the fourth-grade fractions lesson were to rename fractions and find equivalent fractions; the subsequent lesson focused on using models to add and subtract fractions with unlike denominators. The virtual manipulatives called Fraction Equivalence, found on the National Library of Virtual Manipulatives Web site, allowed students to explore the relationship between equivalent fractions. On the Fraction Equivalence applet, students were presented with a partially shaded circle or square and the fraction symbol. They were directed to "find a new name for this fraction by using the arrow buttons to set the number of pieces. Enter the new name and check your answer." To do this, students clicked on arrow buttons below the whole unit, which changed the number of parts. When students had an equivalent fraction, all lines turned red. When a common denominator was identified, students typed the names of the equivalent fractions into the appropriate boxes. They checked their answers by clicking the "Check" button. Each step of the way, the pictures were linked to numeric symbols that dynamically changed with the students' moves (see **fig. 4a**). To help explore the relationship between equivalent fractions, the applet prompted students to find several equivalent fractions. This applet was specifically designed to develop the concept of renaming fractions. Although constrained to one specific objective, the tool allowed for more exploration than do physical manipulatives, such as fraction circles or bars, which are usually limited by the number of fractional pieces. This applet allowed students to equally divide a whole, up to ninety-nine pieces, and generate multiple equivalent fraction names.

The teacher's role

The teacher's role in extending students' thinking during this task was to encourage students to record a list of equivalent fractions, look for a pattern, and generate a rule. For instance, using the applet on a SMART Board, a student found 1/3 = 2/6 = 3/9 = 4/12. As we recorded this on the board, students' eyes started to widen and hands started waving in the air: "Oh, oh, I know the rule!" Some students noticed the additive rule. One student stated, "The denominators are going by a plus-three pattern." Another student echoed, "It is like skip counting." And another voiced, "It is the multiples of three."

To get students to further explore the relationship, the teacher asked them to examine the multiplicative pattern for both the numerator and the denominator in 2/3. Students listed 2/3 = 4/6 = 6/9, and again they quickly saw the additive pattern and the multiples of two for the numerator and three for the denominator. Then the teacher posed the questions, "Are 2/3 and 20/30 equivalent fractions? What about 2/3 and 10/15?" To find a rule beyond the additive rule, students were asked to use the applet and talk to their partners while exploring the relationships between the equivalent fractions and to other fractions. When students came back together as a group, several of them shared their discoveries: "The fractions 2/3 and 20/30 are equivalent, because you multiply both numerator and denominator by ten. And in 2/3 = 10/15, you multiply both numerator and denominator by five." These comments led to a lively conversation about how 10/10 and 5/5 both equal one whole. The teacher connected this idea to the identity property of multiplication by asking, "What happens when we multiply one by any number?" The ensuing discussion reinforced the idea that no matter how you rename the fractions, as long as you multiply them by one or *n*/*n*, you will have an equivalent fraction. To challenge the students, the teacher posed a question: "What would the equivalent fraction be for 1/3 if the denominator were divided into ninety-nine parts?" This type of questioning encouraged students to extend their thinking by making conjectures and testing their rule or hypothesis.

Equity and access for diverse learners

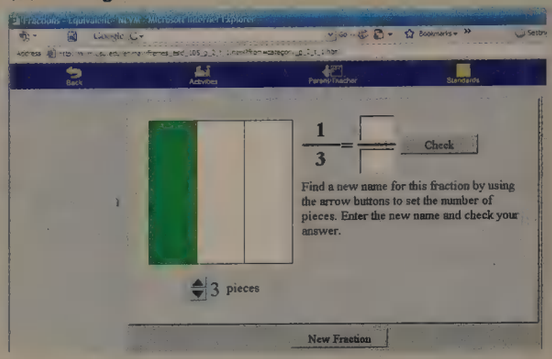
Instead of merely teaching an algorithm, we used the fraction applet to allow all the students to think and reason about the relationships among the opportunity to work with a partner. As the pairs worked together with the applet, they were able to make sense of the mathematics by talking through the processes. The teacher paired limited English-proficient students with students who spoke the same language and could better explain what was happening. The ability to switch to Spanish gave many ELLs better access to the mathematics (see **fig. 4b**). And finally, while other students explored with a partner, the special needs learners worked together in a small group with the mathematics educator, who scaffolded their experience by working collaboratively in front of the SMART Board.

Traditionally, special needs learners are often given direct instruction on how to perform an algorithm using mnemonic devices or procedural steps without being given opportunities to construct

Figure 4

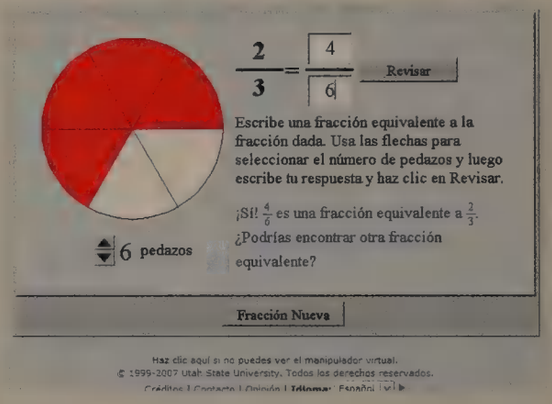
Fraction equivalence applet

(a) in English


1/3 = ___/___
Check
Find a new name for this fraction by using the arrow buttons to set the number of pieces. Enter the new name and check your answer.
3 pieces
New Fraction

National Library of Virtual Manipulatives at Utah State University, copyright 1999–2000. All Rights Reserved.

(b) in Spanish


2/3 = 4/6
Revisar
Escribe una fracción equivalente a la fracción dada. Usa las flechas para seleccionar el número de pedazos y luego escribe tu respuesta y haz clic en Revisar.
¡Sí! 4/6 es una fracción equivalente a 2/3.
¿Podrías encontrar otra fracción equivalente?
6 pedazos
Fracción Nueva
Haz clic aquí si no puedes ver el manipulador virtual.
© 1999-2007 Utah State University. Todos los derechos reservados.

National Library of Virtual Manipulatives at Utah State University, copyright 1999–2000. All Rights Reserved.

conceptual understanding of the procedure. One of the biggest challenges of working with physical manipulatives, such as fractions circles, is that actually manipulating multiple pieces creates so much of a cognitive load on students' thinking processes that they lose sight of the mathematical concept. In our classroom example, working with a virtual applet reduced some of the physical manipulation so that the special needs students could focus more on mathematical processes and relationships among the equivalent fractions. In many ways, the applet gave special needs students access to the mathematics without creating a cognitive overload.

Having visual and numeric representations closely tied together and displayed on the screen helped students make direct connections in relationships among equivalent fractions. Throughout the lesson, the teacher worked with a small group of ELLs and special needs students, who required more teacher support and benefited from small-group interaction. The teacher could re-teach and reinforce skills as needed. The kinesthetic and tactile advantages of the SMART Board also enabled students to grasp greater understanding of the concept as they took turns manipulating the SMART Board and coaching each other through the given task.

Leveraging Technology to Enhance Mathematical Learning

Learning environments that take advantage of virtual manipulatives and applets offer a number of ways for students to develop their mathematical understanding. The authors identify the following as five primary benefits of virtual manipulatives and applets:

1. Linked representations provide connections and visualization between numeric and visual representations.
2. Immediate feedback allows students to check their understanding throughout the learning process, which prevents misconceptions.
3. Interactive and dynamic objects move a noun (mathematics) to a verb (mathematize).
4. Virtual manipulatives and applets offer opportunities to teach and represent mathematical ideas in nontraditional ways.
5. Meeting diverse learners' needs is easier than with traditional methods.

Leveraging Technology in Mathematical Teaching and Learning

As teachers structure their learning environments using technology, the primary focus should be to support mathematical understanding. A number of design and assessment issues are unique to using technology. For example, teachers should consider having students print their work or use a task sheet to record their work, their thoughts, and examples from using the virtual manipulative or applet. By writing and recording their work, students reflect on their own thinking, a metacognitive process, which is essential in problem solving. The task sheet also provides a permanent record that can be used for the teacher's assessment purposes.

Ensuring mathematical discourse with peers and teachers before, during, and after using a technology tool is an important design issue, critical to students' exploration of patterns and relationships. Using appropriate technology in teaching and learning should make learning environments qualitatively different from teaching without technology. That is, integrating technology should not merely add a virtual representation to a lesson; it should enhance teaching and learning by providing opportunities for rich mathematical thinking and discussion. Teachers should consider specific pedagogical issues. In our two classroom examples, we illustrated how using the NCTM Process Standards alongside the unique aspects of the technology tools allowed meaningful learning to take place while meeting the needs of diverse learners.

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Case-Based Tasks in Mathematics Teacher Preparation: What Is Appropriate?

In recent years, the use of case-based tasks in the pedagogical preparation of teachers has gained considerable popularity in teacher education (Broudy 1990; Merseth 2003). Framed by a problem-based orientation to learning and teaching, case-based tasks assumedly enhance teacher preparation on at least three levels:

- They afford real contexts for teachers to learn about the problems of practice (Sykes and Bird 1992).
- They engage teachers in learning more about the subject matter and pedagogy (Walen and Williams 2000).
- They provide for learning about reform-oriented teaching and help future teachers identify features of reform-based instructional strategies that they might use later in their own practice (Merseth 2003).

In light of these perceived benefits, interest is increasing for using such tasks in mathematics content and methods courses designed for teachers (Smith, Silver, and Stein 2005).

The primary goal of this article is to initiate discourse about the merit of these assumptions and the initiative to use such tasks in mathematics content courses designed for teachers. Considering this goal, the article is divided into four parts. First, we offer a brief account of professional perspectives

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that have given rise to an interest in using cases in mathematics teacher preparation. We also outline a few challenges associated with using them. Drawing from these perspectives and research findings, we next describe a framework for designing (or selecting) case-based tasks suitable for professional development of mathematics teachers in order for such tasks to lead to substantial learning. Third, we share examples of cases we have designed and used in content courses we teach. Finally, we give an account of the impact the cases have had on our work and our students' learning.

Why Use Case Studies?

Mathematics teaching is a dynamic process that relies heavily on teachers' intellectual resources, including their problem-solving skills in both mathematical and pedagogical domains. Teachers' problem-solving skills allow them to maintain an adaptive posture (Cooney 1994) in their relationships to mathematics, curriculum, children's thinking, and instruction.

Mathematical problem-solving skills enable teachers to navigate unfamiliar mathematical territories they might experience in and through teaching. A few among many are the following: making sense of new topics that the curriculum covers, responding to unexpected questions that students ask in class, examining solutions and methods that students use and offer, and analyzing children's invented algorithms and choice of representations. Teachers' pedagogical problem-solving skills permit them to productively interact with learners, assess the impact of their instruction on students, make sound decisions about the type and quality of feedback they provide to children, and refine their curriculum and instruction to successfully accommodate students' cognitive and affective needs.

Acting in concert, these two domains of problem-solving skills—mathematical and pedagogical—empower teachers to construct and use instructional representations that are informed by a deep understanding of the discipline, theories of learning and teaching, knowledge of children and their development, and short- and long-term curricular goals and objectives.

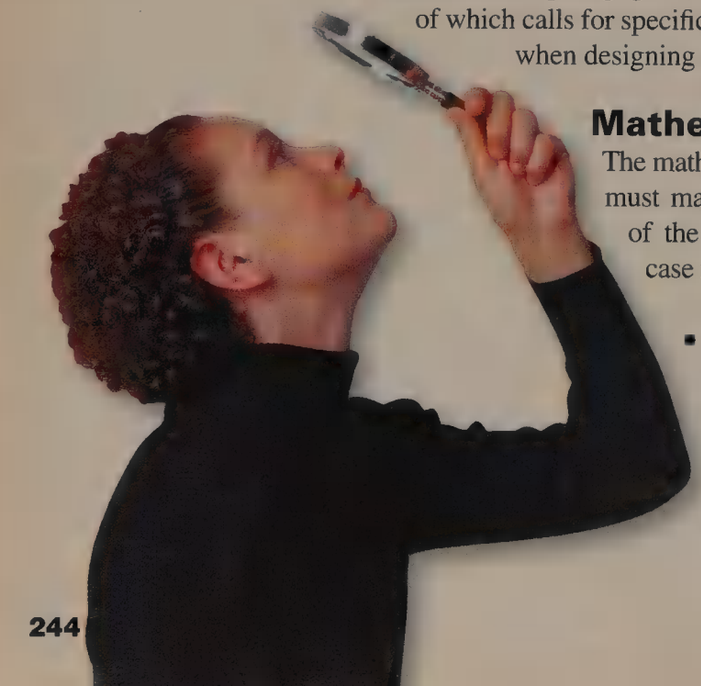
Within the mathematics teacher education community is consensus that courses designed for teachers must ensure support of such problem-solving skills and dispositions for teacher candidates. In an effort to offer such support, Cooney (1994) proposed that mathematics teacher educators should design contexts that engage teacher candidates in identifying constraints they face in teaching and ways in which they might deal with these dilemmas. Moreover, by orchestrating situations in which teacher candidates must assess children's work samples and react to them, teachers learn to translate their knowledge of mathematics into teaching strategies. The move toward providing such contexts to teacher candidates in methods and content courses and using case studies illustrating vignettes of teaching episodes or samples of children's work seems both logical and appropriate. On the one hand, a major goal of content courses is to bridge the traditional gap between pedagogy and content preparation by offering teachers opportunities to reflect on conceptual structures that guide school curriculum from an advance standpoint (NAP 2001). On the other hand, research indicates that opportunities to gain deeper mathematics insights can occur as teachers examine rich ideas that children generate (Schifter 1998).

Certainly, advocates of using case-based tasks in teacher education have provided evidence supporting the potential for raising teacher candidates' critical reflection on professional issues (McWilliam 1992; Richardson 1996; Walen and Williams 2000) and improving their communication skills (Hutchings 1993; Kagan 1993). However, the usefulness of case-based tasks in engaging teacher candidates in analyzing subject matter issues and content-specific problem solving has been subject to debate. Studies on the impact of case-based tasks on mathematical development are rare. However, the few published reports of such research indicate that—even in the presence of subject-specific cases—teacher candidates fail to engage in critical mathematical examination of cases used with them (Lampert and Ball 1998; Manouchehri and Enderson 2003). Lampert and Ball reported that in the course of their investigations of case studies, the teachers with whom they worked rarely addressed mathematics. Expressing similar concerns, Manouchehri and Enderson offered that when their secondary mathematics teacher candidates used case studies, the students repeatedly refrained from considering mathematical issues that were central to understanding children's ideas or analyzing teachers' moves as portrayed in the case. Instead, they focused on effective elements surrounding classroom interactions or teachers' actions. The authors suggested that when using case-based tasks with teacher candidates, facilitators should deliberately stress content analysis as an explicit part

of individual and group investigations by posing specific mathematical questions for students to consider and examine. Teacher educators in other disciplines have made similar suggestions (Elksin 1998; Richert 1992). Such a structure seems vital when case studies are used in content courses designed for teachers. Case-based tasks should provide an intellectual context for teachers' inquiry into mathematics and lead to learning more about the subject. For case-based tasks to secure a legitimate place in content and content-specific methods courses designed for mathematics teachers, issues associated with their content and goals must be carefully examined to assure that the level of analysis elicited of teachers is not compromised.

Designing Case-Based Tasks

Whether used in mathematics content or methods courses designed for teachers, one critical role of case-based tasks is to help teachers realize the intense interactions between pedagogy and subject matter. These tasks must, collectively, increase the teacher candidates' understanding of mathematics, pedagogical content knowledge, and current research-based knowledge about effective mathematics teaching and learning. Further, they must motivate and increase teacher candidates' desire to learn new concepts, help them acquire problem-solving skills, and nurture among them the professional stance that teaching is a proactive process of responding to classroom events and children's thinking—a process demanding sustained curricular and instructional design and redesign. In effect, we argue that three broad categories of goals should guide the content and structure of tasks—namely, mathematical, pedagogical, and professional—each of which calls for specific issues to be considered when designing case-based tasks.



Mathematical goals

The mathematics content of tasks must match the level and scope of the course objectives. The case must—

- address a central mathematical concept or idea and allow for developing insight about a concept or idea's relative significance in the discipline (Wu 2004);
- allow for or lead to developing knowledge of multiple representations and their use in solving different problem types (NCTM 2000);
- provide opportunities for mathematical problem solving and problem posing (Cooney 1994);
- increase understanding about the rules, conventions, and discourse of the discipline (Ball and Bass 2003); and
- provide opportunities for teachers to engage in productive discourse about the subject matter (NAP 2001).

Pedagogical goals

The case must—

- confront teachers' optimistic assumptions about teaching and address problems within instructional planning, curriculum design, or assessment of learning (Manouchehri and Enderson 2003);
- help teachers realize the multiple ways in which children's work and strategies should impact instructional and curricular planning (Ball and Bass 2003);
- increase teachers' ability to listen to and engage learners (Merseth 2003);
- provide opportunities to develop and practice pedagogical reasoning (NCTM 1991);
- provide teachers with enactment strategies that allow them to respond to a diverse student population's mathematical needs (NAP 2001); and
- help teachers gain knowledge about cognitive obstacles associated with learning various mathematical concepts and ways in which these difficulties may be addressed in instruction and through teachers' choices of representations and representational tools (Wilson 1992).

Professional goals

The case must—

- nurture the disposition of inquiry among teachers (Wilson 1992);
- provide teachers with a professional lens through which they can analyze problems of practice (Cooney 1994); and
- help establish the habit of reflective practice and collegial discourse in analyzing professional issues and determining instructional effectiveness (Manouchehri and Enderson 2003).

Additionally, tasks should provide opportunities for productive discourse among teacher candidates while granting teacher educators opportunities for

intervention and instruction—extending the teachers' understanding of both content and pedagogy by sharing new information; introducing new tools; elaborating on mathematical or pedagogical concepts; or challenging their assumptions and beliefs about the subject matter, teaching, and learning. Although no single case alone has the power to address all these goals or foster teachers' development in all these areas, collectively tasks should attempt to nurture all these skills and habits.

What Would You Do?

Figure 1 illustrates an example of a case-based task we designed and used in content courses for teachers of grades K–6. In writing the case, we relied on actual data collected either from interviews with children or from observing classrooms at precollege and college levels. Each case requires teacher candidates to engage in four levels of analysis, described below:

1. Mathematics Content. Teachers are asked to study a problem and identify the various mathematical concepts and skills involved in solving it with the purpose of ensuring that the teachers engage in examining mathematics content and develop an understanding of the problem (if not the solution) before examining children's work.

2. Assessing Children's Solution Methods. Teachers are asked to make sense of children's strategies and solutions to problems and to determine the type of experiences and conceptions (or misconceptions) that could motivate particular approaches. Moreover, they are asked to compare children's solutions and categorize them according to the conceptual structures the children evidently used. To increase their assessments' accuracy and reliability, teacher candidates are then asked to reflect on children's choices of representations, hypothesize about the particular needs of each child, and design questions or tasks to provide additional data on each child's thinking.

3. Relevant Literature. To nurture the habit of grounding their work in professional literature and existing knowledge, each task requires that teachers read a relevant article from a research or professional journal and reexamine their analysis of children's work in light of the reading.

4. Instructional Design. Teachers are asked to write a lesson, design a simulation or exploration

Figure 1

Proportional reasoning: Assessing representations

The problem below was posed to a group of student teachers.

> Five-eighths of the students in a class are boys. After six girls join the class, the number of boys and girls in the class is the same. How many students are in the class now?

Attached is a collection of children's responses to this question. After reading all responses, comment on the following questions:

1. Which solution approach was most familiar to you? Which was least familiar? Explain why the methods were familiar or unfamiliar to you.
2. Discuss, from a mathematics viewpoint, how these responses vary according to the concepts that students used to solve the problem.
3. Identify the various problem-solving strategies students used to solve the problem. How are they similar or different?
4. How are the first and last students' responses similar? How are they different? What would be a suitable follow-up problem to assign to each student to ensure that he or she moves toward higher-level thinking?
5. Write a summary of how you might help the entire group understand and appreciate the utility of each approach. (This may be done through designing tasks in which each of the approaches becomes a powerful tool for solving problems.)
6. Classify students' responses from the least to the most sophisticated. What criteria did you use as the basis for ranking students' responses?
7. Hypothesize about what could be the source of difficulty for the first student. What mathematical procedures would need to be established and clarified with this child?
8. What is your assessment of the second child's method? Design a task for which the use of this problem-solving approach would be most useful.
9. The third student proposes a graphical representation of proportional ratios. How does this approach connect to the study of slopes and linear equations?
10. Study the following article:

Cramer, Kathleen, Thomas Post, and Sara Currier. "Learning and Teaching Ratio and Proportion: Research Implications." In *Research Ideas for the Classroom: Middle Grades Mathematics*, edited by Douglas T. Owens, pp. 159–78. Reston, VA: National Council of Teachers of Mathematics, 1993.

How does your assessment of children's work compare with the ideas presented in the article? In light of what you learned from the article, how might you modify your response to question 7?

11. Study the following article:

Madsen, Anne L., and Kendella Baker. "Planning and Organizing the Middle Grades Mathematics Curriculum. In *Research Ideas for the Classroom: Middle Grades Mathematics*, edited by Douglas T. Owens, pp. 259–79. Reston, VA: National Council of Teachers of Mathematics, 1993.

Now imagine that you are this group's teacher and must proceed with a lesson following your review of students' responses. What would you include in the lesson? How did the recommendations for teaching discussed in the articles that you read guide the development of your lesson?

that would lead to a whole-group discussion, or pose additional questions they would use with each child to gain better understanding of the child's thinking and knowledge base. Furthermore, teachers are asked to elaborate on how instruction may be refined to accommodate the recommendations offered in the articles they read.

We assign such case-base tasks as homework activities and implement a maximum of two a month in each course. Teacher candidates are asked to bring their comments and analysis to class. We devote the following class session to studying the cases. Teachers first share their initial analysis of each case with peers in small collaborative groups. A whole-group discussion of ideas then follows the small-group work.

Some Outcomes

Earlier in this article, we proposed that a current assumption in mathematics teacher education concerns the potential for using case-based tasks for enhancing pedagogical and mathematical teacher preparation. Evidence from our own implementation of such tasks in content courses designed for teachers certainly supports this assumption. Indeed, using case-based tasks, such as those described in this article, granted us more comprehensive knowledge about our students' current understandings and needs in both mathematical and pedagogical domains as well as their problem-solving and communication skills. Additionally, the evidence enabled us to initiate and sustain generative discussions about what it means to know mathematics, how teachers (including us as teacher educators) make decisions about what to teach, and how to choose instructional tools and representations to ensure learners' developing understandings. The tasks successfully engaged our teacher candidates in extensive mathematical problem solving and increased their desire to learn about various representational schemes they could use to communicate ideas with children and with one another.

Case studies heighten teacher interest in subject matter

Perhaps the most striking result of using these tasks was teachers' heightened interest in learning more about the structure of the subject and the "trajectory" of various concepts they studied. They were curious to learn about how a certain concept was treated at different grade bands as well as ways in which the new concepts they learned connected to those they had previously studied. In the past, we had problems raising teacher candidates' interests in learning about these issues. Evidence of learning from cases was also manifested in our students' own mathematical work. They often used children's strategies they had learned from a case to solve novel problems. All these outcomes are central goals of the mathematics teacher education community and certainly support the merit of using case-based tasks in mathematics content courses for teachers.

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HOT Topics

Call for Manuscripts

Teaching Children Mathematics publishes articles on "hot topics" that TCM readers have noted on reader surveys as subjects of high interest. By highlighting issues and challenges facing today's mathematics educators, the Editorial Panel hopes to provide teachers and teacher educators with resources in their efforts to meet the mathematical needs of all students. We invite you to share your classroom experiences and ideas. We especially encourage submission of manuscripts that include photographs and samples of student work or dialogue. The following topics and related questions are a guide; manuscripts addressing related issues are also welcome.

Teachers' knowledge of mathematics

- How can we close the gap between what teachers know and what they need to know to teach mathematics effectively in the twenty-first century?
- What role can a mathematics specialist serve in helping teachers develop content knowledge?
- How can evidence of student improvement based on teachers' increased content knowledge be collected?

Using children's literature in mathematics

- How can children's literature aid in developing mathematics concepts (fractions, time, money, operations, number sense, etc.)?
- How can literature be used to develop problem-solving skills?
- How can language arts and mathematics content be joined to create an integrated curriculum?

Please share your interesting ideas, research, or classroom-tested approaches by writing for the journal. Submit a manuscript to TCM by accessing **tcm.msubmit.net**. For more information, refer to the "Instructions for Submission of Manuscripts" at nctm.org/publications/write.htm. Direct any questions to the Journal Editor at tcm@nctm.org.

Sue McMillen and Beth Ortiz Hernandez

Taking Time to Understand Telling Time

This department features children's hands-on and minds-on explorations in mathematics and presents teachers with open-ended investigations to enhance mathematics instruction. These tasks invoke problem solving and reasoning, require communication skills, and connect various mathematical concepts and principles. The ideas presented here have been tested in classroom settings.

A mathematical investigation is—

- multidimensional in content;
- open ended, with several acceptable solutions;
- an exploration requiring a full period or longer to complete;
- centered on a theme or event; and
- often embedded in a focus or driving question.

In addition, a mathematical investigation involves processes that include—

- researching outside sources;
- collecting data;
- collaborating with peers; and
- using multiple strategies to reach conclusions.

Although "Investigations" presents a scripted sequence and set of directions for a mathematical exploration for the purpose of communicating what happened in this particular classroom, *Principles and Standards for School Mathematics* (NCTM 2000) encourages teachers and students to explore multiple approaches and representations when engaging in mathematical activities. In the November investigation, second-grade students explore seconds, minutes, hours, and the relationships among them. They connect their observations to the movements of a clock's hands and to telling time. This investigation addresses NCTM's Measurement Standard (NCTM 2000).

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Edited by **Jodelle S. W. Magner**, *magnerjs@buffalostate.edu, who teach mathematics and mathematics education courses at Buffalo State College in New York; and* **LouAnn H. Lovin**, *lovinla@jmu.edu, who teaches mathematics methods and content courses to prospective teachers and classroom teachers at James Madison University in Harrisonburg, Virginia. Investigations highlights classroom-tested multilesson units that develop conceptual understanding of mathematics topics. This material can be reproduced by classroom teachers for use with their own students without requesting permission from the National Council of Teachers of Mathematics. Readers are encouraged to submit manuscripts appropriate for this department by accessing* **tcm.msubmit.net**. *Manuscripts must include no more than 3000 words on double-spaced typed pages and two reproducible pages of activities.*

The Investigation

Learning goals, rationale, pedagogical context

Telling time is often taught as a rote skill, without ensuring that children understand the duration of time and its relation to the numbers and hands on a clock (Thompson and Van de Walle 1981). This investigation provides opportunities for children to investigate time units, experience events lasting set amounts of time, and develop personal referents for standard time units. The typical approaches to teaching time do not highlight the differences associated with the two hands of an analog clock. The hour hand simply points to the hour, but the minute hand must be understood in terms of the distance it has moved around the clock from the numeral 12. For this reason, this investigation has students work with two types of one-handed clocks to meaningfully discover both differences and connections between the two clock hands. The sixteen students in the investigation were in a diverse second-grade class at the Samuel S. Gaines Academy in Ft. Pierce, Florida.

Objectives of the investigation

Students will—

- develop personal referents for one second and for one minute;
- discover the relationship between seconds and minutes;
- investigate the connections between the movements of a clock's hands;

- connect the numerals on a clock face to groups of 5; and
- tell time to the nearest minute.

Materials

Lesson 1

1. A clock with a second hand.
2. Each student will need one copy of **activity sheet 1** (activity sheets follow this article).
3. Each activity station will need the materials necessary for the one-minute activities that you choose, such as Unifix cubes, balls, or jump ropes.

Lesson 2

1. Each student will need a slit clock.
2. The teacher will need—

- a slit clock;
- a piece of rope with evenly spaced knots; and
- candles of different widths with evenly spaced notches or lines.

Lesson 3

1. Each student will need—

- 5 connected Unifix cubes;
- a cutout of 5 connected Unifix cubes; and
- a copy of **activity sheet 2**.

2. The teacher will need tape and a clock without numbers.

Lesson 4

1. Each student or pair of students will need a clock with moveable hands that maintain the correct relationships as they are moved.
2. The teacher will need—

- a slit clock;
- a clock without numbers;
- a transitional clock; and
- a regular clock.

Previous knowledge

The students in this investigation could tell time to the nearest hour, and many of them could tell time to the nearest half hour. They had little or no formal experiences with fraction representations, and they had just completed a unit on money.

Lesson 1

The purpose of the first lesson is for students to develop personal benchmarks for one-second and one-minute time spans and discover the relationship between them. Begin the investigation by asking students to think about how long one second is and share what they could do in one second. About half of our class offered reasonable ideas, such as turn on a light, do a front flip, walk one step, snap your fingers, throw a ball, write the numeral 1, do a cartwheel, flip the pages in a book [all at once], count to two, and count back to one. Nicholas connected time to distance by offering, "Move your finger a small amount," as he showed about a one-fourth-inch movement of his finger on his desk. Other students offered less reasonable ideas of one second and shared responses such as "Count to one hundred," "Go on a plane," and "Drink a bottle of water." If time permits, have your class verify how many seconds some of their suggestions actually take.

Students were then asked to put their heads on their desks and to raise their hands when they thought one minute had passed. As we had anticipated, most children underestimated the length of one minute. In our class, two students raised their hands in less than twenty seconds, seven raised their hands between twenty-one and forty seconds, three raised their hands between forty-one and fifty seconds, and four made good estimates of one minute, raising their hands between fifty-one and seventy seconds.

Have the entire class experience some one-minute events, such as running in place for one minute or sitting silently for one minute. Then ask them to predict the result of an activity the teacher

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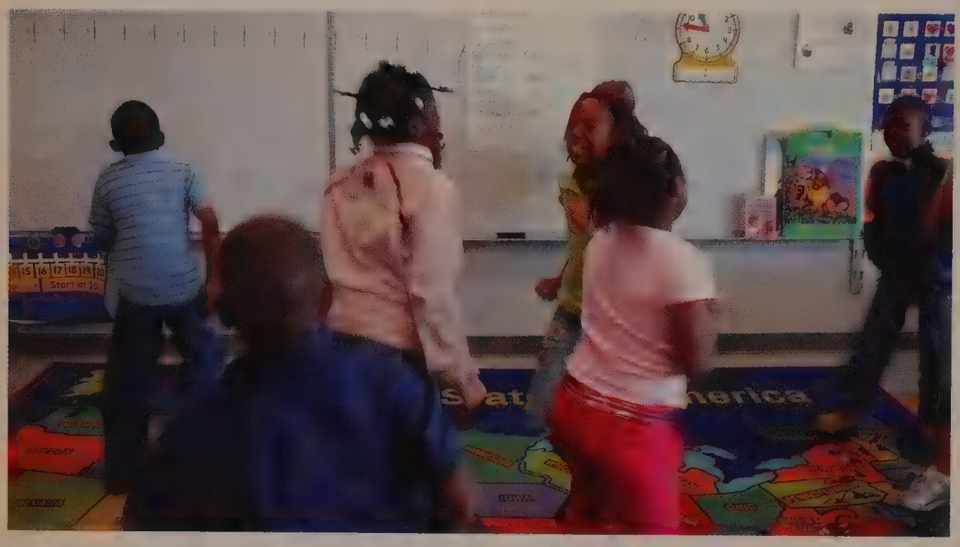
Second graders investigated connections between a clock's two hands.



Students experienced one-minute activities together. They ran in place.

They bounced a ball.

They connected Unifix cubes and jumped rope.

Photographs by Ivan McMillen; all rights reserved

will do for one minute. Our students were familiar with the book *Are You My Mother?* by P. D. Eastman, so they were asked whether or not the entire book could be read aloud in one minute. Twelve of sixteen students predicted that it could be, but the teacher was able to read only one-third of the book during the minute.

Ask students to predict how many times they can repeat an activity during one minute. Give them an opportunity to try each activity for one minute and compare their predictions to the actual results. We gave our students an opportunity to cycle through four activity stations while working with a partner: bouncing a ball, connecting Unifix cubes, jumping rope or doing jumping jacks, and writing their name. The teacher announced the beginning and end of the one-minute time periods. Some students seemed to think that because the allotted time was the same for each activity, the number of repetitions for each activity would also be about the same (see **fig. 1a**). They made predictions that were consecutive numbers or were all very close to the same number. Other students seemed to lack both number sense and a one-minute reference frame and made unreasonable predictions (see **fig. 1b**). You may want to emphasize that they should continue the activity for the entire minute, not stop if they reach their predicted number of repetitions, as some of our students did at first (see **fig. 1c**). Also, some students used stacks of Unifix cubes that were already joined together, resulting in larger numbers than we had anticipated.

Guide students to discover the relationship between seconds and minutes. Be careful not to mention anything about discovering how many seconds are in a minute, because some students will likely already know. Ask the class to watch the second hand and count, as a class, how many times it moves while traveling all the way around the clock. The teacher should indicate when to start and stop counting. Repeat this activity several times, being sure to start with the second hand at different locations on the clock. After several repetitions, students should notice that it always takes sixty seconds. Have a class discussion to ensure that students understand that a minute is a standard measurement unit and must therefore always equal the same number of seconds, namely, sixty seconds. Ask how they can use the clock to tell when one minute passes. If necessary, prompt them to connect to their experiences with the second hand taking sixty seconds to travel the entire way around the clock face.

Figure 1

Students predicted how many times they could repeat an activity in one minute.

(a) Some predicted similar numbers for all activities.

In One Minute

	Predict	Do
Bounces	12	
Jumping Jacks	11	
Unifix Cubes	15	
Write Name	10	

(b) Some made unreasonable predictions.

In One Minute

	Predict	Do
Ball Bounces	1	20
Jumping Jacks	1002	10
Unifix Cubes	3	99
Write Name	3066	1

(c) Some stopped when they reached their prediction, even before the minute was up.

In One Minute

	Predict	Do
Ball Bounces	19	
Jumping Jacks	96	896
Unifix Cubes	24	
Write Name	23	

Next, share some common counting schemes, such as "one Mississippi, two Mississippi …" or "one one thousand, two one thousand …" to approximate one-second intervals. Have one student face away from the classroom clock and use one of the counting schemes to count out an approximation for one minute (sixty seconds) while the other students watch the second hand on the clock to determine the accuracy of the counting scheme.

After counting all the way to "sixty Mississippi," our students came fairly close to one minute and decided these were "good" ways to approximate one minute. At the end of the Lesson 1 investigation, students reflected on their learning as they used their benchmark knowledge of second units and minute units to choose the most reasonable time for certain events (see **activity sheet 1**).

Lesson 2

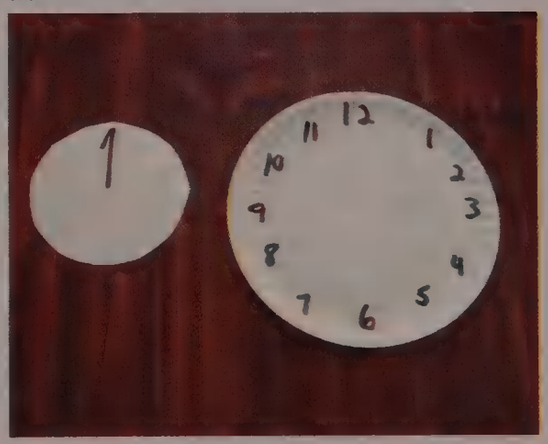
The purpose of the second lesson is for the class to investigate what can be determined about time when a clock has an hour hand but no minute hand. Ask the class how people kept track of time before clocks were invented. Some children may suggest using daylight and darkness. You may want to discuss devices without clock hands, such as sundials and water clocks. In China, people burned ropes with knots at equal intervals. Similarly, some people marked equal intervals on candles. We showed our class candles of different widths, all with marked intervals of the same distance, and asked if it would take the same amount of time for each candle to burn from one mark to the next. Luis said, "No, because the red one is fatter, so it would take a lot longer."

We shared the information that the first clocks were made in the fourteenth century and had only one hand. When asked if we would be able to tell time from a clock with only one hand, most students expressed doubt. We handed out the slit clocks (see **figs. 2a** and **b**) and told the class that the hand represented the hour hand—just like the first clocks from about seven hundred years ago. You may want to demonstrate how to move the slit

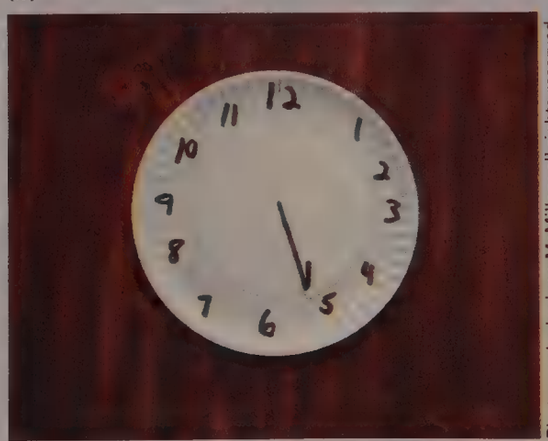
Figure 2

Students worked with one-handed clocks called *slit* clocks.

(a)



(b)



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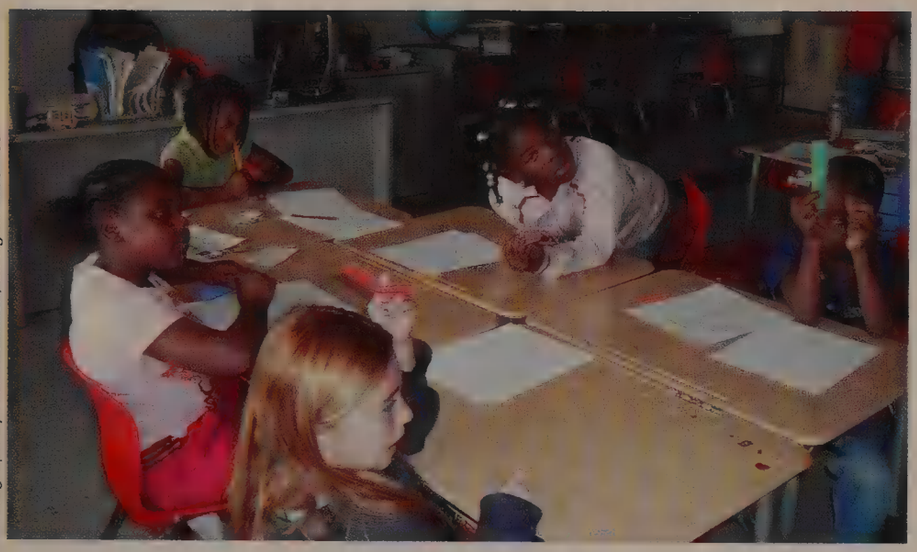

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Number lines made of Unifix cubes helped clarify minutes and hours on a clock as students counted by ones and by fives.

clock: Tell the children to reach behind the clock and turn the circle from the backside with one hand while holding the full plate with the other hand. After allowing them a short time to move the clocks as they wish, ask students to point their hour hands directly at a number. Then ask a student to hold up his or her clock and tell what time it is. Our class quickly agreed that when the hand points directly at a numeral, such as 7, it is "that number o'clock" (i.e., 7:00). They spent a few minutes taking turns, with one partner moving the hand to point directly at a numeral and the other telling the time.

Next, we held up a clock with the hour hand just past the numeral 5 (see **fig. 2b**) and asked what time that could be. Several students told us that it was five o'clock and that the hand had "gone too far." Others responded, "It is five o'clock and some more minutes," and, "It is after five o'clock." With prompting, the class agreed to call this "just past five" or "a little past five." Similarly, when shown a slit clock with the hand just before the numeral 5, they described it as "a little before five o'clock." We also displayed the slit clock with the hand halfway between the numerals 2 and 3. They described it as "between two o'clock and three o'clock" but never offered the idea of being halfway between two and three o'clock (i.e., 2:30). Rather than lead them to the idea of half hours at this point, we let them continue to investigate locating the hand just before or just after a numeral on the clock face. They seemed content to declare each location as closer to one of the two numerals.

As a whole class, students practiced describing several times on slit clocks held up by classmates. The rest of the lesson time was spent on the following five partner activities using the slit clocks. All students worked on the first two activities. Only students who were ready to move on tried the last three activities:

1. One partner sets a time on the slit clock, and the other partner names it using phrases such as "eight o'clock," "a little before seven o'clock," or "a little after three o'clock."
2. One partner describes a time, and the other sets the time on the slit clock.
3. One partner asks a question such as, "When do you get up?" and the other answers by setting the time on the slit clock and naming it.
4. One partner sets a time on the slit clock and asks the other what he would be doing at that time.
5. One partner sets a time on the slit clock, and the other sets the time that is two hours later.

Lesson 3

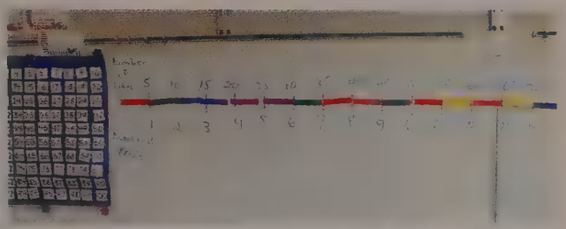
This activity parallels that of Lesson 2, but the class investigates clocks with only a minute hand. The lesson emphasizes connecting groups of five to minutes on the clock face and using efficient counting strategies to tell time. To begin, give each student five connected Unifix cubes and point to children as you go around the room. Have them chant, counting by fives with you.

Then give each student a cutout of five connected Unifix cubes and repeat the counting by fives. Next, have a student tape his cutout cubes to the board. It works best to align the cubes horizontally even though this takes more board space. Draw a vertical line segment at the right end of the cubes. Ask the class how many groups of five cubes are on the board and write the 1 beneath the line segment. Then ask the class how many cubes are on the board and write the 5 above the line segment. Continue in this manner, having students add their cubes to the horizontal number line one at a time. Have the students report both the number of groups of five cubes and the total number of cubes. Some students may notice that the number of groups is the same as the number of students who have put up their cubes.

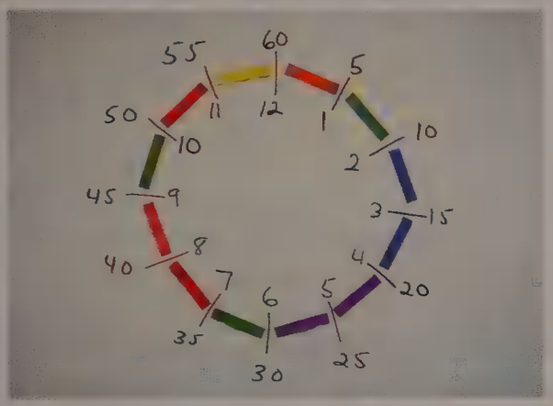
Once the number line is complete (see **fig. 3a**), have the class discuss the two sets of numbers and how they are connected. To start the discussion, we asked questions such as, "How many groups of five cubes make thirty-five cubes?" We also pointed to a specific cube (such as the twenty-third cube) and had the class count up to that cube. Some students counted by fives as long as they could and then by ones, but others counted only by ones.

Figure 3

(a) A completed Unifix number line



(b) A Unifix number line "clock face"



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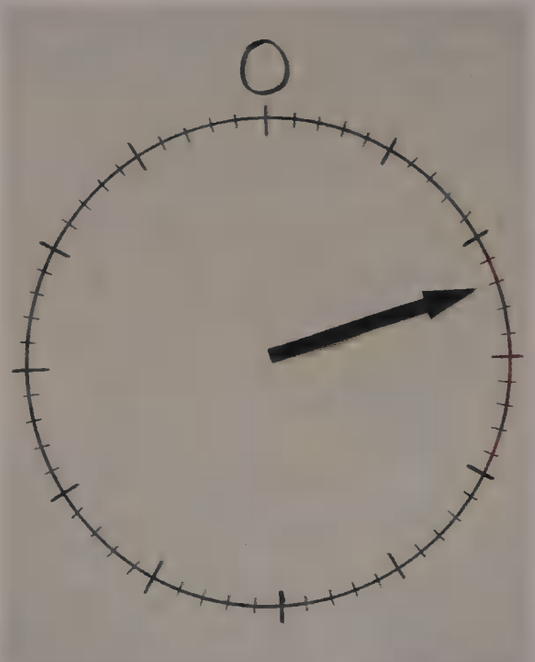
Ask students how many numerals are on the face of a clock and then remove the section of the cube number line beyond twelve groups of five. Have the class discuss whether or not they could use their cube number line to tell time. Some of our students thought it would be too hard because there was "nowhere to go after you get to the twelve," but others pointed out that you could just go back to the beginning and start over again. However, they all agreed that it was easier to tell time on a round clock face, so we rearranged the sections of the cube number line into an approximation of a round clock face. We again labeled with both sets of numbers, keeping the total number of cubes on the outside of the circle (see **fig. 3b**). This created a transitional clock showing numerals for both the number of groups of cubes (hours) and the total number of cubes (minutes).

Help children recall that in the previous lesson's investigation, they used a clock with just the shorter hand. Today they will use a clock with just the longer hand. Ask, "Which numbers should you use to describe where the longer hand points?" Keyontay said it should be the numbers on the outside of the circle because the hand "has longer to go to reach them." Although we referred to the hands only as *shorter* or *longer* in order to preserve the investigative nature of the lesson, some of your students will know them as the *hour hand* and *minute hand* and may use those names in their responses. For example, Roberto said the longer hand "has to be the minute hand, since there is not a thirty-five o'clock." When a couple of our students used this vocabulary, we did not comment on it, but we continued to use the words *shorter* and *longer* in the hope of deepening students' understanding.

We put a clock without numbers on the board next to the cube clock face. When we asked why the zero was written at the top of the clock, class members replied that was because they started the number line with zero cubes. We then asked what the lines on the clock meant. They quickly agreed that each short line was for one cube and that the longer lines were like the lines on our cube number line and signified five cubes or five lines. We asked how many minutes past the hour was shown on the numberless clock (see **fig. 4**). All the students counted by ones to conclude that it showed twelve minutes past the hour. After a couple more examples with a small number of minutes past the hour, the minute hand was positioned to forty minutes after the hour. As anticipated, the class generated several different answers as a result of counting errors. When asked to share their strategies, most indicated that they had counted by ones, but a few shared that they had counted by fives. All agreed that counting by fives was easier, and we then did

Figure 4

A clock without numbers shows twelve minutes after the hour.



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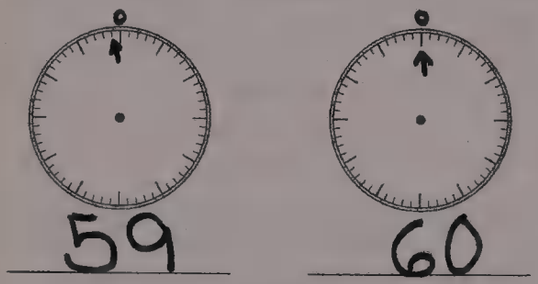
several examples with the number of minutes as a multiple of five. Finally, the long hand was positioned at twenty-eight minutes past the hour. Most students started counting by fives. When they realized that they were not going to end on a multiple of five, some switched automatically to counting by ones, for example, 5, 10, 15, 20, 25, 26, 27, 28. However, many went back to the top and began recounting the entire sequence by ones. During the ensuing examples, we encouraged students to use more efficient counting strategies, but a few continued to count only by ones.

Our students used the remaining time to practice counting the minutes past the start of an hour with their partners and to complete **activity sheet 2**. The sheet also gave them the opportunity to draw a clock hand wherever they wished and then count the minutes. Several children enjoyed creating a large number of minutes past an hour, such as fifty-eight or fifty-nine. Jorge wrote *sixty minutes* and showed the long hand pointing to the top of the clock (see **fig. 5a**). He clearly explained that the hand went all the way around the clock, which was sixty minutes, but when questioned, admitted it did not look any different than zero minutes past the hour. Most of the children were comfortable working without numbers on the clocks, but a couple wrote in the numbers they needed as they completed each problem (see **fig. 5b**).

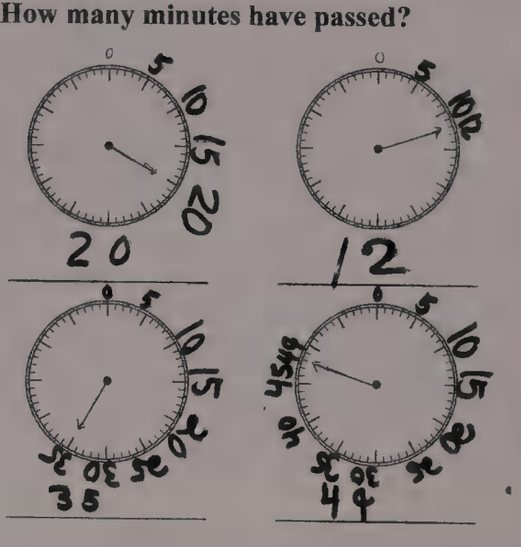
Figure 5

Examples of student work from activity sheet 2

(a) Jorge drew a hand to represent sixty minutes past the hour.



(b) Most of the second graders were comfortable working with numberless clocks, but some wrote numbers to count minutes.



Lesson 4

Today's activity begins by asking students to investigate the motion and location of the minute hand as it relates to the hour hand. As the children investigated, we continued to refer to the clock hands as the *long hand* and the *short hand* until it was time to connect the ideas and practice telling time. We started by asking students to use their individual clock (with hands that maintain the correct relationship), point the shorter hand directly at a number, and then share where the longer hand points. You may want to make a table showing the number for each hand and the resulting time (see **table 1**). Students should conclude that when the shorter hand points directly at a number, the longer hand points to the twelve and the resulting time is a time on the hour (an *o'clock*). Similarly, ask students to place the shorter hand halfway between two numbers and ask where the longer hand is pointing. In this instance, students should conclude that the longer hand will point to the six (or *thirty*) and the resulting times are times on the half hour (the *thirties*).

Table 1

Sample Chart of Clock Hands and Corresponding Times

The shorter hand is pointing directly at	The longer hand is pointing at	The time is
1	12	1:00
5	12	5:00
8	12	8:00
3	12	3:00

Next, ask the students to set the time as 2:00 and then to observe what happens when they move the hour hand to 3:00. Be sure that they move the hands correctly (in the clockwise direction). Have them try this with other times that are on the hour or the half hour. After they determine that the minute hand makes an entire revolution around the clock face in one hour, ask them to explain why. Students should realize that one revolution covers sixty minutes and recall that sixty minutes are in one hour.

It is now time for students to synthesize the individual ideas they have been investigating for the last few days and apply them to telling time on a transitional clock with both hands and two sets of numbers. Have students describe some times displayed on a slit clock. Ask which time unit the shorter hand points to. Next, show a clock without numbers. Ask students to determine the number of minutes past the start of an hour and which time unit the longer hand points to. Finally, set a time on a transitional clock. Ask students how to determine the time.

Our class counted the minutes first, then found the hour, and finally combined them, stating the answer in several ways: (1) It is eighteen minutes past the hour. (2) It is after seven o'clock. (3) It is eighteen minutes past seven o'clock. When asked how to write the answer, children responded with both "seven eighteen" and "eighteen seven" until Nicco reminded the class that the hour must always be written first. After his observation, some students switched to identifying the hour first and then counting the minutes past, but others continued to count the minutes first. Both groups seemed to be equally competent at writing the times correctly. After telling a number of times as a class, students practiced with their partners. One partner set a time on the clock, and the other told the time; then they switched roles. We did not identify minutes before the hour with the class, because all our counting on the clocks without numbers was minutes past the hour. However, if students identified a time as minutes before an hour, we did not ask them to change their response.

The investigation ended by asking the children to use their clocks to make times that seemed funny to them and identify the time. Some made random times, but others created times such as one minute after one, two minutes after two, 2:22, 11:11, and so forth.

Beyond the Lesson

Students can create a timeline for their day and then represent the time of various events using slit clocks, transitional clocks, or regular clocks as they progress through the investigation. The investigation could also be enriched by a discussion of impossible times such as 7:82 or 2:61. Similarly, students could be asked to explain why the minute hand cannot point at the six when the hour hand points directly at the three, or why the hour hand cannot be closer to the three than to the four if the time is 3:45.

Reflections

The activities from this investigation can also be used to address times with minutes before the hour. We deliberately omitted the phrases *quarter past* and *quarter of* and referred to such times only as *fifteen minutes past* and *fifteen minutes of*. The students had just finished a money unit that explained "a quarter" as 25 cents, and we anticipated that they would struggle with a quarter of an hour being fifteen minutes rather than twenty-five minutes. Students have not yet had experiences with different representations of fractions, so we did not attempt to reconcile fifteen minutes with one quarter of a clock face but will return to this concept after we introduce fractions. Similarly, if a class is only telling time to five-minute intervals, all of the activities in this investigation are easily adapted. It is not necessary to wait until the class is ready to tell time to the nearest minute.

This investigation is designed to provide students with a working understanding of time units and their connections to the hands of a clock. It provides opportunities for students to incorporate time concepts into their personal experience. As suggested by *Principles and Standards*, the activities emphasize the development of "concepts of time and the ways it is measured," rather than learning just to tell time (NCTM 2000, p. 104).

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Activity Sheet 1. Taking Time to Understand Telling Time

REASONABLE UNITS OF TIME

Name _____

Circle the most reasonable time.

1. A bus ride to school takes
   10 seconds 10 minutes

2. Peeling an orange takes
   2 seconds 2 minutes

3. Tapping your foot once takes
   1 second 1 minute

4. Yawning takes
   4 seconds 4 minutes

5. Telling someone your name takes
   2 seconds 2 minutes

6. Brushing your teeth takes
   55 seconds 55 minutes

Fill in a number to make the sentence reasonable.

1. It takes our class _____ minutes to walk to lunch.

2. It takes me _____ seconds to stand up and push in my chair.

3. It takes _____ seconds to sharpen my pencil.

4. It takes _____ minutes to do our spelling test.

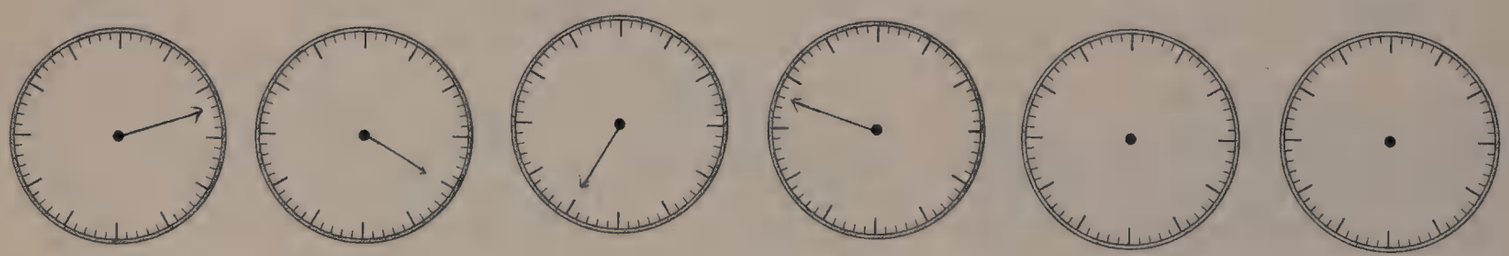
From the November 2008 issue of TEACHING CHILDREN Mathematics

Activity Sheet 2. Taking Time to Understand Telling Time

PRACTICE COUNTING MINUTES

Name _____

How many minutes have passed?



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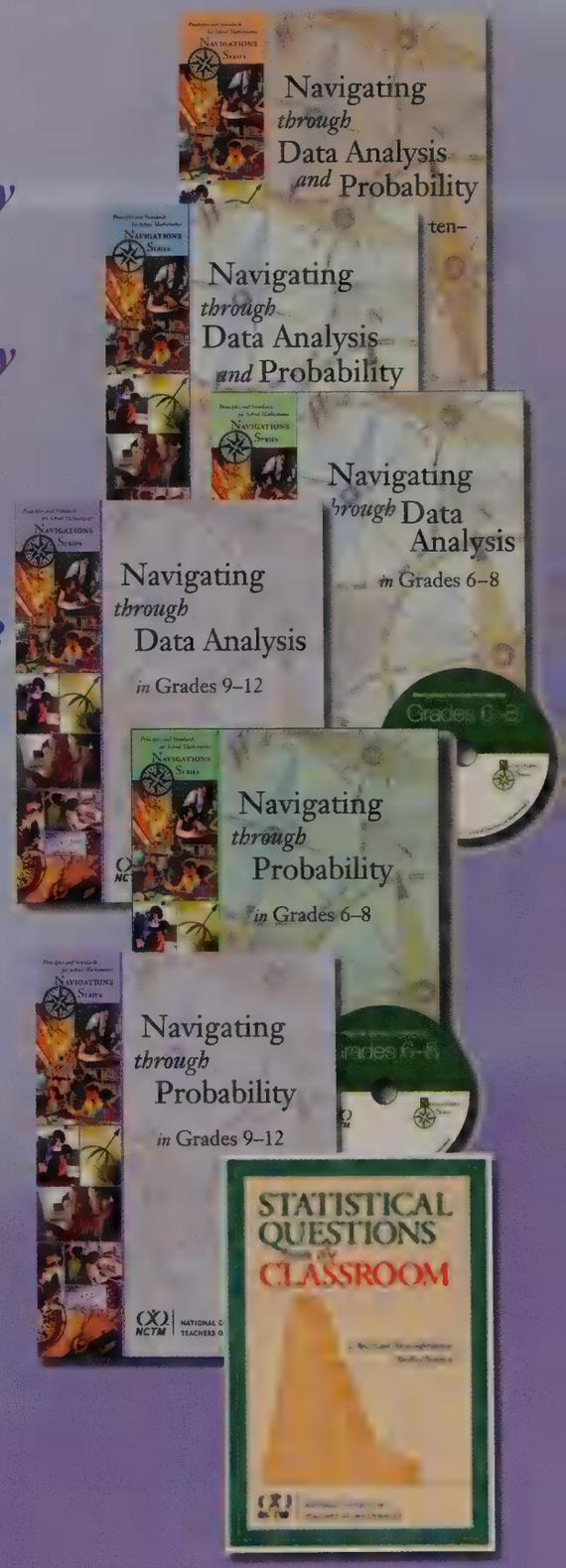
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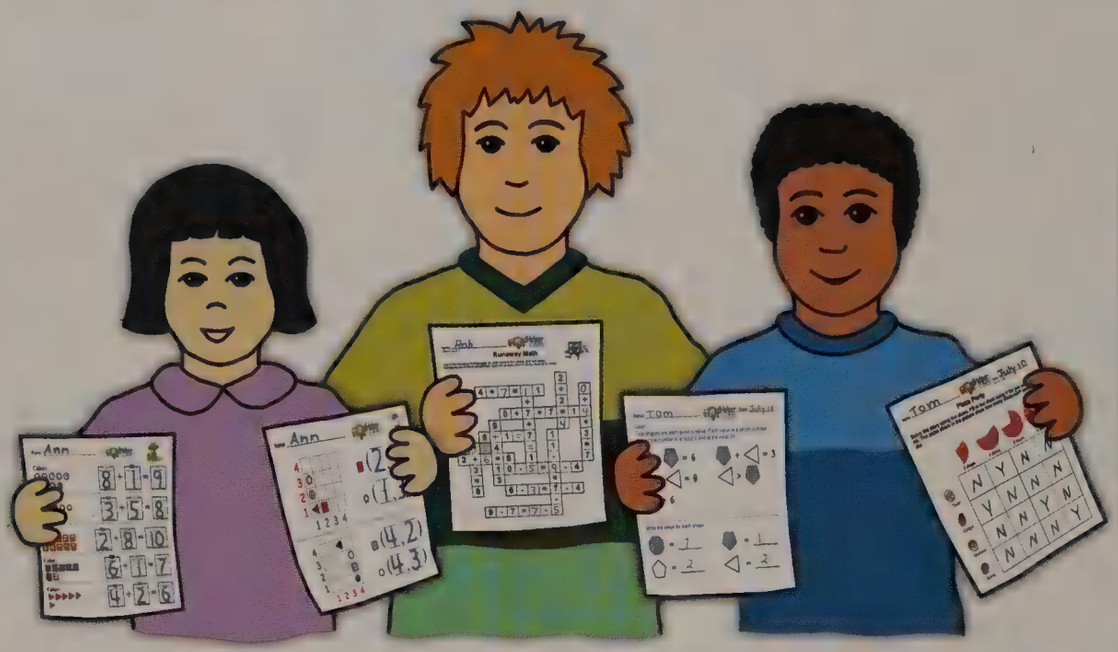
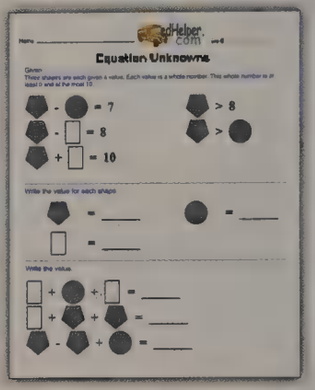


Equation Unknowns



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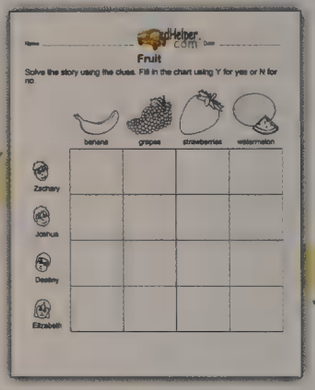

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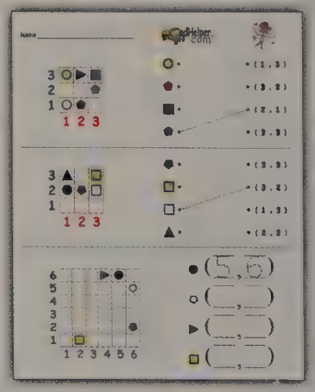

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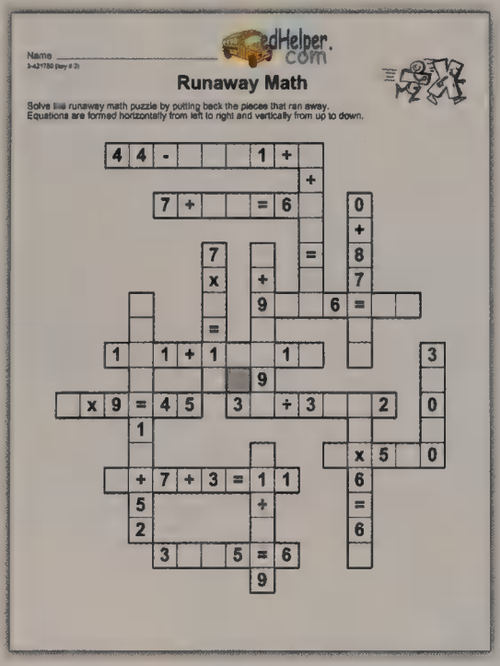
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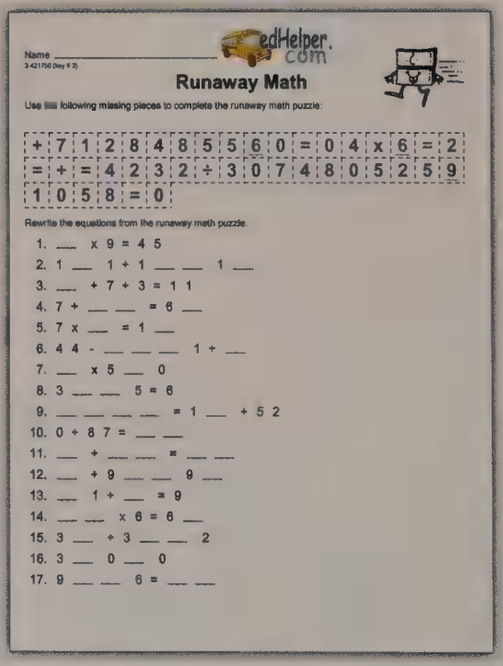
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Mathematics

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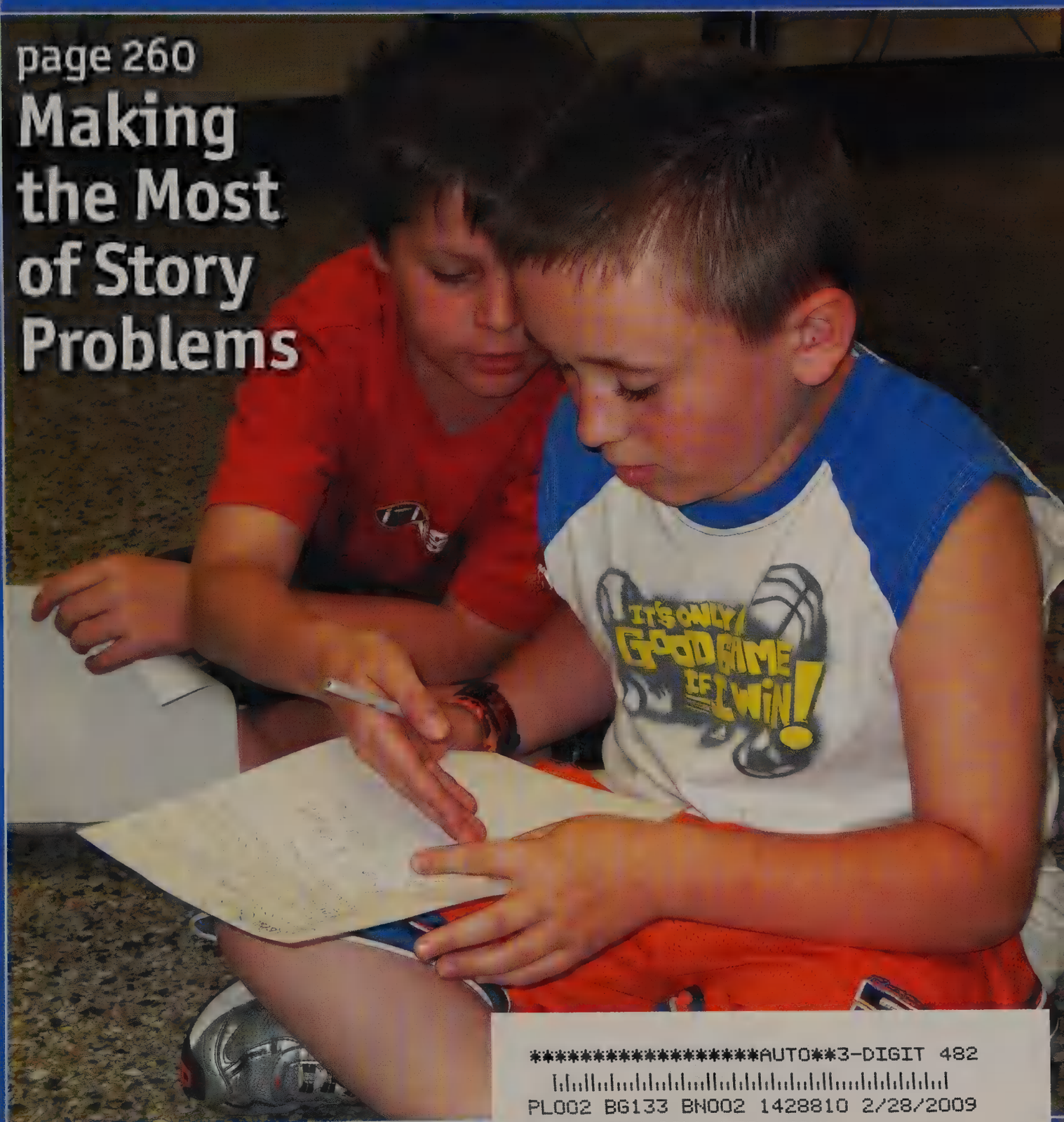
Making the Most of Story Problems



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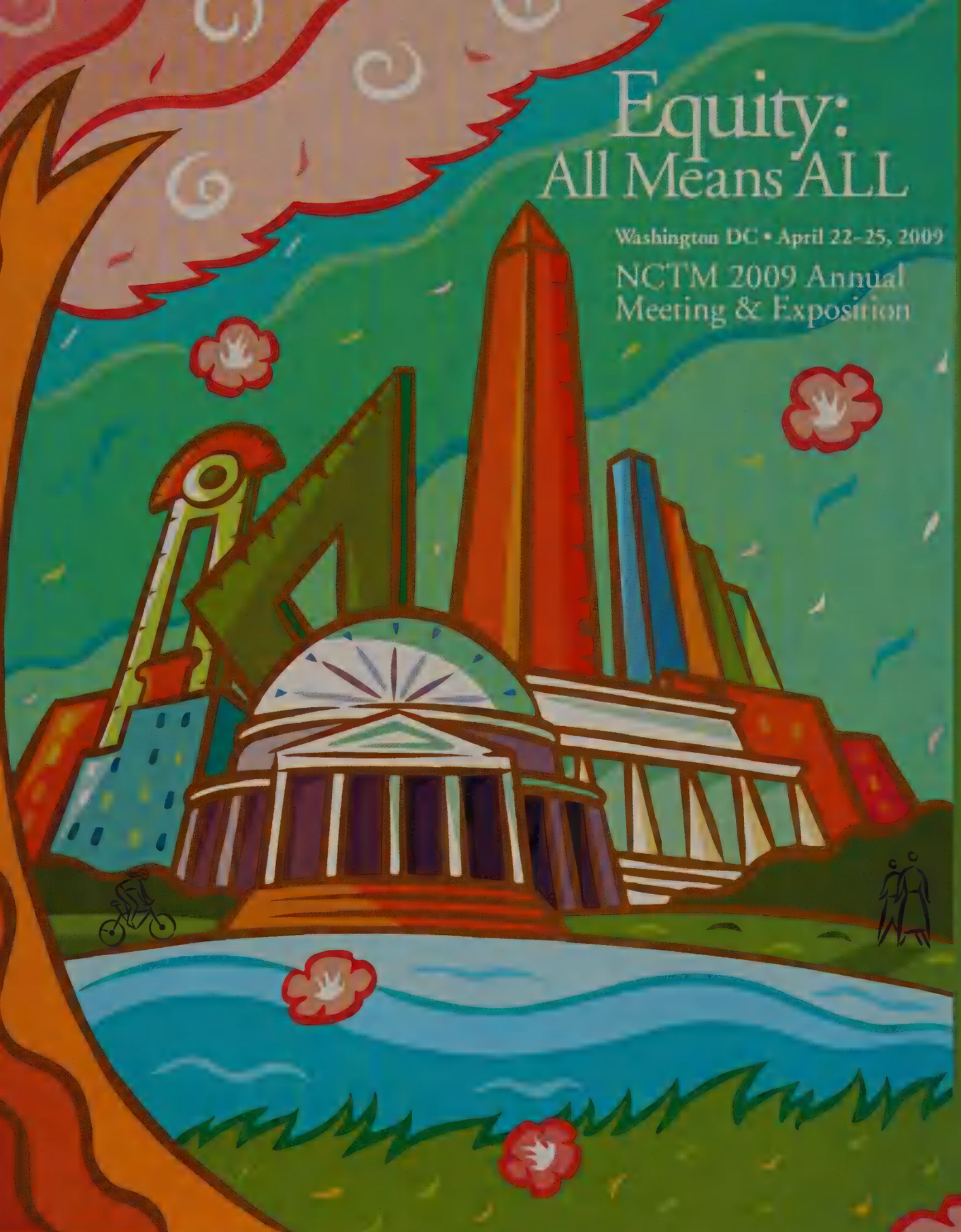

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TEACHING CHILDREN
Mathematics

Volume 15, Number 5,
December 2008/January 2009

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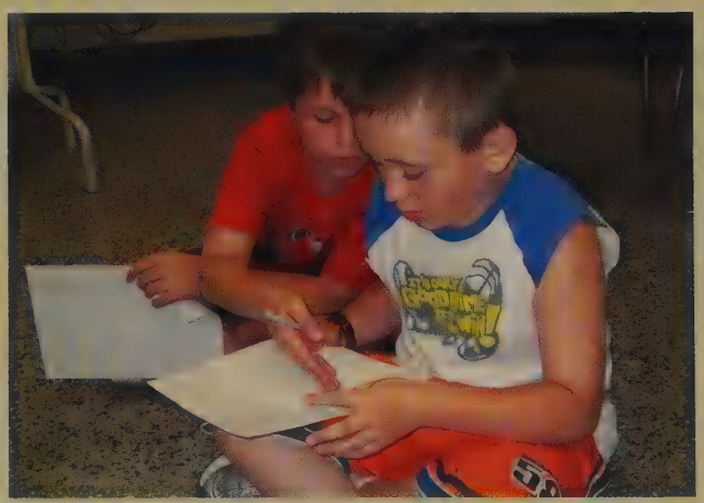

To better understand how teachers can capitalize on the power of story problems, Victoria R. Jacobs and Rebecca C. Ambrose analyzed teacher-student conversations in problem-solving interviews. Their article, "Making the Most of Story Problems," identifies eight categories of teacher moves that, when timed properly, are productive in advancing mathematical conversations.

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CALL FOR MANUSCRIPTS FOR THE OCTOBER 2010 FOCUS ISSUE

Teaching Mathematics in a Flat World

Author Thomas Friedman's "flat world" phrase points to our new century's globally connected nature, exponential technical advances, and instantaneous communication across the planet. Former U.S. Secretary of Education Richard Riley keenly observed, "We are preparing students for jobs that do not exist, using technologies that have not been invented, in order to solve problems we cannot even imagine at this time."

Teaching Tomorrow's Citizens Today

As we hurtle through this rapidly advancing flat world, what does it mean to "know," "do," and "teach" math? Attempting to prepare students for a dynamically changing environment, what mathematical knowledge and essential understanding do elementary teachers need today to help them better educate the workforce and leaders of tomorrow?

The *Teaching Children Mathematics* Editorial Panel is seeking articles to address themes related to the mathematical knowledge of these teachers. We strongly encourage submissions that reflect the value of teacher knowledge directly within the classroom and its influence on student learning and achievement. Moreover, we welcome articles related to preparing preservice teachers as well as practicing teachers. The following topics are to guide—not limit—authors in addressing one or more such issues.

Teachers' Essential Mathematical Knowledge

- Identify the skills, concepts, and dispositions elementary teachers must know to promote both the procedural fluency and the conceptual understanding of their students.
- Provide evidence regarding content areas or specific topics that enable teachers to become more confident and competent in teaching mathematics.
- Describe the *essential* mathematical understandings necessary for elementary teachers and how these may differ from typical mathematics requirements.
- Explain how these essential concepts allow teachers to empower *all* their students with the mathematical skills and innovative problem solving needed in today's flat world.
- Demonstrate the connection between teachers' essential knowledge and the elementary curriculum in state or national standards or within NCTM's *Curriculum Focal Points*.

How to Best Learn

- Provide examples of innovative courses or professional development programs that have been successful in increasing teacher content knowledge and understanding.
- Describe effective, collaborative efforts—among school districts, institutes of higher education, and other agencies—for preparing elementary teachers to teach mathematics.
- Explain obstacles to increasing teachers' mathematical knowledge and how they have been effectively addressed.
- Clarify the importance of "mathematical habits of mind" and pedagogy in relation to learning and teaching mathematics in a flat world.
- Describe technology tools and other resources that support teachers in their work and in their acquisition of essential knowledge.

Impacting the Classroom

- Explicate the effects of an increase in a teacher's mathematical knowledge on the teacher's curriculum selection, teaching practices, and assessment choices.
- Discuss the connections between teachers' mathematical knowledge and student learning and achievement in mathematics.
- Describe specific classroom examples or experiences that demonstrate the importance of teacher knowledge in learning a particular mathematical concept or idea.
- Explore beliefs that teachers hold with regard to the value and type of mathematical knowledge needed in their classrooms.
- Explore the roles of math coaches and elementary math specialists in advancing the understanding of mathematics for both teachers and students.
- Examine the role of teacher collaboration in aiding teachers to reflect on and improve their mathematical knowledge and classroom practice.
- Discuss the level of mathematical understanding required for teachers to successfully address the needs of diverse learners and differentiated instruction.

Manuscripts should not exceed 2500 words; include figures and photographs at the end. Submit completed manuscripts to *Teaching Children Mathematics* by accessing **tcm.msubmit.net**. On the cover page, please state clearly that the manuscript is being submitted for the October 2010 *TCM* Focus issue. Author identification should appear only on the cover page. For manuscript preparation guidelines, visit my.nctm.org/eresources/submission_tcm.asp. ▲

MAKING THE MOST OF

STORY PROBLEMS

Honoring students' solution approaches helps teachers capitalize on the power of story problems. No more elusive train scenarios!

By Victoria R. Jacobs and Rebecca C. Ambrose

Victoria R. Jacobs, vjacobs@mail.sdsu.edu, is a mathematics educator at San Diego State University in California. Rebecca C. Ambrose, rcambrose@ucdavis.edu, is a mathematics educator at the University of California–Davis. They collaborate with teachers to explore children's mathematical thinking and how that thinking can inform instruction.

Story problems are an important component of the mathematics curriculum, yet many adults shudder to remember their own experiences with them, often recalling the elusive train problems from high school algebra. In contrast, research shows that story problems can be powerful tools for engaging young children in mathematics, and many students enjoy making sense of these situations (NCTM 2000; NRC 2001). Honoring children's story problem approaches is of critical importance so that they construct strategies that make sense to them rather than parrot strategies they do not understand.

To explore how teachers can capitalize on the power of story problems, we chose to study

Photograph by Rebecca C. Ambrose; all rights reserved

teacher-student conversations in problem-solving interviews in which a K–3 teacher worked one-on-one with a child. The skills needed for productive interviewing are the same as those needed in the classroom: Teachers must observe, listen, question, design follow-up tasks, and so on. We focused our investigation on interviews because interviews isolate these important teacher-student conversations from other aspects of classroom life.

Supporting and Extending Mathematical Thinking

After analyzing videotaped problem-solving interviews conducted by 65 teachers interviewing 231 children solving 1,018 story problems, we identified eight categories of teacher moves (i.e., intentional actions) that, when timed properly, were productive in advancing mathematical conversations. We separately considered (a) the supporting moves that a teacher used before a student arrived at a correct answer and (b) the extending moves that a teacher used after the child gave a correct answer. We want to be clear that the eight categories of teacher moves we present are not intended to be a checklist that a teacher executes on every problem. Instead, we consider these moves to be a toolbox from which a teacher can draw, after considering the specific situation and instructional goals. In the midst of instruction, the most effective moves arise in response to what a child says or does and, therefore, cannot be preplanned. Because strategically responding to children's mathematical thinking is challenging, we identified our eight categories of teacher moves in an effort to assist teachers in this decision making.

Before a correct answer is given

When a child struggles or has the wrong answer, a teacher must determine how and when to intervene in order to facilitate moving the child forward without taking over the child's thinking. Supporting a student's mathematical thinking requires the teacher to "enter the child's mind" (Ginsburg 1997) as much as possible to determine what the source of difficulty might be. Then the teacher's hypotheses about a child's thinking should drive the choices made. Because "entering the child's mind" can be quite difficult, a teacher needs to be flexible and prepared to explore various supportive approaches. In our analysis, we identified four categories of moves that teachers regularly used to support a child's thinking before the student arrived at a correct answer (see **table 1**).

Table 1

Teacher Moves to Support a Child's Thinking before ■ Correct Answer Is Given

Category	Sample Teacher Moves
Ensure that the child understands the problem.	Ask him to explain what he knows about the problem. Rephrase or elaborate the problem. Use a more familiar or personalized context in the problem.
Change the mathematics in the problem to match the child's level of understanding.	Change the problem to use easier numbers. Change the problem to use an easier mathematical structure.
Explore what the child has already done.	Ask him to explain a partial or incorrect strategy. Ask specific questions to explore how what he has already done relates to the quantities and relationships in the problem.
Remind the child to use other strategies.	Ask him to consider using a different tool. Ask him to consider using a different strategy. Remind him of relevant strategies he has used before.

Ensure that the child understands the problem. A teacher can provide support by helping a child develop an understanding of the problem to be solved. Typical teacher moves include rereading a problem multiple times and asking a child about specific quantities in a problem (e.g., "How many puppies are in the park?"). A twist on this repetition is to ask children to explain problems in their own words. In listening to them describe a story problem in its entirety, a teacher can pinpoint what children do and do not understand.

Rephrasing or elaborating on a story can also help to engage a child. Often, this elaboration involves using a more familiar context or personalization so that the child and her friends are characters in the story. For example, a kindergartner was asked to solve the following problem:

> The teacher has twelve pencils and three baskets. If she wants to put the same number of pencils in each basket, how many pencils should she put in each basket?

The child made a pile of fifteen cubes and kept rearranging them. In response, the teacher, Mr. Reynolds, decided to elaborate and personalize the problem by involving their classroom and making himself the teacher in the story:

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A teacher's most effective teaching arises in response to what a child says or does.

> Let me change it a little bit. Let's try this. Mr. Reynolds has three baskets. I have three baskets, and I have twelve pencils in my hand, and I say, "I've got to do something with these pencils. I can't walk around with them all day! What am I going to do with these pencils? Oh, here's what I'll do. I'll put some in each basket so the kids can come get them." But then I think, "I'd better put the same number in each basket. Because if I put, like, two in one basket and ten in one basket, that's not fair. So I have to put the same number of pencils in each basket." How many pencils would I put in each one of those baskets so that all the baskets would have the same number of pencils inside?

This elaborated story did not change the mathematical structure of the problem but did make the problem more real for the child, and in this case, she solved the problem correctly by using trial and error to create three piles of four cubes each. Elaborating a story may seem counterintuitive because it goes against the traditional approach of helping children identify keywords or irrelevant information in story problems. However, when elaboration is designed to make a problem more meaningful, children are more likely to avoid mechanical problem-solving approaches and instead work to make sense of the problem situation.

Change the mathematics to match the child's level of understanding. When children do not understand a problem, even after attempts to rephrase or elaborate it, changing the problem itself can be productive. One type of change is to use easier numbers. Specifically, using smaller or friendlier numbers (e.g., decade numbers) can help them gain access to the mathematics underlying a problem. After making sense of an easier problem, students generally gain confidence and, in many cases, can then make sense of the original problem.

Similarly, because research shows that children have more difficulty with some problem structures than others, another type of change is to use an easier mathematical structure (Carpenter et al. 1999). For example, a first grader was asked to solve this problem:

> Twelve mice live in a house. Nine live upstairs. How many live downstairs?

Because part-whole problems such as this do not have an explicit joining or separating action, children often do not know how the quantities relate. This student made a set of nine cubes and a set of twelve cubes and joined them to get twenty-one. After several unsuccessful attempts to help the child understand the problem, the teacher chose to change the problem to include an explicit separating action. Specifically, the teacher explained, "Nine of those mice are going to go upstairs and watch TV." In response, the girl separated nine mice from her set of twelve, leaving a group of three. This change in mathematical structure did more than allow the student to solve a problem correctly. By providing her access to an easier but related problem, the teacher created opportunities for discussing the quantities and relationships in both problems. Thus, with further skilled questioning, the teacher could use the child's understanding of the second problem to help her understand the original problem and, more generally, problems with a part-whole structure.

Explore what the child has already done. When struggling with a problem, children can sometimes determine what went wrong if they are encouraged to articulate partial or incorrect strategies. General questions, such as "Can you tell me how you solved it" or "What did you do first?" can be helpful for starting conversations, but follow-up questions require a teacher to ask about the details of a child's

strategy and thus cannot be preplanned. For example, a first grader was asked to solve this problem:

> One cat has four legs. How many legs do seven cats have?

The child (C) put out seven teddy-bear counters. He saw teddy-bear counters as having two legs and two arms and, therefore, counted only two legs on each teddy bear, answering "Fourteen." The teacher (T) recognized that his confusion was linked to the counters he had chosen, and she posed questions to clarify how his work related to the problem context:

T: How many legs on a bear?
C: Two.
T: How many legs on a cat?
C: Four.
T: How many did you count? How many legs *each* did you count?
C: Two.
T: Is that how many legs cats have?
C: No, cats have four, and bears have two.
T: OK, could you do that again for me?
C: First I get one cat [*puts out one teddy-bear counter*], and then I get a bear [*puts out another teddy-bear counter*], and this cat has four legs, and the bear has two legs.
T: Are there bears in the story?
C: No, there's cats.

This dialogue continued for some time before the child solved the problem correctly by counting four legs on each bear and then again by using a different tool. The support the teacher provided began with what the child had already done, and through specific questioning, she helped him make sense of how his initial strategy was related (and not related) to the problem. Note that she could not have preplanned this conversation, because it grew out of her careful observation of his way of using the teddy-bear counters.

Remind the child to use other strategies. Sometimes students get lost in a particular strategy, and instead of abandoning that strategy for a more effective one, they persist in using it in unproductive ways. A teacher can help by nudging them to think more flexibly and to try alternative approaches. A simple suggestion to try a different tool or a different strategy can sometimes give a child permission to move on and self-redirect. At times, a teacher may also find that suggesting a particular tool or reminding a child of strategies used in the past is beneficial. For example, a first-grade student was asked to solve the following problem:

> Let's pretend we're out at the snack tables, and four seagulls come to the snack tables. And then seven more seagulls come to the snack tables. How many seagulls are at the snack tables?

The child first counted to four, raising one finger with each count. She then put those four fingers down. Next, she counted to seven, raising one finger with each count. At this point, the child was baffled, staring at her fingers. The teacher suggested, "Want to try it with cubes?" The child immediately made a stack of four Unifix cubes and a stack of seven Unifix cubes and then counted them altogether to get an answer of eleven. She was confident and efficient once she started using the Unifix cubes. The teacher did not tell the child how to solve the problem but did encourage her to consider using a tool that was more conducive to representing both sets; the child did not have enough fingers to show seven and four at the same time. This support reflected the teacher's understanding of children's

Table 2

Teacher Moves to Extend a Child's Thinking after a Correct Answer Is Given

Category	Sample teacher moves
Promote reflection on the strategy the child just completed.	Ask her to explain her strategy. Ask specific questions to clarify how the details of her strategy are connected to the quantities and mathematical relationships in the problem.
Encourage the child to explore multiple strategies and their connections.	Ask her to try any second strategy. Ask her to try a second strategy connected to her initial strategy in deliberate ways (e.g., more efficient counting or abstraction of work with manipulatives). Ask her to compare and contrast strategies.
Connect the child's thinking to symbolic notation.	Ask her to write a number sentence that "goes with" the problem. Ask her to record her strategy.
Generate follow-up problems linked to the problem the child just completed.	Ask her to solve the same or a similar problem with numbers that are more challenging. Ask her to solve the same or a similar problem with numbers that are strategically selected to promote more sophisticated strategies.

direct-modeling strategies in which they represent both sets before combining them.

After a correct answer is given

Solving a story problem correctly using a valid strategy is an important mathematical endeavor. However, we view problem solving as a context for having mathematical conversations, and this conversation need not end when the correct answer is reached. Instead, a teacher can pose additional questions to help students deepen their understanding of completed work and connect it to other mathematical ideas. We have identified four categories of moves that teachers regularly used to extend children's thinking after arrival at a correct answer (see **table 2**).

Promote reflection on the strategy just completed. Once a student has correctly solved a problem, a teacher can ask for a strategy explanation or for clarification about how the use of a particular strategy makes sense with the quantities and mathematical relationships expressed in the problem. Articulating these ideas can reinforce a child's understanding and give a teacher a window into that understanding. Again, attention to detail matters. Similar to the supporting questions intended to explore children's partial or incorrect strategies, teachers' extending questions were most productive when they were specific and in response to the details of what a student had already said or done. For example, a second grader was asked to solve this problem:

> This morning I had some candy. Then I gave you five pieces of candy. Now I have six pieces of candy left. How many pieces of candy did I have this morning before I gave some to you?

The student quickly solved this problem mentally and explained, "Five plus five, if you took one away, is ten and then one more is eleven, so you had eleven." Children often provide correct answers to problems with this structure, in which the initial quantity is unknown, without really understanding what they are finding. In this case, the teacher probed the child's thinking in relation to this issue:

T: So how did you know to add them together?
C: I don't know. I just added them, I guess.
T: Well, think about it. Why does that make sense for the problem?

The child thought about this question for some time and eventually used Unifix cubes to act out the story and convince himself (and the teacher) that eleven was the correct answer *and* made sense with the story. By asking him to reflect further on his strategy, the teacher ensured that he was making sense of the mathematics.

Encourage the child to explore multiple strategies and their connections. Children need opportunities to not only solve problems but also explore the mathematical connections among multiple strategies for the same problem. One approach is to ask them to generate a second strategy—any strategy—to a problem they have already solved. Another approach is to ask for a second strategy that is connected to their initial strategy in deliberate ways. For instance, a third grader using base-ten blocks to represent 12 pages of 10 spelling words per page put out 12 tens rods but counted all 120 blocks by ones! The teacher built on this initial strategy by asking her to count the blocks another way. The child responded by counting by tens and even shared that this second strategy was easier.

Another way a teacher can deliberately build on an initial strategy is to ask for a mental strategy that is an abstraction of work with manipulatives. For example, a third grader was asked to solve the following problem:

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Although a teacher may initially support a student's step-by-step recording of a strategy, he should diligently support the *child's* thinking.

> There are 247 girls on the playground and there are 138 boys on the playground. How many children are on the playground?

The student initially represented both quantities with base-ten flats, rods, and single cubes. Next he combined the hundred flats (3), combined the ten rods (7), combined some of the single cubes to make 10, traded the 10 single cubes for 1 tens rod, making a total of 8 tens rods, and finally counted the remaining single cubes (5) to answer 385. The teacher then asked, "Doing just what you did with the materials, could you solve that problem in your head?" The child looked at the numbers and abstracted what he had just done with the cubes. Specifically, he explained that he could add 100 to 200 to get 300 and then add 30 to 40 to get 70. Next he put 2 from the 7 with the 8 to get another 10, which made 80, and had 5 ones left, so the answer was 385. When executing this mental strategy, the child articulated the underlying mathematical idea of both strategies: combine like units and, when necessary, regroup (i.e., decompose the 7 into 5 and 2 so that the 2 can be combined with the 8 to make a new 10).

Through experiences with multiple strategies, children can gain the ability and flexibility to change strategies when one is unsuccessful. A teacher can also use multiple strategies to highlight underlying mathematical ideas by asking students to explicitly compare and contrast strategies. At times, a teacher may even ask a child to compare a successful strategy to a previously unsuccessful attempt, because, in many cases, the child will discover the reason the strategy failed.

Connect the child's thinking to symbolic notation. When solving a story problem by drawing, using manipulatives, or computing mentally, students may not use any symbolic notation. A teacher can encourage students to connect their work with mathematical symbols by asking them to either generate a number sentence that "goes with" the problem or record the strategy used to solve the problem. Although requesting a number sentence that "goes with" the problem is perhaps the more typical request, asking for a strategy representation can be powerful. Young children often begin recording their strategies in unconventional ways that include a mix of symbols and drawings. They might draw pictures of manipulatives they used and then add number labels to parts of those pictures. Over time, children's recordings become progressively more abstract until they are completely symbolic.

Generating a symbolic representation of a strategy can help children develop meaning for, and facility with, mathematical symbols because the representation is linked with their interpretation of the problem. For example, a second grader solved the problem about the number of legs on seven cats by first putting out seven tiles (cats). Next he moved two tiles to the side and said, "Four plus four equals eight." He then moved another tile to the side and said, "Eight plus four equals twelve." He continued moving one tile at a time until he had used them all, each time adding four more to his running total. When asked to write a number sentence to show what he had done, he wrote the following:

$$4 + 4 \rightarrow 8 + 4 \rightarrow 12 + 4 \rightarrow 16 + 4 \rightarrow 20 + 4 \rightarrow 24 + 4 \rightarrow 28$$

Unlike the number sentence that "'goes with" the problem ($7 \times 4 = 28$), his symbolic representation reflects how this student thought about and solved the problem. Note that his use of arrows instead of equal signs avoids the incorrect use of the equal sign between expressions.

Requesting links between strategies and symbolic notation is important so that children see the mathematics done on paper as connected to solving story problems. Moreover, once children become facile with symbolic notation, the notation itself can become a tool for problem solving and reflection. We offer a final note of caution: A teacher may initially need to support a student in recording each step of a strategy so that parts are not omitted. However, a teacher needs to be vigilant in providing support to record the child's—not the teacher's—ways of thinking about a problem.

Generate follow-up problems. By carefully sequencing problems, a teacher can create unique opportunities for mathematical discussions. Although we recognize the importance of practice, we are suggesting something beyond simply assigning additional problems to solve. We advocate that, in the midst of instruction, a teacher can consider a child's existing understanding and then modify the initial problem or create a new problem to add challenge or to encourage use of more sophisticated strategies. A first grader was asked to solve this problem:

> The Kumyeey woman was collecting acorns. She had nine baskets, and she put ten acorns in

each basket. So how many acorns did she have altogether?

The child quickly responded, "Ninety," explaining that he had counted, "Ten, one," putting up ten fingers and then one finger. He continued, "Twenty, two," again putting up ten fingers and this time, two fingers. He continued with this pattern of counting and finger use: "Thirty, three; forty, four; fifty, five; sixty, six; seventy, seven; eighty, eight; ninety, nine." The teacher then decided to extend the child's use of ten by posing a related problem and asking him to consider the connections:

T: So that's how you got ninety. What if she had nine baskets, but she put eleven in each basket instead of ten? [*Child thinks for a while.*] Could you use some of the work that you've already done—that we did during the afternoon—or would you have to start all over again? She still has nine baskets, and there are still ten acorns in each basket, and then she puts in one more so that each basket has eleven.
C: Ohhhhh! I get it. Well, there's already ten in each basket, so that's ninety. So I count up nine, one more nine. I mean nine ones. I'm going to add nine ones. So there's already ninety, so ninety-one, ninety-two, ninety-three, ninety-four, ninety-five, ninety-six, ninety-seven, ninety-eight, ninety-nine.

By strategically selecting numbers and by drawing attention to the link between problems, the teacher was able to further this child's base-ten understanding by helping him recognize and use the ten in the number eleven.

Summary

Our project builds on previous work on teacher questioning (see, for example, Mewborn and Huberty 1999; Stenmark 1991), which provide lists of potential questions. These lists can be important starting points for eliciting a student's thinking, but we hope that our eight categories of teacher moves (four designed to support children's thinking and four to extend it) can help teachers further customize questioning to make the most of story problems. These moves do not always lead to correct answers, and we reiterate that not all eight are intended to be used in every situation. However, together they form a toolbox from which teachers can select means to help students solve problems and explore connections among mathematical ideas. Engaging in mathematical discussions about story problems is challenging; we offer three final guidelines:

- **Elicit and respond to a child's ideas.** The most effective teacher moves cannot be preplanned. Instead, they must occur in response to a student's specific actions or ideas. Thus, expertise is tied less to planning before a student arrives and more to seeding conversations, finding the mathematics in children's comments and actions, and making in-the-moment decisions about how to support and extend children's thinking.
- **Attend to details in a child's strategy and talk.** Research on children's developmental trajectories shows that subtle differences in children's strategies and talk can reflect important distinctions in their mathematical understandings (NRC 2001). A teacher can customize instruction on the basis of these distinctions, and by attending to details of a child's explanations and comments, a teacher also communicates respect for a child's ideas.
- **Do not always end a conversation after a correct answer is given.** Important learning can occur after students give a correct answer if the teacher asks them to articulate, reflect on, and build on their initial strategies.

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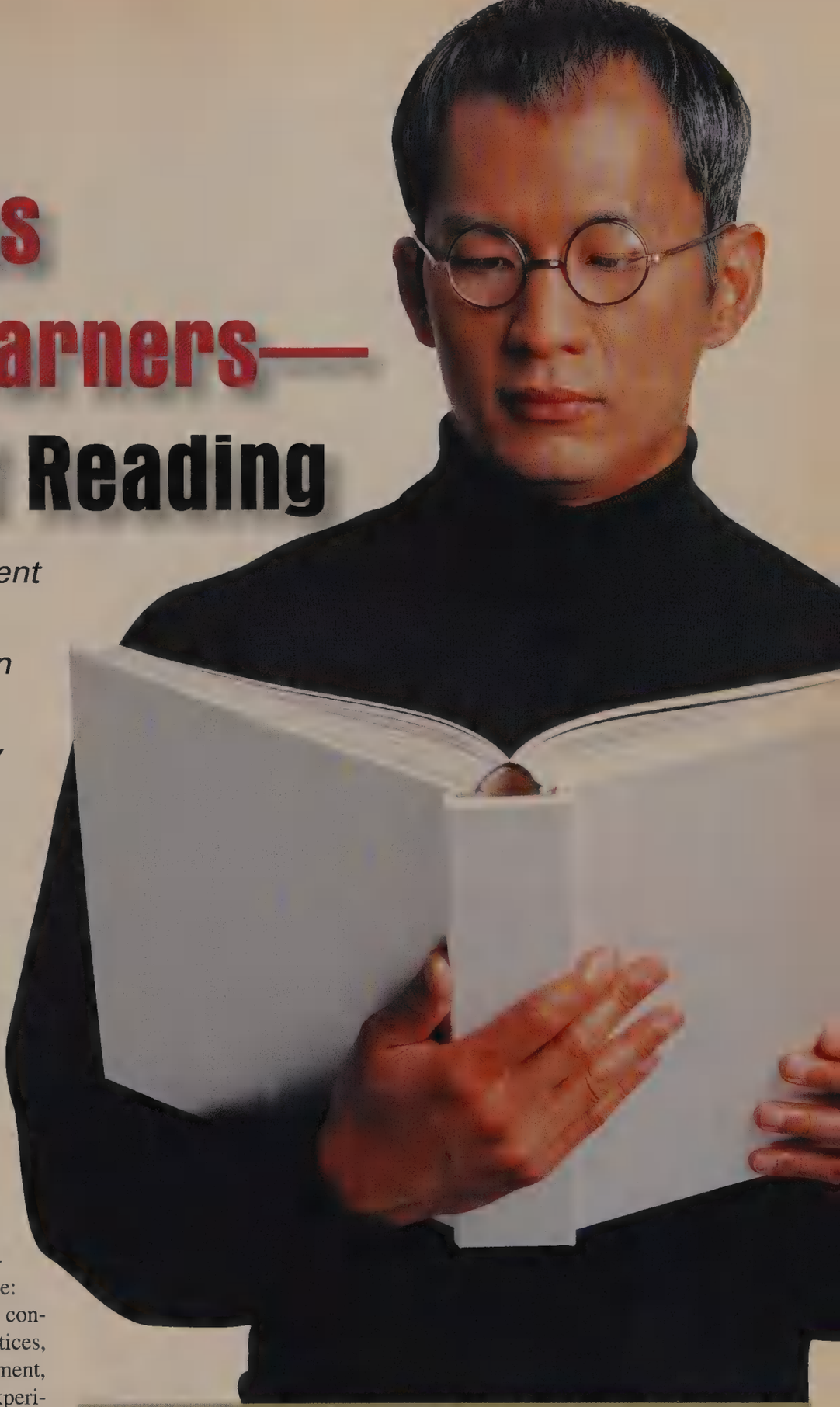
Teachers as Lifelong Learners—The Role of Reading



Why is it so difficult for student teachers to consistently understand, apply, and retain important content from professional literature? They must be taught strategies to read in this specific genre.

As a teacher educator, I realize that "effective teaching requires continually seeking improvement" (NCTM 2000, p. 19). In working with prospective elementary teachers, I strive to equip my students with tools and resources that will support their ongoing professional growth beyond the college classroom. Professional journals such as *Teaching Children Mathematics* are convenient, readily available resources that teachers can rely on to enhance mathematical and pedagogical content knowledge. When read carefully, articles in these journals provide a variety of ways for teachers to improve: interesting new ways to look at mathematical content, research on promising pedagogical practices, insight into students' mathematical development, and resources to enrich classroom learning experiences. For these reasons, preparing future teachers to use this valuable resource is a major goal in my mathematics education courses.

In the past, I tried to engage my students in the articles' mathematical and pedagogical content through whole-class discussions, journal prompts, article critiques, small group "book" clubs, and an annotated bibliography project. Regardless of

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the method I used, however, my students typically could not connect with the writing at anything beyond a literal level. They could accurately identify facts from the article but had a hard time analyzing or retaining an article's ideas for any length of time. We would have a great whole-class discussion on a reading, but in a few weeks many of the students could not recall the key ideas we had talked about. I was at a loss. Why was it so hard for my students to consistently comprehend, apply, and retain important mathematical and pedagogical concepts from these articles?

Coincidently, as I was struggling with this question, I was also reading the current research regarding reading comprehension. I stumbled across a possible answer to why my students labored to comprehend the professional journal articles I assigned. *Mosaic of Thought* (Keene and Zimmerman 1997) describes how a lack of experience with reading in a particular genre can lead to difficulty in comprehending and retaining information. The challenge they described seemed to match my students' difficulty; although my students have the mechanics necessary to read professional journal articles, they are often unaware of the strategies required to read effectively in this specialized area (Tovani 2004).

Reading Comprehension Strategies

What are these effective reading comprehension strategies? Based on the proficient-reader research synthesized by Keene and Zimmerman (2007), the strategies are a small set of metacognitive processes that enable a reader to independently make sense of challenging text. Effective readers use the strategies fluently and flexibly, but learning them requires a great deal of intentionality. These strategies (see **table 1**), which support comprehension regardless of age or grade level, are identified as the following:

- Monitoring meaning
- Using and creating schema
- Asking questions
- Inferring
- Creating mental images
- Determining importance
- Synthesizing information

Although the comprehension strategies represent natural mental processes, using them in an intentional manner is a learned skill. Readers develop effective use of the strategies through focused, explicit, long-term practice (Harvey and Goudvis 2000); this does not mean that readers encounter or learn these strategies in isolation. Consider the metaphor of a spotlight on a stage filled with players (comprehension strategies) interacting with one another but with the spotlight beam focusing our attention on a single one. Watching the highlighted player helps us learn the role and its place in the overall act. All the interrelated workings of the comprehension strategies are too numerous to address in this article, but the resulting combination provides a powerful set of processes that readers can access as they try to comprehend difficult text.

Metacognitive Comprehension Strategies

As I reflected on the research in reading comprehension, observed what proficient readers do in an elementary classroom, and applied these comprehension strategies to my own reading, I noticed a change in the way that I read. The strategies affected the way I made sense of information as I

Table 1

Reading Comprehension Strategies
(Anyone interested in these strategies is directed to the references at the end of this article. Each of these seven strategies requires metacognitive ability—the ability to think about your own thinking.)

Monitoring meaning occurs when the reader can stop and think when a text does not make sense or be aware of and acknowledge when it does.

Using and creating schema involves activating the known to make sense of the new. Many comprehension lessons address three categories of connections: text-to-self, text-to-text, and text-to-world (Keene and Zimmerman 1997).

Asking questions refers to a reader's ability to question the text as she reads. These may be questions about words, author's intent, or possible applications—all of which can propel the reader forward.

Inferring occurs as the reader make conjectures about the text based on prior understanding. Teachers often demonstrate inference through think-alouds by stopping and predicting what might happen next or what is going on behind the scenes (Miller 2002).

Creating mental images is essentially the ability of readers to place themselves into the text. Readers who connect to sensory input from texts are more engaged in making sense of what they are reading.

Determining importance requires a reader to prioritize information through purpose and experience. Readers who ask, "Why am I reading this?" have an easier time discerning what to attend to in the text (Tovani 2004).

Synthesizing information is the most sophisticated strategy in that it combines elements from each of the others. Synthesis is more than summary; it is the reader's ability to make meaning from the text and integrate it into her understanding.

connected my experiences and current knowledge with new ideas presented in a text. I wondered if my prospective teachers could apply the same processes in reading *Teaching Children Mathematics* articles. So, I decided to explicitly teach my students some of these metacognitive strategies as a way to enable them to become more proficient readers in this genre. I trusted that through this process, they would discover the relevance of this resource, for both teaching and understanding mathematics.

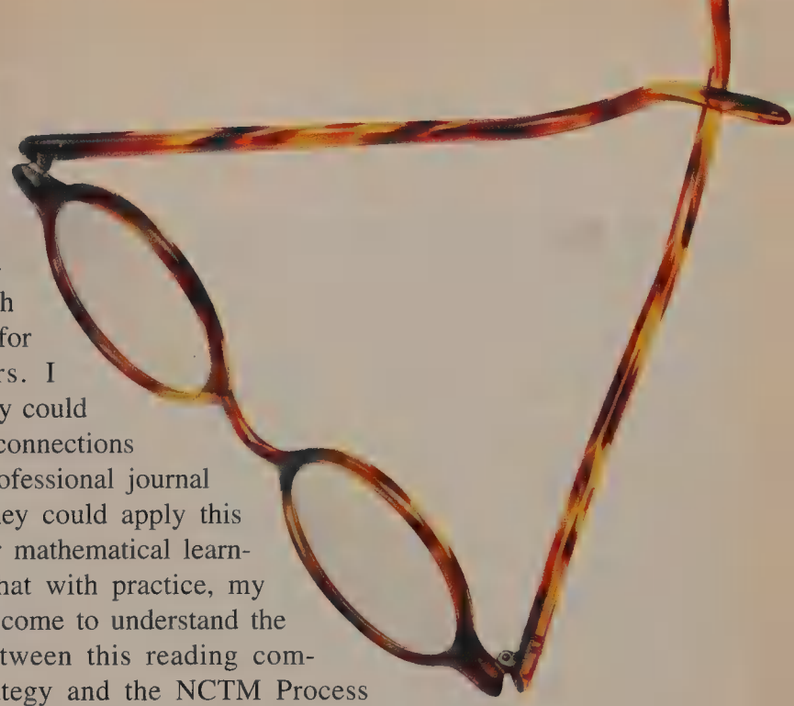
Using and creating schema (making connections)

A question emerged: "Which strategy should I focus on first?" Research in reading comprehension does not indicate a particular order for teaching these comprehension strategies; instead teachers should make well-informed choices of instruction based on the needs of the students, the demands of the text, and the students' previous experiences with these reading strategies (Keene and Zimmerman 2007). I decided to spotlight the metacognitive process of using and creating schema because it has many parallels with NCTM's Connection Standard (2000), and we were specifically studying the Connection Standard as a way to understand *how* we do, learn, and teach mathematics. In addition, authors such as Keene and Zimmerman, in their first edition of *Mosaic of Thought* (1997), and Miller (2002) recommended this strategy as a reasonable starting point. This strategy emphasizes that in order to engage with and make sense of a text, readers—

- purposefully recall relevant prior knowledge before, during, and after they read and learn;
- use current schema to make sense of new information as they read and then store this information with related knowledge in their memory; and
- assimilate information from the text and other learning experiences into their prior knowledge (i.e., connect new understanding with prior knowledge) and then change their schema accordingly by deleting inaccurate information, connecting knowledge, and adjusting or building on existing schema. (Keene and Zimmerman 2007)

Although this strategy focuses on reading, it echoes much of the cognitive activity underlying the Connection Standard (NCTM 2000); both emphasize connecting new knowledge with previous knowledge and applying it in new situations.

My goal was to demonstrate the power that making connections has for learning by intentionally highlighting this metacognitive approach in my courses for future teachers. I knew that if they could learn to make connections as they read professional journal articles, then they could apply this strategy in their mathematical learning. I trusted that with practice, my students would come to understand the relationship between this reading comprehension strategy and the NCTM Process Standards, realizing that this strategy applies not only to reading but also to making sense of and learning new ideas in mathematics, in teaching, and in general.



A Framework

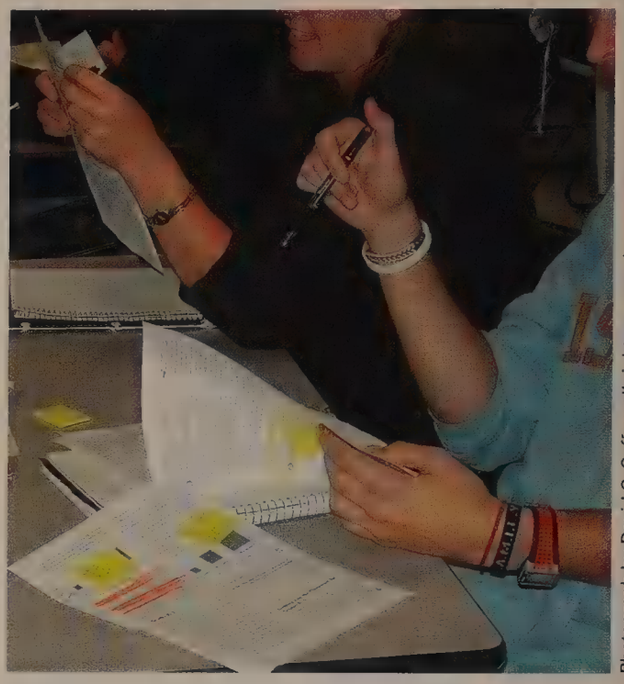
As I reflected about ways to explicitly teach this metacognitive process, I hypothesized that my students needed some type of concrete framework to help them identify, categorize, and organize their schema so that they could make connections and relate what they are reading to what they already know. In-service teachers can read an article from *Teaching Children Mathematics* and relate its ideas to their own practice; most of my preservice students do not have the type of classroom experiences (either as students or teachers) that allow them to connect with and make sense of the scholarly reading I was assigning. As a result, they struggle to gain much from the reading because they do not know what experiences to build on; how to purposefully recall relevant prior knowledge before, during, and after they read; or how to connect ideas from the article with their previous knowledge and experiences. Once again, I drew from the reading comprehension research for ideas; by explicitly identifying text-to-self, text-to-text, and text-to-world connections before, during, and after the reading experience, a reader learns to link previous knowledge with concepts and ideas presented in a text. When I identified a link or relationship between a reading comprehension strategy and the Connection Standard, I made what some reading comprehension authors categorize as a text-to-text connection (Harvey and Goudvis 2000; Keene and Zimmerman 1997). As the names imply, these categories represent a way for readers to organize and categorize their thinking; however, the reading

Table 2

Annotated Bibliography Text Connections

As you read each article, identify passages that relate to the questions provided in the categories below. You should use sticky notes or write directly on the article to answer the questions and mark the text that connects to your thinking. Ultimately, this will aid you in developing your schema about teaching mathematics in elementary classrooms.

Text-to-Problem **Doing Math**	Text-to-Standards **Teaching Math**	Text-to-Teaching **Teaching in General**
Where is the passage you are making a connection to?	Where is the passage you are making a connection to?	Where is the passage you are making a connection to?
What problem are you reminded of?	Which Principle or Standard (from *Principles and Standards*) did this passage remind you of?	What does this portion of the article say about promising practices in teaching mathematics?
How did this problem remind you of the other problem?	How are this passage and the Standard related?	How do you see this impacting your math teaching in the elementary schools?
Why is this connection important to your understanding of the article's content?	Why is this connection important to your understanding of the article's content?	Why is this connection important to your understanding of the article's content?



Photograph by David C. Coffey; all rights reserved

Small-group discussion helped teacher candidates realize that the ultimate goal is not to identify the *type* of connection but to develop intentional habits of recognizing and seeking out particular connections.

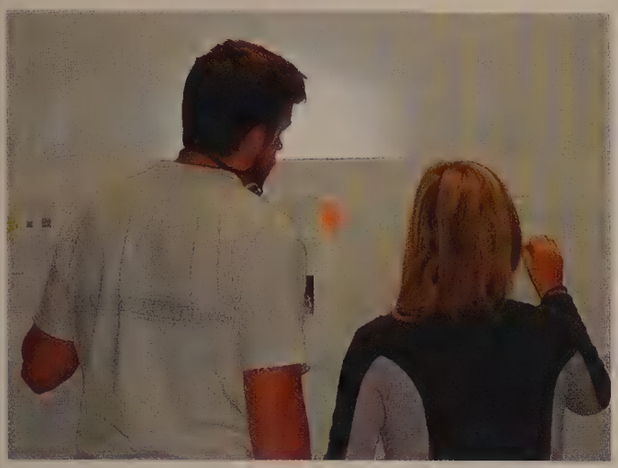
categories themselves are not as important as the actual connections the reader makes. Ultimately, the goal is not to identify the type of connection made but rather to develop intentional habits of recognizing and seeking out particular connections that will enhance the reading experience.

The annotated bibliography project

Using this framework, I decided to overhaul and refocus an assignment I had previously used, a project requiring prospective teachers to develop an annotated bibliography of articles addressing mathematics and its teaching in elementary schools. The articles are from *Teaching Children Mathematics* and are chosen because they relate to mathematical topics addressed in my course. Instead of having students summarize the key mathematical and pedagogical points as I had required in the past, I now wanted my students—on the basis of what we had previously studied in class—to identify specific connections that related what they already knew with ideas presented in the articles. I developed categories of text-to-problem, text-to-Standards, and text-to-teaching connections (see **table 2**) to match my students' experiences in our course curriculum and to provide a framework for identifying and categorizing schema that would help students make sense of and explicitly make connections with the new material before, during, and after the reading process.

Harvey and Goudvis (2000) suggest explicitly modeling each reading strategy. To completely and accurately model these strategies, teachers must share both actions and thoughts using "think-aloud" lessons (Miller 2002). So, I first carefully read through the first article I assigned, "Lessons from the TIMSS Videotape Study" (Geist 2000), and identified and categorized the connections I had made while reading the article. For example, as I read the article, I looked for text-to-problem connections. In order to identify and organize these connections, I asked myself questions: Did anything in the article remind me of a problem we had already solved or discussed in class? In what ways? Why or how is this connection (to a previous problem or strategy) important for understanding the content of this article? Similarly, I made text-to-Standards and text-to-teaching connections using the **table 2** questions to guide me.

After assigning the same article for my students to read, I spent part of the next class period "thinking aloud" about the connections I had made while reading the article so that my students could hear and see how I had identified these connections. This led to a discussion of how my *process* of making con-



Photograph by David C. Coffey; all rights reserved

Group members posted their connections on one copy of the article they had all read.

nections as well as the actual connections I made compared with my students' processes and how I had made sense of the article by explicitly making these connections. Because I was expecting my students to eventually make their own connections as they read articles and formally wrote up these connections, I also modeled how I might choose and then write about the connections I had made in a formal annotated bibliography entry, making it explicit that I wanted my connections to clearly and concisely represent my current understanding of the article and that I was free to revise them later. On the basis of this demonstration, I wrote my annotation of the article (see **fig. 1**) and posted it on our course Web site for discussion and feedback.

My next instructional step was, as a class, to collectively look for connections before, during, and after the reading process. I selected the article "Experiences with Patterning" (Ferrini-Mundy, Lappan, and Phillips 1997) because we had spent the past few class periods exploring algebraic thinking and children's reasoning through the context of patterns. After students independently read the article, they were instructed to write the strongest connection they made in each category on a sticky note. They were also to note the type of connection. Coding text-to-problem connections as T-P, text-to-Standards as T-S, and text-to-teacher as T-T made it easier to see how people reading the same passage might make very different types of connections.

Students met in groups to discuss the connections they had made while reading the article as well as to describe how these connections helped them better understand ideas presented in the article. After each student shared, group members supported one another by asking questions, discussing the connections, and then selecting what they felt to be the strongest connections identified during their discussion. Meanwhile, I had physically taped the pages of the article on the board at the front of the classroom. As students identified connections, a group member placed the sticky notes with the strongest connections next to the corresponding passage on the posted article. Then, we engaged in a whole-class discussion to compare and reflect on the connections they had made and how these connections helped us activate and develop our understanding of the article and what we were studying in class. We discovered collectively that students had the most difficulty identifying text-to-Standards connections. Consequently, we spent some time talking about how to apply this same cognitive reading strategy to the reading and making sense of *Principles and Standards* (NCTM 2000). We realized that as we made connections between the article and the Standards, our understanding of the Standards would also deepen.

Figure 1

The instructor modeled his process for writing an annotation example.

Geist, Eugene A. "Lessons from the TIMSS Videotape Study." *Teaching Children Mathematics* 7 (November 2000): 180–85.

Connections
by David Coffey

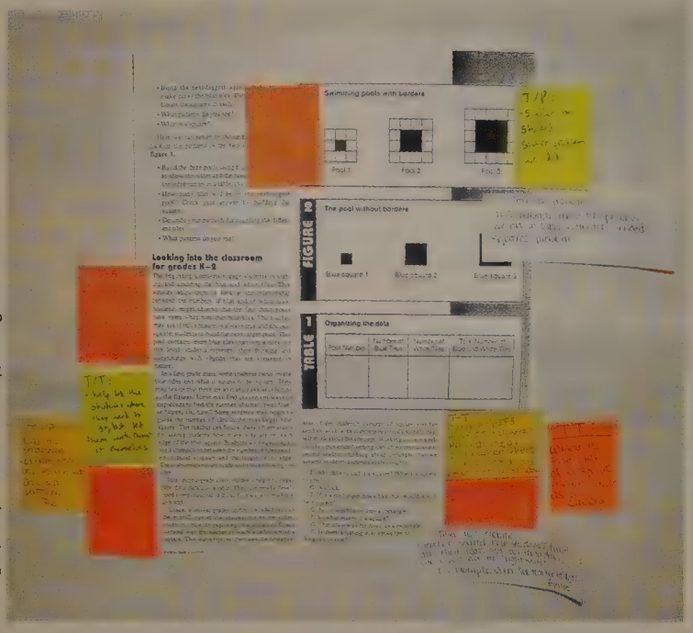
Text-to-Problems
On page 181, under the first characteristic of mathematicians, I thought the comparison between mathematicians working on a problem for a long time and U.S. students working on a lot of questions in a short time reflected our experience with the Charlie Brown problem. Rather than practice a lot of exercises over and over (as we saw on the overheads from the sixth-grade textbook), we worked on a few modified problems that challenged us to use the problem-solving process. This really helps me to think about the difference between a problem and an exercise. I thought that the comment about "making errors as stepping stones to solutions" (p. 184) connected to the Learning Principle. In particular, students develop greater understanding in mathematics by building on prior experiences. I see that even if an experience leads to an incorrect result, it is a learning opportunity.

Text-to-Standards
I thought that the comment about "making errors as stepping stones to solutions" (p. 184) connected to the Learning Principle. In particular, students develop greater understanding in mathematics by building on prior experiences. I see that even if an experience leads to an incorrect result, it is a learning opportunity.

Text-to-Teaching
A comment under the sixth characteristic on page 183 really struck me. The idea that allowing students to think for themselves can be as engaging as playing a video game or putting together a puzzle is intriguing to me. I now put in a lot of time designing classroom activities, and practicing teachers point out that they do not have that kind of time for planning. It makes me think, "How can I shift my teaching so that the focus is on thinking rather than the bells and whistles I now spend so much time preparing?"

Figure 2

(a) At the end of the class period, the instructor collected the connections.



Photograph by David C. Coffey; all rights reserved

(b) The instructor documented the connections to post on the class Web site.

Ferrini-Mundy, Joan, Glenda Lappan, and Elizabeth Phillips. "Experiences with Patterning." *Teaching Children Mathematics* 3 (February 1997): 282–88.

Connections
by MTH 221, section D, Fall 2006

Text-to-Problems

⇨ Page 112, 2 column, last paragraph: This reminds me of the Share Square problem. The representation looks almost identical, but the contest is different.
⇨ Page 113, 1 column, 1 paragraph: The swimming pool with borders was very similar to our Shaded Square problem.
⇨ Page 113, 2 column, figure 2: Use different representation methods to help understand problem.
⇨ Page 113, 2 column, table 1: Reminds us of the problem in class we did about shaded boxes.
⇨ Page 115, 2 column, 1 paragraph: Observations are important for finding patterns like we did in class at our table.
⇨ Page 115, 2 column, quote: This reminds me of how Dara asks us leading questions to help us see patterns and relationships.

Text-to-Standards

⇨ Page 114, 1 column, 4 paragraph: One part of the Communication Standard talked about using definitions.
⇨ Page 114, 1 column, dialog portion: Using representations with tiles to help solve a problem; organize their mathematical thinking through communication with teacher.
⇨ Page 114, 2 column, figure 3: This reminds me of the Communication Standard that students should be able to communicate their mathematical thinking.
⇨ Page 115, 1 column, 3 paragraph: Encourages students to reason.
⇨ Page 115, 2 column, quote: Communication can get kids going ... get on the right track.
⇨ Page 116, 1 column, figure 4: Representation
⇨ Page 116, 2 column, dialogue portion: In order to justify the pattern-equation found, one needs good communication skills.

Text-to-Teaching

⇨ Page 112, 1 column, 1 paragraph: As teachers, we can hinder or help our students.
⇨ Page 113, 2 column, last paragraph: Good question to begin problem solving.
⇨ Page 114, 1 column, 4 paragraph: Working in open-ended setting can often reveal unexpected student thinking about concepts assumed understood.
⇨ Page 114, 2 column, dialogue portion: As a teacher, what do I do when my plans go awry?
⇨ Page 114, 2 column, 2 paragraph: Examples of using questioning to eventually lead to her learning almost on her own without someone simply telling her.
⇨ Page 115, 2 column, table 2: Ask questions, organize data, make charts, use blocks.
⇨ Page 116, 2 column, dialogue portion: The interviewer really gets the students to explain how they got the answer.
⇨ Page 117, 2 column, last paragraph: When your students catch on, you can increase the demand and have students look at patterns to predict.
⇨ Page 118, 1 column, 3 paragraph: Teacher suggests exploration with graphs of these patterns expanding on knowledge of patterns.

At the end of the class period, I collected all the sticky-note connections the students had posted on the displayed article (see **fig. 2a**) so that I could record and post them on our class Web site (see **fig. 2b**). Reading through the entire set of connections also provided an opportunity for me to gauge the types and depth of connections that the students were making. For the next class period, students were to read through the Web-posted connections, select one for each category, and then expand on it and complete an annotation entry. Through this process, students had yet another opportunity to reflect on the article as well as the connections they and others had made. The process also provided

a chance for students who were still struggling to identify and make relevant connections to do so with the support of their peers.

For the next article assignment, I had groups of students collaborate during and after the reading process. They collectively identified text-to-problem and text-to-Standards connections and formally co-wrote these connections for their annotated bibliography entry. However, after discussing text-to-teaching connections, each student was asked to identify a connection that was personally relevant and that related to his or her own emerging vision of teaching.

At this point, I wanted to provide support for students in a way that would facilitate independent reading and making sense of the articles. During the class period after I had assigned a reading and we had discussed specific connections in class, I would ask for a volunteer to write the article's annotation entry. This way, students could choose articles that interested them, and those who had yet to develop a clear understanding of the project or who continued to struggle to access and develop their own schema through making these types of connections had additional opportunities to observe other students' connections and write-ups. When asked, I offered individualized feedback and suggestions throughout this write-up process. **Figure 3** provides an example of the type of annotated bibliography entry that a student, Elizabeth, independently created after reading an article. Because I sometimes choose an article for its mathematical connections and at other times for its pedagogical connections, the types of connections students make in one category are not necessarily as vital as another; the extent and strength of the connection depend on the types of previous knowledge (schema) that the student can access, activate, and then assimilate.

Each student, over the course of the semester, has the opportunity to write at least one individual entry as we collectively create a class bibliography. Through this process of explicitly using and creating schema, students become engaged and enthused about the connections they are making as they read these articles, and many volunteer to write about the connections. For example, after discussing pentominoes in class, Jenna approached me after class and asked if anyone had yet volunteered to write an entry for the Whitin (2004) article. She was excited about how the article's ideas on problem posing related to the Problem-Solving Standard and our class exploration on filling different-sized arrays with pentominoes. On learning that another student had already volunteered to write the entry, Jenna was disappointed but continued to speak enthusiastically about the connections she had made and how she had even explored some of the mathematical ideas presented in the article.

By framing our discussions and ways of reading to emphasize identifying and making specific connections, students began a shift; the primary motivation for reading an article was no longer as a means to fulfill a course assignment for a grade. Instead, many students began reading the articles for their own instruction (not to meet the expectations of me, the instructor). Explicitly identifying connections helped them discover purpose and personal relevance for reading these articles and also provided a framework to help them make sense of a different type of text. My hope is that students will begin, in time, to identify their own categories for making connections.

Concluding Remarks

As I reflected at the end of the semester, I determined that I was on the path to answering my own question of how to help my prospective teachers learn to use professional journals as a resource for their professional development. When asked what they learned through this process of reading articles

Figure 3

A student's individual annotated bibliography entry

The article by Frykholm (2001) discusses practices in learning the concept of "chance" or the likelihood of events occurring.

Text-to-Class
We have yet to focus on probability in class; however, we had one pink handout with a problem called "A Dicey Situation."

Text-to-Standards
Principles and Standards states that instructional programs should foster question formulating (p. 48). On page 113, Frykholm emphasizes that "helping young children think of probability in terms of chance is fruitful. Children must learn to ask, 'What is the chance that some event will occur?' A teacher must foster this questioning. Mathematics is an integrated subject, the more connections made, the more successful the student will be."

Text-to-Teaching
When problem solving, *Principles and Standards* addresses that students in pre-K through grade 12 develop and evaluate predictions (p. 48). Frykholm uses vocabulary to facilitate his students' learning probability. Phrases such as "more likely" or "less likely" are better choices (p. 113) than asking children prematurely to view probabilities as numbers, calculations, and ratios. Pertaining to the example given in the text, "Are you more likely to see a white marble or a purple marble?" is a better question than "What is the probability of seeing a white marble?"

and completing the annotated bibliography projects, students typically responded as this one did:

> Working on my annotated bibliography and making my own connection to each article, I learned a great deal about what kind of teacher I hope to be. These articles taught me many tactics in teaching, such as working off students' prior knowledge to teach them new knowledge.

Others also commented about the process we used to make sense of the article: "It made me feel great that I was able to make all of those connections.... I know I learned how to make the connections [as] well, which I think was an important part of the assignment." My students were now making connections, and consequently these articles became more meaningful.

Students who engage in the metacognitive process of using and creating schema reap multiple benefits. First, connecting text to experiences increases their motivation for reading. By being intentional about seeking connections to their experiences and beliefs, what they read becomes more interesting. Second, their schema offers a conceptual network on which they can better analyze difficult text. Recalling previous learning experiences allows them to connect challenging new texts with current understandings by looking for themes and patterns. Finally, being aware of prior knowledge provides a framework on which they can build new ideas. The continuous development of their understanding requires an awareness of what they know so they can add new understanding. Readers who make conscious connections when encountering difficult texts are more likely to be successful in comprehension because they are interested, analytical, and aware. One student summed it up nicely:

> I have learned a valuable reading skill as I worked on this project, which I will carry with me throughout my life. I have learned about the importance of connecting what I read to existing knowledge. The connections I make in my reading will help me remember new ideas.

The annotated bibliography project represents a small step in meeting the demand to improve teachers' ability to engage in professional development through reading. Other proficient metacognitive strategies also require attention so teachers can access them as necessary when encountering difficult text. It is important to remember, however, that the goal is to develop teachers as proficient readers, not simply as vessels of proficient reading comprehension strategies. Teachers who can adapt the strategies to meet their styles and needs will emerge as literate professionals—teachers capable of interpreting education articles, such as those found in *Teaching Children Mathematics*, for innovative uses in their classrooms.

***Editor's note**: Readers are encouraged to apply the technique developed by Coffey and Billings as they read articles in this issue of* Teaching Children Mathematics. *Then tell us how you used this professional learning strategy as part of your professional development. Share your insights, connections, and reflections with your fellow subscribers. The Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Letters may be submitted to* Teaching Children Mathematics *at* **tcm@nctm.org**. *Please include "Reader's Exchange" in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit may be subject to abridgment. Letters are also edited for style and content.*

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CALL FOR MANUSCRIPTS

Supporting Teacher Learning

The Editorial Panel of *Teaching Children Mathematics* is interested in manuscripts that discuss teacher learning. Anyone who plans, facilitates, or supports preservice or in-service teacher learning is invited to share his or her experiences with issues such as the following:

- The roles of coaches and mentors and the ways they support teacher learning
- Examples of effective activities used during preservice experiences, including field placement
- Creative approaches to support induction of new teachers
- Productive ways to promote deep understanding of elementary school mathematics topics for teachers
- Descriptions of a lesson-study cycle
- Effective use of student work to enhance teacher knowledge
- Approaches to address and support the diversity of teachers in relation to the diversity of children in their classrooms
- Innovative professional development to support teachers' use of assessment
- Innovative ways of using technology, as in videotape case studies for professional development sessions

Manuscripts should be no longer than 2500 words; include figures and photographs at the end. Submit completed manuscripts to *Teaching Children Mathematics* by accessing **tcm.msubmit.net**. Author identification should appear on the cover page only. For manuscript preparation guidelines, visit www.nctm.org/publications/content.aspx?id=7696.

M. Lynn Breyfogle and Lauren E. Williams

Designing and Implementing Worthwhile Tasks

Teachers often need to alter mathematical tasks that they find in their district-adopted set of curriculum materials or develop new ones if none is present on a particular topic. However, how to best go about this work is not always clear. How do you make effective decisions about alterations? What should you keep in mind as you consider developing tasks to help your students with a particular idea or misconception? These and other questions were central in our minds as we developed a task to help students learn about elapsed time.

In preparation for the Pennsylvania System of School Assessment (PSSA) in mathematics, a fourth-grade class of twenty students in rural central Pennsylvania regularly used a practice book that covered a variety of mathematical concepts with a format similar to the state assessment. While observing this class, a student teacher noticed that many students exhibited frustration and a limited understanding of elapsed-time calculations. Because elapsed-time calculation is identified as a core standard in Pennsylvania (PDE 2008), teachers were concerned about this observation and therefore decided to facilitate a lesson that developed the concept of elapsed time in a meaningful way for the students.

Teaching elapsed time integrates easily at a point in a chapter that discusses regrouping in both addition and subtraction. Often it is taught at this point to encourage students to make the connection with time and extend the connection by using a number system other than base ten. Frequently, though, what happens is that students mistakenly assume that time *is* a base-ten system and arrive at erroneous answers. For example, given the problem, "Find how much time has elapsed between 9:33 a.m. and 11:08 a.m.," students follow a procedure of subtraction with regrouping and arrive at the nonsensical answer below:

$$\begin{array}{r} {}^{\backslash}1{}^{0}{}_{\backslash}{}^{1}08 \\ -\ 9:33 \\ \hline 1:75 \end{array}$$

Or, instead of approaching the problem by focusing on regrouping, teachers show a particular approach, such as missing-addend or counting-up strategies, similar to counting back change as a cashier does. Given the same problem, the solution procedure might be "from 9:33 a.m. to 10 a.m. is twenty-seven minutes, from 10 to 11 is one hour, and then eight more minutes, giving us an answer of one hour, thirty-five minutes." Although this approach has potential for helping students find elapsed time more successfully, the procedure pushes students to use a particular method that may or may not allow them to make sense of elapsed time.

Given the fact that the textbooks used in this classroom focused only on one particular procedure, the teachers decided to design a different set of tasks. This article describes the tasks they designed, discusses how the tasks aligned with NCTM's 1991 recommendations for worthwhile tasks, and further explains the importance of implementation and reflection strategies in helping students retain the challenging nature of the tasks (using the context of the fourth-grade elapsed-time lesson).

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Edited by **Kate Kline**, *kate.kline@wmich.edu, who is an associate professor of mathematics education at Western Michigan University. "From the Classroom," dedicated to the practicing elementary teacher, is a forum for sharing knowledge that is daily generated and used in classroom settings. Readers are encouraged to send submissions to this department by accessing* **tcm.msubmit.net**. *Manuscripts must not exceed 2000 words.*

Creating Tasks

One of the most important and yet difficult aspects of designing a lesson is choosing or creating the worthwhile mathematical task a teacher wants students to engage in (Lappan and Friel 1993). The teacher must consider how well the task provides the opportunity for students to investigate the mathematics content in an open but structured way and how well the task

connects with students' existing knowledge while pushing them deeper. When considering the design of a worthwhile task for elapsed time, Breyfogle and Williams kept this in mind by using the following questions as guideposts for the design:

- What goal is this to serve?
- Does it allow my students to make connections to content they already know?
- Does it allow for multiple solution methods or approaches?
- Does it encourage students to reason about mathematics and allow for them to communicate mathematically?

In the end, they arrived at a task requiring students to create their own schedule of a school day. Each student was to complete a blank schedule (see **table 1**) by arranging the eight provided classroom activities (e.g., math, reading, science) and assigning the starting and ending times for each. The students were also given specific parameters their schedule must meet (see **table 2**). They chose these parameters to increase students' cognitive demands by forcing them to go beyond simplistic solutions, such as using only whole-hour or half-hour increments.

For the second part of the task, students were placed into heterogeneous groups and instructed to discuss one another's schedules and choose one schedule to copy onto a poster for future reference. Students were to evaluate one another's schedules using the parameter list as a checklist to be sure the schedule met the criteria. Then they chose the schedule with the greatest variety of activity lengths. Finally, as a small group, they were to answer six analysis questions (see **fig. 1**) that included a comparison to a schedule the teacher had created with varied starting and ending times. As a whole group, students then had the opportunity to share their selected schedules and describe how they arrived at them as well as how they responded to the analysis questions.

What makes a task worthwhile?

A worthwhile task is a project, question, problem, construction, application, or exercise that engages students to reason about mathematical ideas, make connections, solve problems, and develop mathematical skills (NCTM 1991, pp. 24–25. On the basis of this definition, this task is worthwhile in that it (1) allows for connections, (2) incorporates multiple approaches and solutions, (3) requires higher-level thinking, and (4) facilitates reasoning and communication.

Table 1

Student Worksheet for Part 1 of Task

New School Schedule Name _____

Activity	Start Time	Stop Time
	9:00 a.m.	
		2:42 p.m.

*** Reminder**

- Lunch starts at 12:12 and is 32 minutes long.
- You must have gym, recess, science, math, reading, social studies, and art. You must spend at least ten minutes on each activity and at most one hour and ten minutes.

Table 2

Parameters for Part 1 of Task

Parameters for New School Day Schedule	
1.	The day must start at 9:00 a.m. and end at 2:42 p.m.
2.	Lunch must occur from 12:12 to 12:42.
3.	All eight daily activities must be used.
4.	Activities must last for at least ten minutes and no more than one hour and ten minutes.

1. A worthwhile task allows for connections. One identified aspect of a challenging task is its ability to connect to students' previous knowledge, experiences, and interests. The challenge presented in the elapsed-time lesson succeeds in connecting to the learners by capitalizing on a real-world experience (e.g., following a school schedule) familiar to each

of the students, therefore making the prompt accessible to all the students in the class. The task also capitalizes on a common student desire to change the school schedule. In addition to connecting with students' experiences and interests, the task makes connections between elapsed time and prior knowledge, such as addition, subtraction, time telling, and more-than and less-than concepts.

2. A worthwhile task incorporates multiple approaches and solutions. A second identified attribute of a worthwhile task is that it can be approached in more than one way or has more than one legitimate answer. This attribute generates a shift in focus from the importance of providing the "correct answer" to an emphasis on the problem-solving process and the development of associated skills. Additionally, because multiple approaches offer flexibility for students with varied prior knowledge and experiences to engage with the task at their own level, differentiation is conveniently built into the task and fosters equity in the instruction.

The open-ended format of the new school schedule task allows for numerous student approaches and solutions. The activity could result in the class creating twenty different, yet legitimate, school schedules. It also prompts discussion of the different approaches and solutions to the same challenge with the potential for students to share numerous methods they used to complete the task. For example, one student could compute how many hours are available in the entire day, subtract minutes for lunch, and then split her time among the activities as equally as possible. Another student could recognize the parameter of a specific lunch time as a benchmark to split her thinking. She could start at lunch and work backward to the beginning of the day and then work from lunch forward to the end of the day. A third student could begin by giving each of her favorite activities one hour and ten minutes and then divvying up the remaining time to the other activities. Each of these different approaches uses slightly different thinking. It is important for teachers to consider the different methods their students could use.

3. A worthwhile task requires higher-level thinking. A third attribute of a worthwhile task is its demand for students to use higher-level thinking. The elapsed-time task effectively uses limiting parameters to increase cognitive demands. Limiting the length of time spent on daily activities (at least ten minutes and at most one hour, ten minutes) causes students to reconsider the duration of school subjects. For example, when this task was implemented, one student asked, "If I spend an hour and ten minutes on all of my favorite subjects, I won't have time for social studies. What should I do?" which then prompted a useful class discussion on how to resolve this problem. The time restrictions, coupled with the constraint of a designated lunch time increases the use of higher-order thinking skills as students reevaluate their schedules against the parameters.

4. A worthwhile task facilitates reasoning and communicating mathematically The last key attribute is that the task ought to facilitate the need for students to reason and communicate mathematically. Including tasks that provide opportunities for valuable classroom discussion is therefore critical. The second part of the elapsed-time challenge provides an opportunity for students to collectively reason through analysis questions (see **fig. 1**), communicate their ideas with group members, and then share them with the entire class. The point of this whole-group discussion is for students to have the opportunity to publicize their thinking about the situation's mathematics—not why they chose how much time to assign to art but rather how they thought about the concept of elapsed time and determined the mathematics of the schedule.

Figure 1

Student groups received questions during part 2 of their task.

Group Questions

1. Which group member's schedule had the class members spend the least amount of time on science? How much time did they spend?
2. Which group member's schedule had the class members spend the most amount of time on science? How much time did they spend?
3. Did any member's schedule have the class spend more time in gym than in reading? If so, whose? How much more time?
4. Did any group member's schedule have the class spend less time in math than Ms. Williams's schedule? If so, whose? How much less time?
5. Look at Ms. Williams's starting time for art and your starting time for art. Which group member's schedule started art class at a time farthest away from Ms. Williams's starting time for art? How much time elapsed between those two starting times?
6. Look at Ms. Williams's ending time for reading and your ending time for reading. Which group member's schedule ended reading at a time farthest away from Ms. Williams's ending time for reading? How much time elapsed between those two ending times?

Implementing the Elapsed-Time Task

After Williams briefly led the entire class through scheduling and planning a party for her eleven-

year-old niece as an example, students independently created their own school-day schedules. They were eager to find solutions that not only fit the parameters but also satisfied their individual interests. Teachers enjoyed watching students actively engage with the activity, share ideas with neighboring students, ask questions, and ultimately reach a solution that they were excited to communicate.

What *keeps* it worthwhile?

Even such an engaging activity requires diligence on the teacher's part to keep the task at a high cognitive level. Features of the implementation—such as scaffolding, honoring a variety of solutions, and pressing students for explanation—are fundamental to maintaining a high learning value because they keep students, rather than the teacher, doing the reasoning. To support the students' development of mathematical problem-solving skills, teachers must prevent themselves from taking over the thinking (Herbel-Eisenmann and Breyfogle 2005).

1. Scaffold student thinking. During the implementation of this activity with a fourth-grade class, scaffolding proved to be a successful tool in promoting student reasoning through the elapsed-time challenge. The following exchange occurred as a student calculated the elapsed time from 11:10 to 12:02:

John: I can't figure out how much time I spent on math.
Teacher: Well, what is one hour from 11:10?
John: 12:10 ... [*pausing*] so, it's less than one hour.
Teacher: Yes, now we need to find out how much less than one hour.
John [*pausing*]*:* Eight minutes.
Teacher: Ok, then how much time has elapsed from 11:10 to 12:02?
John: Sixty minutes minus eight is [*pausing*] fifty-two! Fifty-two minutes!

By suggesting that the student think about a one-hour chunk as a benchmark, Williams allowed John to do some of his own thinking. She also allowed think time for him to answer the questions she posed. When John answered, "12:10," she did not immediately jump in to ask another question. She provided a series of scaffold questions or statements that did not funnel his responses; he could use them as models in the future.

2. Honor a variety of strategies. Allowing for a variety of strategies is another way to maintain the level of the task. Students used many strategies to determine their schedules. For example, in contrast to John's approach of thinking about adding an hour and then subtracting to get to a benchmark number, Amanda used an additive approach. She calculated the duration of recess, from 12:44 to 1:10, by first counting on one minute from 12:44 to 12:45 (which can be considered a benchmark number or time), and then counting on using five-minute intervals until she reached 1:10. Amanda also used the addition problem 5 + 42 = 47 to calculate the elapsed time from 1:55 to 2:42. In this calculation, she arrived at her answer of forty-seven minutes by considering what it would take to get from 1:55 to 2:00 (five minutes) and then adding the number of minutes to 2:42.

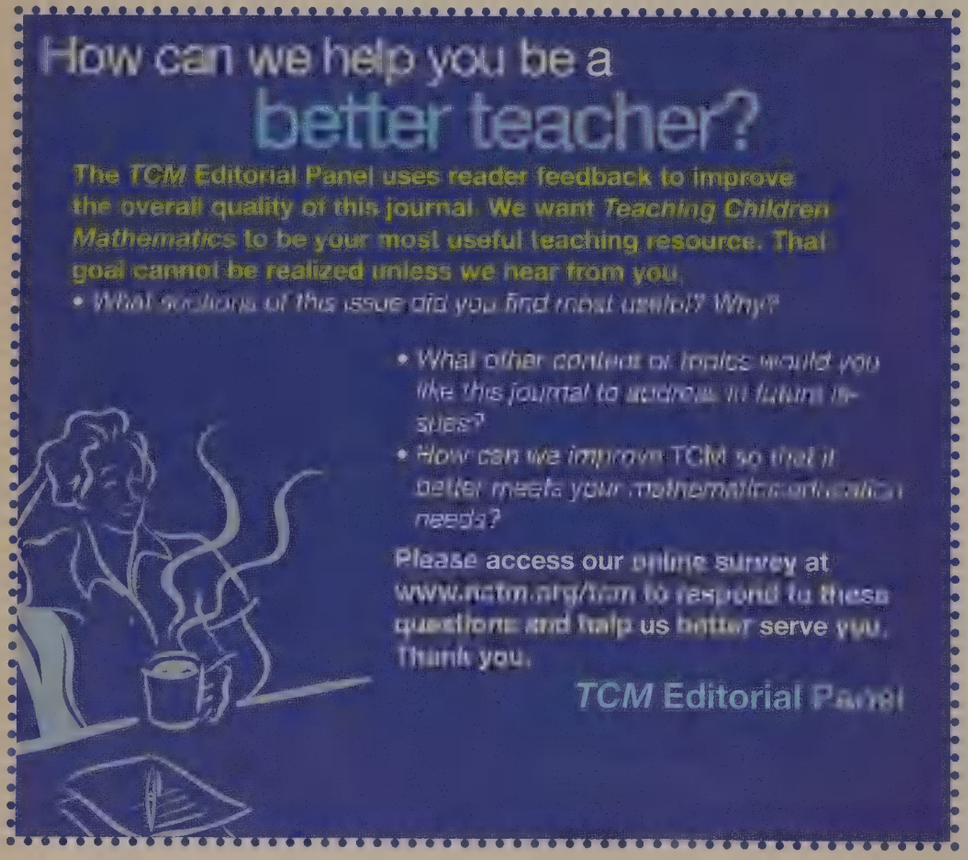
3. Press for explanations. Continuous teacher questioning and pressing for explanation and justification is another aspect of effective implementation of a worthwhile task (Stein et al. 2000). This technique keeps the emphasis on solution strategies and reasoning (rather than the correct answer) and plays an important role in supporting mathematical reasoning (NCTM 1991). During the lesson, one student correctly calculated the elapsed time from 1:54 to 2:42 by providing the answer of forty-eight minutes to the class. Rather than acknowledging the correct answer and progressing with the lesson, Williams asked the student to explain his solution process:

Teacher: How did you find that answer?
Tom: Well, one hour from 1:54 is 2:54. Then I did fifty-four minus forty-two and got twelve.
Teacher: Why did you do fifty-four minus forty-two?
Tom: To figure out how much less than an hour it was. It was twelve minutes less than an hour. Then I did sixty minus twelve and got forty-eight, so the answer is forty-eight minutes.

A brief conversation requiring students to explain their thought processes can provide valuable insights to all. The process of communicating mathematically will benefit the student with the opportunity to clarify his thinking. The teacher benefits because she gains insight into this student's thinking and thought processes, which could guide further instruction. The class members gain from these exchanges because they are able to hear other methods that possibly differ from their own thinking.

Reflecting on the Implementation

The reflection that occurs after the lesson is also critical in making improvements so that teachers can become more effective at keeping the tasks' cognitive demands at a high level. In this case, as Breyfogle and Williams reflected collaboratively on the implementation of the elapsed-time tasks, they came to several observations. First, in the exchange with John described previously, it might have been possible to gain an even greater insight into the child's mind and provide a richer learning experience. If Williams had instead asked questions such as, "What are you thinking?" or "Do you have an idea of how you could proceed?" she could have come to understand what John already knows in order to guide, rather than lead, him to developing his own way to solve the problem. Additionally, in the second exchange, Williams might have pressed Tom's thinking more by asking exactly what he meant when he said, "Fifty-four minus forty-two," because many of the other students in class may have been thinking, "Fifty-four take away what will get me to forty-two?" Having Tom clarify his interpretation of "fifty-four minus forty-two" as the difference between forty-two and fifty-four would have been helpful to the other students. Continuing this conversation to involve some discussion of the different ways you can interpret subtraction, accompanied by appropriate representations, could have helped clarify Tom's thinking and perhaps provided more opportunity for other students to understand it.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better teacher?
The TCM Editorial Panel uses reader feedback to improve the overall quality of this journal. We want Teaching Children Mathematics to be your most useful teaching resource. That goal cannot be realized unless we hear from you.
• What sections of this issue did you find most useful? Why?
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• How can we improve TCM so that it better meets your mathematics education needs?
Please access our online survey at www.nctm.org/tcm to respond to these questions and help us better serve you. Thank you.
TCM Editorial Panel

Conclusion

The point of creating this worthwhile task was for students to develop ways to accurately and flexibly determine elapsed time. Student work showed that each was able to successfully complete the task. For example, at the beginning of class, Tony quickly responded, "Four o'clock," when students were asked to calculate the elapsed time of a party beginning at 1:30 and ending three hours later. At the end of class, Tony was able to successfully complete this task. The task challenged the entire class to make connections to prior knowledge, think critically, and communicate mathematically. It not only encouraged them to make sense of elapsed time but also resulted in a noticeably positive change in student engagement and learning. By incorporating the key elements of a worthwhile task and effective implementation and reflection techniques, the elapsed-time task created a desirable learning environment that is an essential foundation for understanding mathematics.

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CALL FOR MANUSCRIPTS

Assessing to Learn and Learning to Assess

In support of one of the strategic initiatives set by the NCTM Board of Directors for the Council, the Editorial Panel of *Teaching Children Mathematics* is seeking manuscripts related to assessment. To avoid the implication that we are referring to one true test (a commercial, standardized test), note that we avoid saying "the assessment." *TCM* seeks manuscripts that reflect an understanding of assessment and that use a variety of strategies, including alternative assessment methods, techniques for interpretation and impact on diagnosing student performance, and planning for instruction. The assessment process can be thought of as the following four interrelated phases:

- ✔ Planning Assessments
- ✔ Gathering Evidence
- ✔ Interpreting the Evidence
- ✔ Using the Results

Manuscripts should include at least one of the following assessment standards and may also include assessment ideas that address the No Child Left Behind legislation:

The Mathematics Standard
- What mathematics is reflected in various assessments?
- How does assessment engage all students, including those of underrepresented minority groups, in realistic and worthwhile mathematical activities?

The Learning Standard
- How does assessment contribute to each student's learning of and attitudes toward mathematics?
- How does assessment provide opportunities for students to evaluate, reflect on, and improve their own work—that is, to become independent learners?
- How does assessment provide opportunities for teachers to evaluate, reflect on, and improve their teaching?

The Equity Standard
- What opportunities has each student had to learn the mathematics being assessed?
- How does the assessment design enable all students to exhibit what they know and can do? Are there specific assessment techniques or considerations that allow for more reliable evaluation of students from a variety of underrepresented groups?
- What assumptions are made by educators using assessments relative to diverse students?

The Openness Standard
- How are teachers and students involved in choosing tasks, setting criteria, and interpreting results?
- How is the assessment process open to evaluation and modification?
- How do educators identify and control for possible assessment bias?

The Inferences Standard
- How is professional judgment used in making inferences about learning?
- What efforts are made to ensure that scoring is consistent across students, scorers, and activities?

The Coherence Standard
- How does assessment match the curriculum and instructional practice?
- How can assessment practice inform teachers as they make curriculum decisions and determine their instructional practices?
- What additional aspects of child development can be assessed, and how can this information be used to enhance mathematics instruction?

Manuscripts should not exceed 2500 words. Attach figures and photographs at the end. Manuscript preparation guidelines describing how to include photographs and figures, the use of resources and quotations, and other procedural information are available online at **my.nctm.org/eresources/submission_tcm.asp.** Submit completed manuscripts online at **tcm.msubmit.net.**

Randall I. Charles

Do State Content Standards Promote Excellence in Teaching and Learning Mathematics?

The NCTM vision for school mathematics includes a call for curriculum that is "coherent, focused on important mathematics, and well articulated across the grades" (NCTM 2000, p. 14). In an effort to improve student learning of mathematics, a major focus over the past two decades has been on the dual goals of identifying curriculum standards and aligning curriculum materials and assessments with the standards. This strategy was further promoted by the No Child Left Behind legislative mandate for states to adopt "challenging academic content standards" in mathematics, reading and language arts, and science that (a) specify what children are expected to know and be able to do; (b) contain coherent and rigorous content; and (c) encourage the teaching of advanced skills in these areas.

States, as well as local school districts across the country, have been engaged in discussions and implementation of new grade- and course-level standards. Additionally, national organizations such as the College Board, Achieve, and the American Statistical Association have weighed in with suggested standards for K–12 mathematics.

The three companion articles in this month's issues of *Teaching Children Mathematics (TCM)*, *Mathematics Teaching in the Middle School (MTMS)*, and the *Mathematics Teacher (MT)* examine issues related to standards in mathematics in the United States. The article by Randall I. Charles in *TCM* examines the question, "Are state content standards promoting excellence in teaching and learning mathematics?" In *MTMS*, Aladar Horvath and colleagues raise the question, "Are middle grades' state content standards a constraint or an opportunity?" Finally, in the related article in *MT*, Eric W. Hart and W. Gary Martin discuss current developments with regard to high school mathematics standards, exploring the next steps in the evolution and use of standards to improve school mathematics.

"There is a substantial and critical difference between standards as a vision of excellence and standards as narrow and rigid requirements for mastery" (Clements 2004, p. 8). During the past year, I have had the opportunity to visit a number of classrooms throughout the United States. I have observed a growing trend to post state mathematics content standards somewhere in the classroom, visible to all. Generally, not all standards are posted; only the standard of focus for the day's lesson. As state content standards have become linked to high-stakes assessments, standards are now the major influence on what mathematics is taught and what is not taught in schools, what is emphasized and what is not, and what students and teachers believe and do not believe about mathematics. If state mathematics content standards are influencing in real ways what teachers do when they close the classroom door, it is essential to ask if the standards being used are the right ones. The litmus test is whether the standards promote excellence in teaching and learning mathematics.

State curriculum standards are intended to communicate to students, teachers, administrators, and parents what mathematics students should know and be able to do. Both conceptual understanding and procedural fluency, two strands of mathematical proficiency (NRC 2001), are important, but the reality is that most

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"Contemporary Curriculum Issues" provides a forum to stimulate discussion on contemporary curricular issues across a K–12 audience. NCTM plans to publish sets of three articles, focused on a single curriculum issue. Each article addresses the issue from the perspective of the audience of the journal in which it appears. Collectively, the articles are intended to increase communication and dialogue on issues of common interest related to curriculum. Manuscripts on any contemporary curriculum issue are welcome. Submissions can be for one article for one particular journal, or they can be for a series of three articles, one for each journal. Submit manuscripts at the appropriate Web site: tcm.msubmit.net, mtms.msubmit.net, *or* mt.msubmit.net, *or contact editors Barbara Reys, reysb@missouri.edu, for* TCM*; Glenda Lappan, glappan@math.msu.edu, for* MTMS*; or Chris Hirsch, christian.hirsch@wmich.edu, for* MT. *Manuscripts should not exceed 2500 words.*

state standards communicate only the mathematical skills that students should master (procedural fluency) and not the ideas and relationships that students need to understand mathematics (conceptual understanding). **Table 1** shows examples of typical skill-mastery standards found in many state curriculum documents.

This article has two purposes. The first is to argue that skill-mastery standards like those shown in **table 1** can have a negative impact on the teaching and learning of mathematics. The second is to illustrate how the current emphasis on standards can be used as a springboard for a new kind of standard—one that emphasizes thinking and promotes excellence in teaching and learning mathematics.

The Impact of Skill-Mastery Standards

Do you want your students to understand mathematics? Do you teach mathematics for understanding? Teachers generally answer each of these questions with a resounding yes! However, the TIMSS Video Study concluded that "many teachers would like their students to understand the mathematics they study, but when asked to specify the goal for a particular lesson, most U.S. teachers … talked about skill proficiency; few mentioned understanding" (Hiebert 2003, p. 60).

The history of curriculum materials in this country might explain teachers' inclination to focus on skill mastery. For over thirty years, U.S. textbooks have stated one or more "objectives" for each mathematics lesson. The easiest learning objectives to write are statements of skills that students should master, and most learning objectives are of this type. Most state mathematics standards have grown out of this tradition, so it is not surprising that most state standards are also skill-mastery statements. A teacher posting a statement of a skill-mastery standard is today's version of a teacher in the past posting the skill objective of the lesson.

Standards like those shown in **table 1** communicate what students are expected to be able to do, but they do *not* communicate what students should understand. In recent observations of classrooms and school districts, I have often noted the following significant negative influences of skill-mastery standards on teaching and learning mathematics and on students' and teachers' beliefs about mathematics:

- Curriculum and instruction is focused exclusively on developing skills.
- Instructors teach skills that they believe help students get correct answers without regard for whether procedures are grounded on understanding.
- Teachers and students believe that learning mathematics means only developing skill proficiency.
- Assessments evaluate only skill mastery.
- Students are uninterested in understanding mathematics because "understanding" is not encouraged, valued, taught, or assessed (Lambdin 2003).

Table 1

Sample State Skill-Mastery Standards

Intermediate-Grades Topic: Comparing and Ordering Fractions	
State and Grade	Standard
Virginia Grade 4	The student will identify, model, and compare rational numbers (fractions and mixed numbers), using concrete objects and pictures (2002).
Georgia Grade 5	Use <, >, or = to compare fractions and justify the comparison (2006).
Arizona Grade 6	Compare two proper fractions, improper fractions, or mixed numbers (2003).
Florida Grade 3	Compare and order fractions, including fractions greater than one, using models and strategies (2007).
Primary-Grades Topic: Counting Whole Numbers	
State and Grade	Standard
Oklahoma Grade 1	Count as many as 100 objects by ones, twos, fives, and tens (2002).
California Grade 1	Count, read, and write whole numbers to 100 (2005).
West Virginia Grade 1	Count forward to 100 and backward from 20 with and without objects (2008).

Figure 1

A teacher's presentation of a single solution method can negatively influence students' beliefs about mathematics.

$$\frac{3}{8} \quad \frac{5}{7}$$

$$3 \times 7 = 21$$

$$8 \times 5 = 40$$

$$21 < 40$$

$$\frac{3}{8} < \frac{5}{7}$$

An example from an intermediate-grades lesson illustrates the negative influence skill-mastery standards can have on teaching and learning. The standard of the day was posted in the front of the room: "Compare two proper fractions, improper fractions, or mixed numbers." The teacher wrote the fractions 3/8 and 5/7 on the board and told the students that they were going to learn how to determine which of two fractions is the greater fraction. The teacher then illustrated a technique for finding the answer (see **fig. 1**). First she drew an X to indicate cross multiplication: $3 \times 7 = 21$ and $8 \times 5 = 40$. She then wrote and explained that because 21 is less than 40, 3/8 must be less than 5/7. She continued with several more examples and then had the students practice similar problems.

Putting aside issues related to the instructional approach this teacher used, I asked why she chose that particular method for comparing fractions rather than, for example, finding common denominators or estimating their values using number sense. She replied, "It always works. Students know how to get the correct answer if they do not make a computational error."

This classroom episode illustrates what happens in too many classrooms when skill-mastery standards drive instruction. In other conversations, this teacher acknowledged the importance of having her students understand mathematics. However, the desire, or perhaps the pressure, to explicitly teach the state standard in a way that she believed gave the highest probability of her students getting correct answers led her to teach a specific algorithm (devoid of understanding) for solving this type of problem. Classroom episodes such as this are not hard to find.

In spite of the skills-mastery focus of most mathematics standards, many teachers do focus on conceptual understanding, skill proficiency, and problem solving for all students. However, if skill-mastery standards and their related assessments continue to be the norm, even the best teachers will have difficulty avoiding the negative influences noted above.

Table 2

Sample Primary-Grades Skill-Mastery Standard and Related Essential Concepts

State Standard	Count, recognize, represent, name, and order a number of objects up to 30.
Essential Understandings	When counting, the last number word said tells the number of items in the set. Counting a set in a different order does not change the total. The nature of the objects being counted and their physical arrangement does not affect the number of objects counted. There are a number word and a matching symbol that tell exactly how many items are in a group.

Table 3

Sample Intermediate-Grades Skill-Mastery Standard and Related Essential Understandings

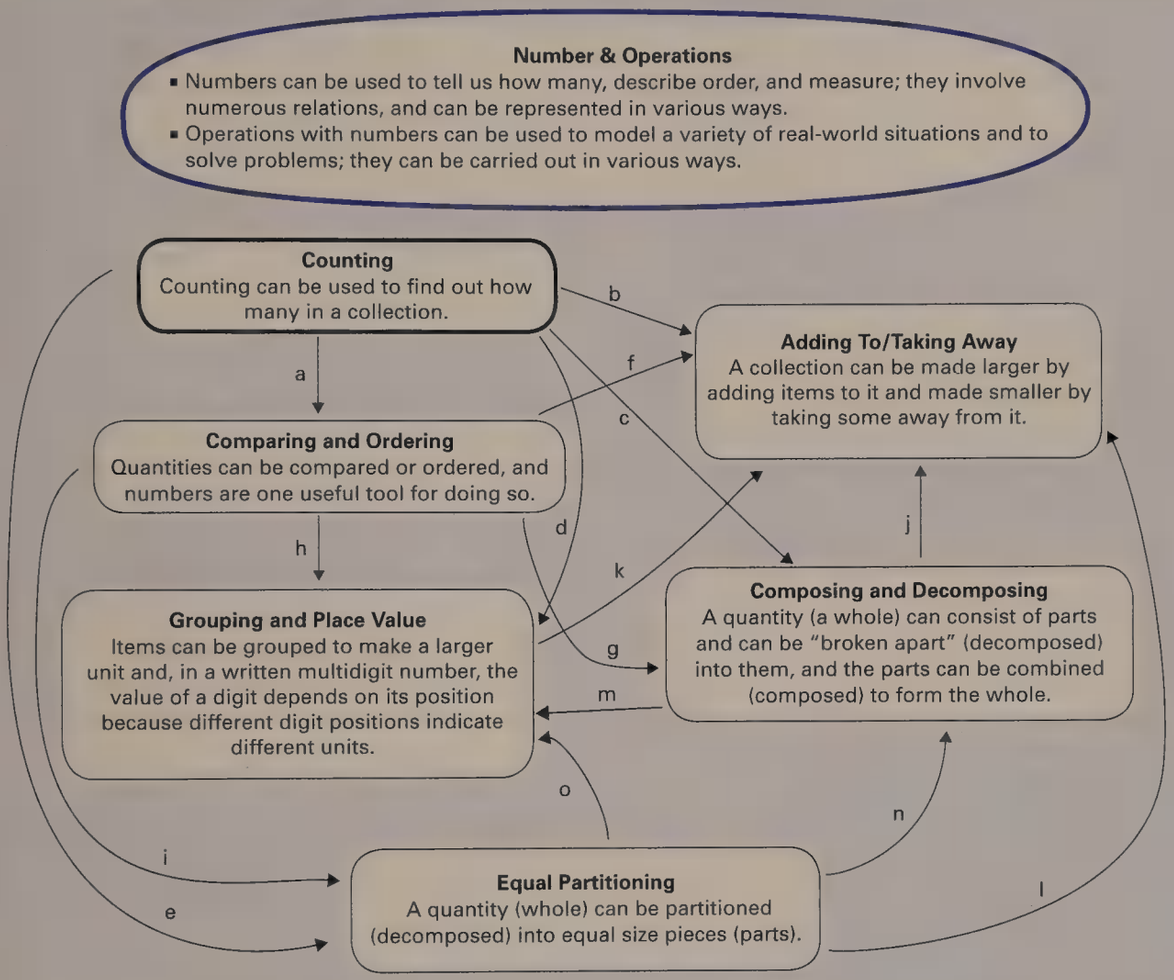
State Standard	Compare two proper or improper fractions.
Essential Understandings	The location of any fraction on the number line can be approximated. Some pairs of fractions can be compared by showing their relative location on the number line and showing that the one to the right on the number line is the greater fraction. If two fractions have the same denominator, the one with the greater numerator is the greater fraction. If two fractions have the same numerator, the one with the lesser denominator is the greater fraction. Any fraction can be compared to one-half by using number sense. Some fractions can be compared by determining how each compares to one-half. One or both fractions can be changed to equivalent fractions with like denominators or like numerators and then compared. Fraction-decimal benchmarks and relationships can be used to compare fractions.

Beyond Skill-Mastery Standards to Excellence

Most state curriculum documents acknowledge the need for understanding mathematics, and some use the term *understanding* within statements of standards. For example, "Students understand and describe simple additions and subtractions" in kindergarten (CDE 2005). However, using *understand* in this way does not necessarily communicate the mathematical ideas and relationships associated with a standard. If state mathematics standards are to promote excellence in teaching and learning, they must make explicit the "essential mathematical understandings" that should be developed and assessed. By "essential understandings," I am referring to the mathematical ideas and relationships that students should acquire in order to learn mathematics with understanding. (No consensus

Figure 2

Some essential understandings related to number and operations (Clements 2004, p. 18)


Number & Operations
▪ Numbers can be used to tell us how many, describe order, and measure; they involve numerous relations, and can be represented in various ways.
▪ Operations with numbers can be used to model a variety of real-world situations and to solve problems; they can be carried out in various ways.
Counting
Counting can be used to find out how many in a collection.
Adding To/Taking Away
A collection can be made larger by adding items to it and made smaller by taking some away from it.
Comparing and Ordering
Quantities can be compared or ordered, and numbers are one useful tool for doing so.
Grouping and Place Value
Items can be grouped to make a larger unit and, in a written multidigit number, the value of a digit depends on its position because different digit positions indicate different units.
Composing and Decomposing
A quantity (a whole) can consist of parts and can be "broken apart" (decomposed) into them, and the parts can be combined (composed) to form the whole.
Equal Partitioning
A quantity (whole) can be partitioned (decomposed) into equal size pieces (parts).
a
b
c
d
e
f
g
h
i
j
k
l
m
n
o

exists in the mathematics education literature on the meaning of the phrases *big ideas* and *essential understandings*. See Charles 2005, Clements 2004, and Van de Walle 2004 for illustrations of different ways to think about these terms. In the end, the differences seem to be mostly in language, not content.) **Tables 2** and **3** illustrate some essential understandings that can be attached to two typical skill-mastery standards (Charles 2005; Van de Walle 2004). Notice these important points: Essential understandings—

- are statements about content, not pedagogy. They do not promote a particular instructional approach.
- provide a coherent set of ideas to guide instruction and assessment.
- are based on research when possible, the logical structure of mathematics, and teacher experience.
- can be translated to rich instructional tasks.
- can be translated to rich assessment tasks.

Figure 2 shows another way to communicate essential understandings. This representation shows not only essential understandings but also relationships between these ideas. Essential understandings connected to skill-mastery standards provide key ingredients for rigorous mathematics standards, curriculum materials, and assessments.

Too often, standards and curriculum materials are identified as "rigorous" because the grade levels at which certain skill-mastery standards are developed is lower than what traditionally has been the case. The grade placement of standards certainly influences rigor. However, an equally important factor in determining whether standards or curricula are rigorous should be the extent to which they are grounded on a rich set of essential understandings.

Instruction and assessment driven by essential understandings are also rigorous. **Figure 3** shows an example of an assessment task that could be used if instruction were driven by the essential understandings related to comparing fractions from **table 1**. Obviously the student who learned only the cross-multiplication algorithm can determine which fraction in each pair is greater. But it is unlikely that this student could choose an appropriate method for making the comparisons. The student who can flexibly choose and use different methods of comparing fractions would certainly be considered to have a deeper understanding of fraction comparison than one who could only use the cross-multiplication algorithm. Instruction and assessment driven by tasks such as the one in **figure 3** are more rigorous than instruction and assessment driven by memorizing an algorithm.

Conclusion

One criticism of curriculum and instruction in the United States that has been repeated many times is the "mile-wide and inch-deep" characterization. The main purpose of *Curriculum Focal Points* (NCTM 2006) was to illustrate one way that curricula can be organized and sequenced to provide more focus and depth (Schielack and Seeley 2007).

Whereas *Curriculum Focal Points* provides a model for consideration, it was not meant to replace the current long lists of skill-focused standards with a short list of skill-focused standards. It is not enough to simply address the "mile-wide" curriculum. We must also help teachers address the "inch-deep" issue.

Teaching and learning a few important mathematical topics grounded on essential understandings, as well as skills, can be a step toward addressing both the mile-wide and the inch-deep curriculum. Communicating to teachers not just the skills to be mastered but also the ideas and relationships students need to know and understand will help teachers go beyond the surface level of teaching skills to develop deeper understandings of mathematics. "The standards movement in the United States is, in part, an effort to set learning goals for students. The contribution of this activity to improving teaching depends on whether the standards are expressed clearly enough that teachers can use them for planning instruction" (Hiebert and Stigler 2000, p. 13).

Classroom teachers and other leaders in mathematics education do not have to wait for a new generation of content standards that include essential understandings to begin making a difference in the classroom. We can begin now to include essential understandings in daily instructional planning and revisions of our school or district curriculum guidelines. Three sources for essential understandings appear at the end of this article. Additionally, read NCTM journals and books with an eye toward identifying essential understandings. Attend sessions at state and NCTM conferences with a goal of identifying essential understandings for particular topics. And, if an administrator requires that the "standard-of-the-day" be posted in the room, go beyond the skill to be learned and communicate to students in their language the ideas and relationships, the essential understandings, they will be developing. There is no need to keep secret from children or teachers the essential understandings that need to be part of teaching and learning mathematics. For too long, we have heard complaints that mathematics education may not be making significant differences in what students understand. Maybe it is difficult to make a real difference in what students understand until we make that "understanding" explicit in the classroom and explicit in our state standards.

Finally, many teachers cannot teach for understanding unless they know what understandings to teach. Furthermore, teachers cannot teach for understanding unless *they* understand the mathematics to be taught. Building standards, curriculum, instruction, and assessment on essential understandings can also serve as a vehicle for developing every teacher's mathematical content knowledge.

Figure 3

This assessment task can be used when instruction on comparing fractions is driven by essential understandings.

Task: For each pair of fractions, choose one of the methods below to find the greater fraction. Explain your choice of a method.

5/8 and 3/8 2/3 and 6/7 7/8 and 7/9 3/8 and 5/7 1/8 and 3/16

A. Compare each fraction to the benchmark of 1/2.
B. Compare numerators when fractions have the same denominator.
C. Compare denominators when fractions have the same numerator.
D. Find equivalent fractions with a common denominator.
E. Find equivalent fractions with a common numerator.

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Additional Resources for Essential Understandings

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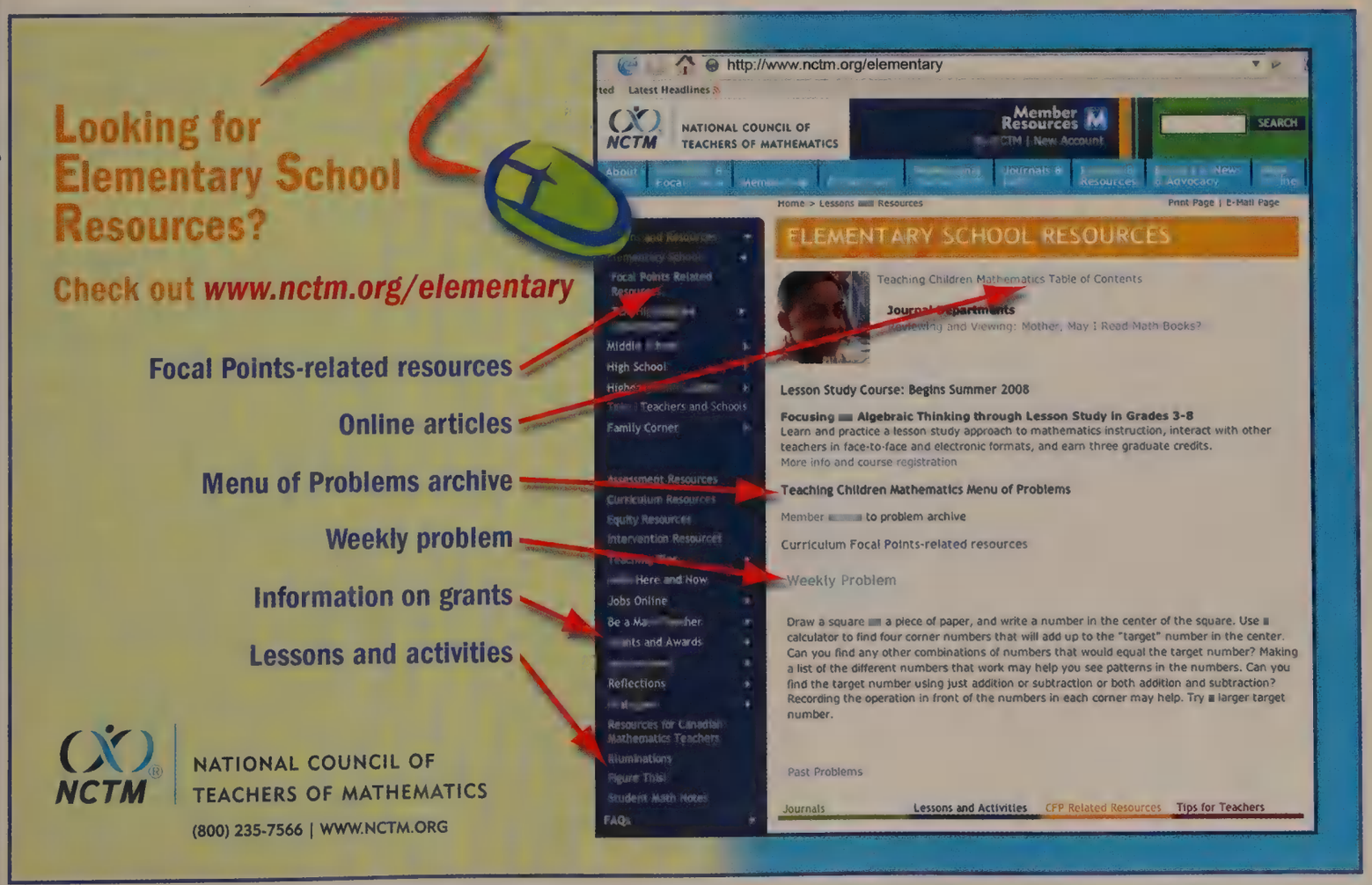
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Christine Gallego, Dora Saldamando, Grace Tapia-Beltrán, Karolyn Williams, and Thomas C. Hoopingarner

Don't Burst My Bubble

"Math by the Month" activities are designed to engage students to think as mathematicians do. Students may work on the activities individually or in small groups, or the whole class may use them as problems of the week. No solutions are suggested, so students look to themselves for mathematical justification and develop confidence to validate their work.

The following activities are designed to help you and your students explore bubbles and spheres through mathematical investigations involving number sense and operations, data collection, pattern and algebra, measurement, communication, and representation. ▲

WEEKLY ACTIVITIES

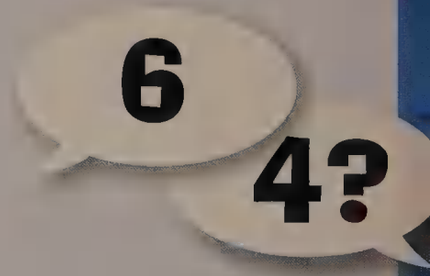
DON'T BURST MY BUBBLE: K–2

DECEMBER 2008/JANUARY 2009

5 **Bubble shapes** are called *spheres*. Where have you seen spheres? Where have you seen bubbles? Draw and label a picture of five other objects that have a similar shape.

12 **To make bubble wands,** ask your teacher for pipe cleaners to bend into different closed shapes. Which shapes make the best bubbles? Experiment to test your idea. Draw a picture and write what you discovered about the shape of the wand compared to the shape of the bubbles it makes. Were your findings surprising?

19 **Hold that thought.** The shapes that float above cartoon characters when they speak are called *thought bubbles*. On a small piece of paper, write a number inside of a thought bubble and then fold your paper in half. Have your classmates try to guess the number. Other students may ask you if the number is even or odd, greater than or less than a specific number, or they may guess a specific number. You may answer their questions with only a yes or no reply. When someone guesses your number, show it to the class. The person who correctly guessed your number writes the next number in a thought bubble.



26 **Another game** using thought bubbles is for a large group of students to each write a number less than 100 inside a thought bubble drawn on a small piece of paper. Get into groups of 3 and compare your number with your group members'. Arrange the thought bubbles in order from lowest to highest. Were any numbers the same? The person with the lowest number now groups with 2 other people from 2 other groups who also had the lowest number in their group. The person with the number in between the lowest and highest now groups with 2 other people from 2 other groups who also had the middle number. The person with the highest number in the group now joins 2 other people with the highest number from other groups and compares numbers. Were any numbers the same? What was the lowest number in the class? What was the highest number?

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Edited by **Dana Islas**, *who teaches kindergarten at Pueblo Gardens Elementary School in Tucson, Arizona. Readers are encouraged to submit problems to be considered for future "Math by the Month" columns to the department editor at* **dana.islas@tusd1.org**. *Receipt of the problems will not be acknowledged; however, those selected for publication will be credited to the author. Readers are also invited to submit creative solutions and adapted problems to* **tcm@nctm.org** *for potential publication in "Reader's Exchange." Please include "Reader's Exchange" in the subject line. Because of space limitations, responses beyond the 250-word limit are subject to abridgement. Submissions are also edited for style and content.*

DON'T BURST MY BUBBLE: 3–4

DECEMBER 2008/JANUARY 2009

5

Sum bubble gum: Begin by chewing a piece of your favorite bubble gum. Using a timer and tally marks, your partner will keep time and count how many bubbles you can blow in 2 minutes. Count only bubbles that are a complete sphere. How many bubbles did you blow? Repeat the experiment 3 times. If this pattern continued, how many bubbles would you blow in 4 minutes, 10 minutes, and 30 minutes? Now time your partner for 2 minutes. Compare your bubble-blowing data. How many more or fewer bubbles did you blow than your classmate? Try the same thing with another brand of bubble gum. Did the second gum generate more or fewer bubbles in 2 minutes? Create a graph using your bubble gum data. What data could you give the bubble gum companies about their gum?

12

If bubble gum costs $.70 per pack, how many packs could you buy with three $1 bills, 5 quarters, 5 dimes, 3 nickels, and 10 pennies? Will you have any change left over? How much?

19

Blow bubbles by the cup. You will need a Styrofoam or plastic cup, a straw, a clock with a second hand or a stop watch, and a flat surface. Pour 1/2 cup of bubble solution into your cup. One at a time, you and a partner will time each other to see how long you can make a bubble last. Insert the straw into the bubble solution and gently blow. How long did the bubble last before it popped? Create a chart showing your bubble-by-the-cup data, using the 10 longest-lasting bubbles. Share and compare data with the class. How long did the longest bubble last? What was the shortest bubble time? Was there an outlier? What was the class mode? The class range?

26

Bubble impossible Using bubble solution and a bubble wand, can you join 2 same-sized bubbles together? What shape do they make? What other observations can you make? Can you join 2 different-sized bubbles together? Does the shape stay the same? Try to blow bubbles to create a 3-bubble configuration. What do you notice about this grouping? Continue blowing and joining bubbles. How many different multiple-bubble groupings can you create? Share your data. What did you discover? Do your classmates' data match yours? Explain.

DON'T BURST MY BUBBLE: 5–6

DECEMBER 2008/JANUARY 2009

5

Measuring bubbles can be a little tricky! Ask your teacher for a straw, a shallow container, bubble solution, string, and a ruler. Pour some bubble solution into the container. With your straw in the solution, slowly blow a bubble through the straw as large as you can until it pops. Once it pops, use string and a ruler to determine the bubble's diameter and circumference by measuring the bubble residue. Record the measurement data on a group chart. Repeat this many times. What do you notice about the measurements of each bubble's diameter and circumference? When you finish, graph your data and compare your graph with your classmates' graphs.

12

How much air can your lungs hold? You will need a shallow pan with some bubble solution, a straw, a ruler, and some string. To measure your lung capacity, take a deep breath and blow slowly through the straw to create a bubble. Continue blowing gently until you run out of breath. Wait for the bubble to pop and then measure the diameter (in centimeters) of the circle left by the bubble. Use the chart to estimate the amount of air your lungs can hold. Make a bar graph showing the lung capacities of the students in your class. What do you notice about the data?

Ring Diameter (in cms)	Lung Capacity (in liters)
14	0.7
15	0.9
16	1.1
17	1.3
18	1.5
19	1.7
20	2.1
21	2.4
22	2.8
23	3.2

19

Count bubbles by working with a partner to see how many bubbles you can blow in 1 minute. You will each need bubble solution and a straw. One student blows bubbles into the air while the second student counts the bubbles. When the minute is up, record the number of bubbles on a chart. After each student from your group has had a turn blowing bubbles, find the total number of bubbles for the group. On the basis of this data, how many bubbles could you and your partners blow in an hour, a day, and a week? How many bubbles did your classmates blow altogether? How many bubbles could the entire class blow in an hour, a day, and a week?

26

Make some bubblecombs. You will need a clear, zip-top bag, bubble solution, and a straw. Put 2 tablespoons of bubble solution into the bag. Close the bag and spread the solution around so it covers all areas of the bag. Open the bag and place a straw inside to where the solution has pooled. Blow into the straw; keep blowing until your bubbles are oozing out of the bag. Slowly push down, releasing the air in your bag until it is about 1/2 inch to 3/4 inch thick. You will have a variety of bubble sizes. What do you notice about the bubbles and the walls of the bubbles? What have you discovered about the vertices? Does this remind you of anything? Why do you think this happens? Why do you think bees form these same shapes (cells) when making honeycombs? Are they mathematicians? What can we learn from bees?

On-the-Spot Assessments

Informal evaluations informed this teacher's instruction, facilitating her young students' prediction skills throughout a series of number investigations.

This article presents a series of number-sense activities that encourage young students to use their estimation, prediction, and observation skills. The activities explore the relationships between the number of objects in a container and (1) its fullness and (2) the size of objects in the container. The objects used in these activities were three different sizes of sweets, but counters or small blocks of different sizes could be used as an alternative to sweets. Informal, on-the-spot assessments informed the teacher's instructional planning and decision making throughout the activities.

Identifying the Stumbling Block

During an investigation of the numerical contents of small, white, translucent, sealed film canisters filled with brightly colored sweets, the teacher of a class of seven- and eight-year-olds realized that her students' observation skills were a stumbling block to their ability to estimate quantity. The students had been asked to predict the numerical contents of the canisters without opening them. Each pair of children had a filled canister and access to a variety of resources for this investigation, including extra candies, an empty canister, a ruler, and a calculator. Following their explorations and predictions, the students opened their canisters and counted the contents. What concerned the teacher was that many students' predictions were wild guesses that seemed unrelated to their observations. Just as students need to learn how to count, they also need to learn how to use observations to inform their predictions of quantity.

On the basis of her assessment of the students' key difficulty, the teacher designed a second investigation to focus on enhancing students' observation and prediction skills. In this investigation, the teacher purposefully interspersed her instruction with assessment as she sought to build a strong relationship between students' observations and predictions.

Constructing a scaffold

The second investigation spanned two lessons during which students predicted the numerical contents of four filled and sealed canisters labeled A to D. Canister A was identical to the canisters used in the first investigation. Canisters B, C, and D were purposefully filled differently (see **fig. 1**). This investigation commenced with students creating a number line from the first investigation's canisters. The canisters were labeled with the children's names and the number of candies each one contained. The horizontal number line ranged from nineteen to twenty-four (i.e., the number of candies in the students' canisters). Canisters that held the same

By Carmel Diezmann

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number of candies were arranged vertically to show the frequency distribution. Students were asked to think about the number line they had created and to predict how many candies were in canister A:

Teacher: Here's canister A [*holding it up*]. Look at what you've written down for your guess for canister A. Use the information on the floor to help you [*indicating the number line made by numerically ordering the canisters*].

The teacher then asked students to indicate in turn with a show of hands whether they predicted fewer than nineteen candies, more than twenty-four candies, or between nineteen and twenty-four candies. All students except Eddie predicted that the number was between nineteen and twenty-four. The presence of the canisters lined up in number order appeared to help most students formulate a reasonable prediction. Thus, the canister number line acted as a concrete scaffold for students' thinking. Eddie predicted the number in canister A was thirteen but was unable to explain why. The teacher monitored Eddie's thinking closely throughout the investigation.

Students were then to predict how many candies were in canister B. They were not alerted to the fact that canister B was only partially filled, and not surprisingly they made predictions within the range on their number line. Coincidentally, canister B held thirteen candies, which matched Eddie's earlier prediction. The teacher encouraged students to think about why there were only thirteen candies in canister B when previously the fewest number of candies in the canisters was nineteen. She encouraged them to reconcile the fact that there were thirteen candies in canister B with their previous experiences of filled canisters containing between nineteen and twenty-five candies:

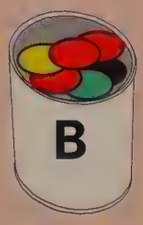
Teacher: Eddie said thirteen before. Well, I'd better write down thirteen on that [card]. I'm going to put a question mark because it seems far, far away from all the other answers [in the number line range]. Talk to your partner for a minute and see if you can think why it might have been only thirteen.

The students' various responses indicated that they had not thought about the fullness of the canister. The teacher also used this opportunity to put Eddie on notice to be accountable for his reasoning:

Leanne: Because it might have been the last jar left, and there weren't any candies left.
Teacher: Ah. The last one in the production line. Eddie, start thinking, because you had thirteen before. Shaun, what's your idea?
Shaun: Umm, because they could be in different places [in the can], and there could be more space or more room.
Teacher [*picking up can B*]*:* He [Shaun] said that last week, you know. He said that the way they're packed in [the canister] makes a difference because sometimes there's lots of space [between objects]. You know one thing I've been thinking about? Is it possible to ever make a mistake in counting?
Students: Yes.
Teacher: So, what's something we really should test out?
Robert: Count it again.
Teacher: Count it again. Out you come, Robert. It's your idea. [*Robert approaches the table where the thirteen candies lay.*]
Teacher: They're already sitting out there. I want him to count them back in here [*pointing to canister B*] because maybe it was just a mistake in Cara's counting.

Figure 1

Three of the four candy canisters for the second investigation were purposely filled differently from canister A.

			
Canister A was full of regular candies.	Canister B was about half full of regular candies.	Canister C was full of giant candies.	Canister D was full of sweets resembling mini candies.

Robert [*holding canister B, picks up the candies and drops them into the canister one at a time*]: One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen.
Teacher: Was it a mistake in her counting?
Students: No.
[*Robert picks up canister B from the desk, looks at it, and returns it to the table.*]

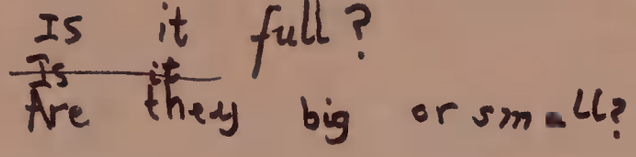
Another Assessment

Although Robert was handling the candies and the other students appeared to be watching him closely, no one noticed that canister B was only partially full. This reinforced the teacher's view that the students needed to further develop their observation skills. Her on-the-spot assessment of the students' observations skills provided direction to the lesson. The teacher scaffolded students' thinking by highlighting where they needed to direct their attention; she hinted to them to closely examine canisters A and B:

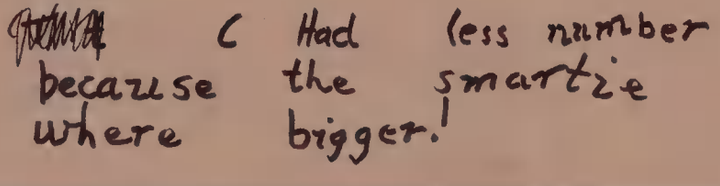
Teacher: I'll tell you what. Here's canister A, Robert, and there's canister B. I don't know; see if you can find out anything that's unusual about canisters A and B.
Robert [*picking up one canister in each hand*]: Well, this one here [*shaking canister B*] is not as full as this one [*indicating canister A*].
Teacher: What do you mean "not as full"?
Robert: Well … [*shaking canister A*].
Teacher: Take the lid off A and take the lid off B.
Robert: Umm. Ok. Ah.

Figure 2

Students recorded their observations in their mathematics journals.
(a) One student questioned both the fullness of the canisters and the size of the units in the canisters.



(b) Another student explained the relationship between the size and the number of the units.



Teacher: Test your idea. He says that this [*pointing to B*] is not as full, but I'm not sure I believe him.
Teacher: Take it around. No? Out you come, Caroline. She's the assistant. Have a look in there. Do you agree?
Caroline [*looking in the canisters*]: Yes.
[*Robert moves them around and nods his head.*]
Teacher: So, B had thirteen, but look inside…. It's only about half full. What would be a question you should ask me when I ask you to write down the number of candies? Is the canister [*pause*] what?
Robert: Full.
Teacher: Is the canister full? Because it makes a difference [to the number of candies].

Although the students had been slow to notice that canister B was not full, they were very interested in this investigation, and one student spontaneously asked about canister C:

Gemma: What about C?
Teacher: Well, what about C? All right. Here's C [*holding up the canister*]. Does anybody want to ask me a question before you guess this one? What was the question from B that was most important? Is the canister [*pause*] what?
Various students: Is the canister full?
Teacher: Is the canister full? Well, I'm not going to answer. I'll let Leanne look in the top [*offers Leanne the canister*] and tell you. Is the canister full, Leanne?
Leanne: Yes.
Teacher: There you are. Believe what she says. Write down your guess. How many are in canister C?

Although canister C was full, it was full of giant candies, which are much larger than regular candies. The size of the candies was not obvious to the students because the canister was only semitransparent. After the teacher had explored the range of students' responses, she invited Elise, who had predicted twenty-three candies, to count the candies.

Teacher: Twenty-three. Ok, off you go [*moving away from the table*].
Elise [*placing each candy into a row and counting*]: One, two, three, four, five, six, seven, eight, nine, ten.
Students [*observing Elise's counting and expressing surprise at the final count*]: Oh!
Teacher: It was a full canister, but it only had ten. [Ten is] less than thirteen.

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You Tricked Us!

Spontaneously, Elise began counting again, this time putting the candies back into the container one at a time. The rest of the class again closely watched her counting. Suddenly, one student noticed the size of the candies—the importance of which was quickly understood by most other students. Thus, there had been a change in students' levels of observation. Previously, students had failed to notice the fullness of the container, and the teacher had responded by scaffolding their observations. However, now the students were alert to the effect of the size of the sweets on the number that could fit in a container. Thus, students' alertness to size as a variable was an additional payoff from the teachers' earlier scaffolding of observations of fullness.

Gemma [*pointing at the candies*]*:* But they're *big* candies!
Leanne [*looking closely at the candies*]*:* They are! You [*pointing at the teacher*] tricked us!
Students: They're *big* candies!
Gemma [*standing*]*:* They're big candies [*looking perplexedly at the teacher*].
Teacher [*moving toward Elise*]*:* What are they?
Students: They're *big* candies.
Teacher: Oh..., they're the giant kind. Does that make a difference?
Students: Yes.
Teacher: Ok. Here's your job. If somebody asks you to guess how many candies are in a candies canister, write down two questions you should ask them.

Students wrote questions in their mathematics journals (see **fig. 2a**). The teacher then encouraged the students to discuss their questions.

Teacher: So, imagine we had canister D ... [*drawing a canister on the white board*] ... canister D ... and it contained some candies, and here are some questions. Before you guessed [how many candies are in] canister D, let's see how many different questions we can find.

During the discussion of questions, Toby highlighted a key understanding of mathematical measurement regarding the relationship between the size of a unit and the number of units. The teacher did not follow up on the relationship at the time because she was trying to highlight the critical variables of fullness and size, and some students were becoming fatigued. However, the teacher noted Toby's statement and followed up in the subsequent lesson:

Teacher: Toby, up in the back. What's one question you would ask the person?
Toby: How big are the candies?
Teacher: So, he's going to ask, "How big are the candies?" Toby, why?
Toby: Because if they are big, [you] won't be able to fit as many in the canister.
Teacher: All right. So, you get [fewer] big candies. Would that be a good reason?
Students: Yeah.
Teacher: That would help your thinking, wouldn't it? Ok.

In the follow-up lesson, the students recounted what they had learned about predicting the number of candies in canisters. They then suggested questions about canister D before predicting the numerical contents, and they discussed the appropriateness of their questions. Canister D was filled with small sweets that looked like mini candies, but again the students could not easily see the actual size of the sweets because of the nature of the sealed canister. Students established that the candies were small and made predictions up to the fifties. Some students spontaneously used the term *between* and predicted a range of numbers, which was a qualitative change in thinking from predicting a single number.

Once the canister was opened, students were encouraged to revise their predictions. They were again encouraged to revise their predictions when one-quarter, one-half, and three-quarters of the contents had been counted. Throughout the process, students watched intently and repeatedly revised their estimates. It became apparent that the students' observations were informing their predictions. Thus, the teacher did not discourage students from changing their answers. Although the counting process was lengthy, the students maintained a high level of interest. Following the counting, students were encouraged to record what they had learned about predicting the number of candies in a canister. A discussion of the variables of size and fullness was followed by a request from the teacher to explain in writing why canister C from the previous lesson had so few candies. One student's response highlighted the inverse relationship between the size of a unit and the number of units (see **fig. 2b**). Students further

discussed this size-number relationship, a key understanding in mathematics and particularly important for sense making about conversions between units. For example, if the age of a ten-year-old is converted to months, the number of months will be greater than the number of years because a month's duration is less than a year's duration. This inverse relationship between the size of a unit and the number of units has widespread applications in mathematical conversions, such as from dollars to dimes and vice versa.

Conclusion

The teacher's assessment of students' mathematical knowledge in the first investigation and her ongoing assessment in the second investigation resulted in quality learning opportunities for students. In the first investigation, the teacher identified students' lack of attention to using observations to inform numerical predictions. She prepared a subsequent investigation to focus on the interrelationship between observation and prediction. Throughout the second investigation, the teacher continually assessed students' progress and scaffolded their thinking to encourage them to use and hone their observation skills and to make reasoned predictions. Additionally, the teacher's assessment also identified strengths in the students' knowledge that could be capitalized on, namely, an awareness of the interrelationship between the size of a unit and the number of units. Thus, this teacher's assessment following the first investigation was complemented by her on-the-spot assessments during the second investigation, which immediately informed the instructional process. Such assessment could be characterized as one of the "*profound changes* in what and how mathematics is taught and assessed" (Baroody and Coslick 1998, p. vii, emphasis in original) that are necessitated by inquiry approaches.

Reference

Baroody, Arthur J., and Ronald T. Coslick. *Fostering Children's Mathematical Power: An Investigative Approach to K–8 Mathematics Instruction*. Mahwah, NJ: Lawrence Erlbaum Associates, 1998.

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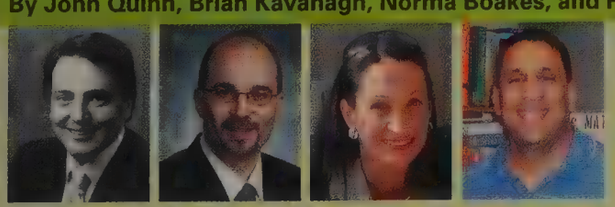
Two Thumbs Way, Way Up

Index Card Recap and Review

When fourth graders in an urban district—where standardized testing is traditionally a struggle—learned to peer review and recap their daily mathematics lessons in an easy-to-implement index card format, 100 percent passed the end-of-year assessment, 73 percent at the advanced level.

By John Quinn, Brian Kavanagh, Norma Boakes, and Ronald Caro



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The famous television duo Siskel and Ebert (then Ebert and Roeper) made a cottage industry of reviewing films. With a simple gesture of thumbs up or down, an audience knew their movie critique: Movies that they deemed destined for good fortune were said to get "two thumbs up, way up."

Movie reviews continue to be popular in our culture. Most newspapers have reviews that include five-star ratings for the latest blockbusters. In the same way that we depend on movie reviews, we may go to an online site to find how others rated a book we are considering for purchase. Reviews are often

the final arbiter of whether or not we buy the book. Many of us also love to scroll through the list of other purchases readers made, searching for our next buy. Most of us tend to be interested in rating things and seeing how others rate the same item. Who among us does not like to share a personal opinion?

Children share our curiosity about reviews and ratings. The index card recap and review project uses student summaries and peer review of those summaries as a means of reviewing material, communicating mathematical ideas, and assessing student content knowledge.

Background

Brian Kavanagh teaches fourth grade in an urban school district with a large proportion of at-risk students. A nontraditional teacher with a background in finance, his desire to build his knowledge of mathematics instruction as well as how to teach it led him to enroll in a graduate education program at Richard Stockton College of New Jersey. The graduate course guided students to consider various alternative assessment strategies that evaluate how well students apply their new knowledge in context. Can they discuss what they have learned in an understandable way? Are they able to communicate their ideas effectively? *Principles and Standards* (NCTM 2000) has much to say about how valuable such assessments are in the classroom.

One day, Kavanagh and John Quinn began brainstorming ideas concerning authentic assessments for Kavanagh's fourth-grade classes. They came up with an idea for a culminating activity for each class that included student summaries along with peer reviews of those summaries. This article describes the project and its results.

The Communication Standard

Today's teachers look for ways to get their students to organize and consolidate their mathematical ideas in order to communicate those more clearly to teachers, peers, and others. Additionally, we want our students to be able to evaluate others' ideas and the precision with which they communicate those ideas.

Using student summaries and assessing them through peer reviews can improve communication among students and allow students to actively manage their learning and increase their time on task and their level of communication (Liu and Carless 2006, Topping and Ehly 1998), while giving them timely feedback on their summaries.

The Assessment Principle

Teachers often consider various new ways of assessing student understanding of concepts. For instance, they supplement single tests with rich, diverse modes of authentic performance assessments. As professionals, we investigate alternative, ongoing, and creative ways to assess student learning.

The index card recap and review gives students the opportunity to investigate problems instead of memorizing and repeating algorithms with little thought. Students explain a problem situation based on the classroom work of the day. This approach gives teachers another way besides standard tests and quizzes to assess student progress. Finally, the recap and review project gives clear indications to the entire class about what students need to know and be able to do.

This assessment strategy can complement other alternative assessment techniques, can be integrated into mathematics portfolios of student work, and can provide a useful tool for math journaling as a way to continually assess student progress.

Index Card Recaps

Kavanagh's classroom offers many exploration and collaborative activities. He was looking for a culminating activity that would pull together key ideas from an activity, was easy to implement, and could continue throughout the year. Kavanagh and Quinn developed a culminating assessment activity that could be instituted in the last ten minutes of a lesson. Each student is given a three-by-five- or four-by-six-inch index card and asked to summarize the key concepts of the lesson or to complete an open-ended or constructed response problem to assess their mastery of the day's concepts. More details about the process appear below.

Number sense using money

The fourth-grade class was engaged in a lesson on money. The lesson objective was for students to investigate all the ways to combine various coins to get $0.88. Toward the end of class, Kavanagh asked if one combination both equaled $0.88 and had the fewest coins. He asked students to summarize the main ideas from the activity on an index card (see **fig. 1**). Although these student examples are

rather terse and the class was just one month into the fourth-grade year, the first card (see **fig. 1a**) communicates the sense of the class objective. The student does not complete full sentences but uses descriptive phrases such as *least amount* and a drawing of the combinations, making clear that the combination with the fewest number of coins contains three quarters, a dime, and three pennies. The student also includes other combinations that require more coins. This response demonstrates that the student understands coin denominations and combining to a specified sum. Further, the student is able to communicate in writing, providing key vocabulary to indicate the reasoning behind the conclusions.

The second card (see **fig. 1b**) features a table, which can be an effective device for communicating mathematical ideas. However, the lack of verbal cues leaves the reader in a position of having to interpret the table. Although the student shows several combinations, only one produces the correct total amount. This student may know that different coins can be combined to a specific sum, but a problem exists in the calculations. The second and third combinations of coins add up to \$0.80 and \$0.90, respectively, not the desired \$0.88. Additionally, no clear strategy emerges to identify the fewest number of coins that could be used. The first student did a better job of communicating the mathematical idea of the lesson and an understanding of the material presented. The second student's work is incomplete. She also failed to indicate any higher-order thinking of what was the least amount or how one might determine it. Both students seem to have some sense of how to determine coin value and a sum, but neither is consistent in her examples. Whereas a teacher would find such information useful in terms of assessment, Kavanagh's goal was that students draw more from the activity. Illustrating the coin combinations clearly was a valuable exercise for students, and their verbal descriptions provided windows into understanding their thought processes. The teacher's desire that his students understand these benefits led to the next step. If students could see one anothers' work, they could see what needed to be communicated, what strategies worked best, and how problems are solved.

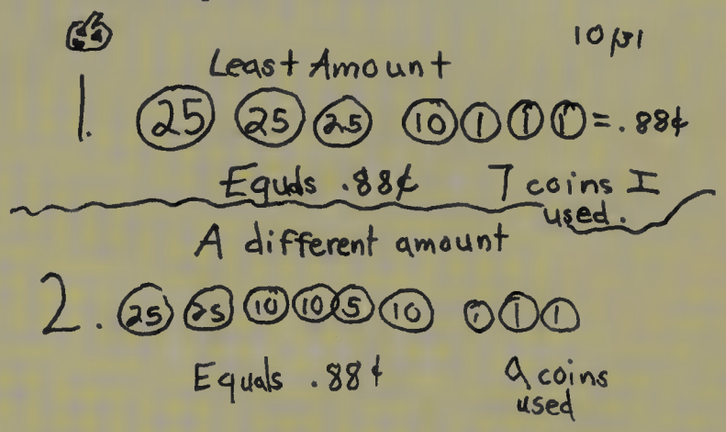
Peer Reviews of Recap Cards

The recap cards were collected and given to other students in the class for peer review using a simple scale:

Figure 1

The teacher asked students to summarize the main idea of the activity on index cards.

(a) This student demonstrates understanding of coin denominations, combining to a specified sum, written communication, and providing key vocabulary to indicate the reasoning behind the conclusions.



(b) A table can be an effective device for communicating mathematical ideas, but this student's work is incomplete.

Q	D	N	P
2	3	1	3
2	3		
3	3		

☆☆☆ Three stars were given to a recap that was clear, accurate, grasped the main ideas of the lesson, and showed how the problem was solved.

☆☆ Two stars indicated a satisfactory job of getting the main ideas, but the response was either unclear to the reader or contained a simple arithmetic error.

☆ One star showed an attempt at the problem but was hard to read, not very clear, or had several errors. The response showed only a basic understanding of the ideas involved.

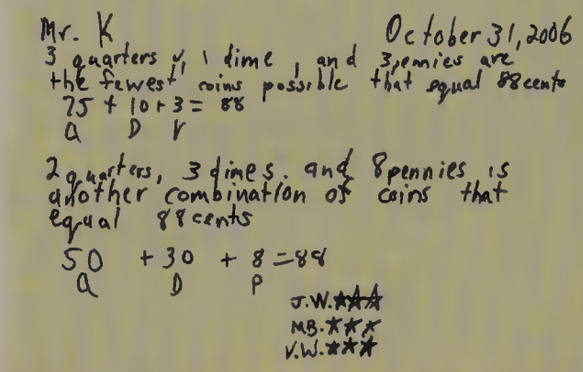
Zero stars showed major errors in understanding the ideas.

Classmates who rated the first card gave the recap three stars, indicating that they liked the review and thought it captured the lesson pretty well. Peers gave the second recap one star, indicating that it lacked clarity and precision.

In the beginning, Kavanagh continually modeled the difference between one-, two-, and three-star

Figure 2

This is a three-star rated recap card with peer reviews.

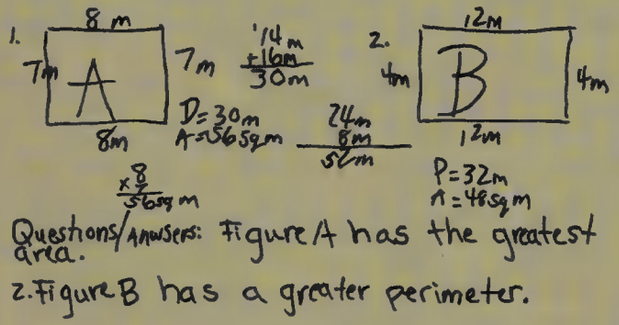

Mr. K
October 31, 2006
3 quarters, 1 dime, and 3 pennies are the fewest coins possible that equal 88 cents
75 + 10 + 3 = 88
Q D P
2 quarters, 3 dimes, and 8 pennies is another combination of coins that equal 88 cents
50 + 30 + 8 = 88
Q D P
J.W.★★★★
M.B.★★★
V.W.★★★

responses. Over time, students and teacher talked about the ratings, and a classroom norm began to emerge of what a three-star recap should look like. Students who began the assessment strategy by regularly turning in one-star recaps learned how to improve their mathematics communication skills.

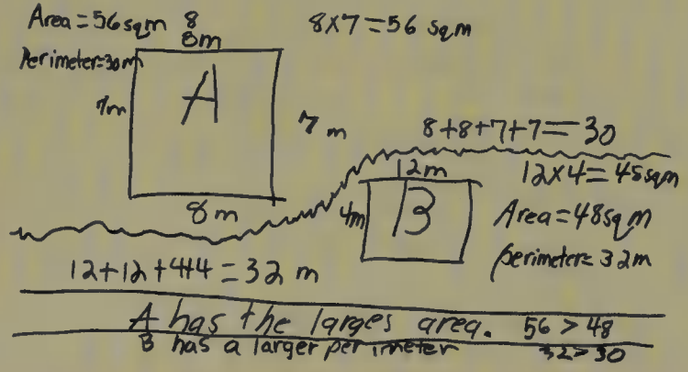
Figure 3

This lesson objective was to compute area and perimeter for rectangles.

(a) This card got three-star ratings from all four classmates.



(b) Although this student is less linear in his presentation, his card also received three-star ratings from classmates.



The most obvious reasons for these improvements is that students who had been struggling every day to understand the rating criteria were now able to instantly see what excellent index card recaps looked liked. Second, the process lent itself to peer-to-peer teaching. Often, before Kavanagh even had the cards back in his hand, a student's misconceptions from the lesson were cleared up by other students. For example, a student received a rating of two stars and asked why he did not get three stars. Students informally shared strategies for improvement before handing in their cards. The rating was not changed, but keys to the correct solution were exchanged. (Examples of this peer-to-peer interaction appear later in this article.) This peer-to-peer teaching increased the retention of the students who were doing well and improved the understanding of those students who did not grasp the concept as fully. Students no longer had to wait for the teacher to pull a small group together the next morning to clear up their misconceptions about the lesson. They received the instruction they needed from their peers. See **figure 2** for another three-star recap with the peer rankings on the card.

Over the year, students became focused on succinctly summarizing the mathematics content of a lesson. When a student returned from an absence, Kavanagh used the opportunity to share with him or her a three-star recap card or to have a peer use the three-star recap card to review the main ideas covered in the missed lesson.

Geometry using area and perimeter

The next example is from a class that focused on geometry. The lesson objective was for students in cooperative groups to compute area and perimeter for rectangles. Students were also expected to write their answers in appropriate units. During class, students found the areas and perimeters of a variety of rectangles. Toward the end of class, students were given a performance assessment that began with two rectangles. The first rectangle was seven meters by eight meters, and the second rectangle was twelve meters by four meters. Students were asked to determine which rectangle had the greater area and which rectangle had the greater perimeter. They were to show the rectangles and all their work, writing all answers in appropriate units.

The card shown in **figure 3a** got three-star ratings from all four classmates, revealing that this student performed the task satisfactorily. At

fifty-six square meters, the first rectangle has a larger area than the second rectangle's forty-eight square meters. However, the second rectangle has a perimeter of thirty-two meters, whereas the first one has a perimeter of thirty meters. The performance assessment counters a misconception that the larger area will always be associated with a larger perimeter. The student showed all his work, his calculations are clear and accurate, the figures are clear, and the questions are answered directly. **Figure 3b** shows a response that is also three-star quality. This student is less linear in his presentation, yet all the calculations are there. The answers use the greater-than symbol.

These two three-star responses are typical of the students who shoot up their hands to respond to review questions at the end of the period. Novice teachers will often hear from two or three of these students and incorrectly assume that the entire class understood the lesson.

Figure 4a shows a card that records the area of the first rectangle as fifty-six meters. The card received two-star ratings from peers. It shares several similarities with the earlier examples: This student's calculations are correct, the figures are clear, and the conclusions are proper. We might conclude that it was a simple oversight to fail to list the units as *square* meters. However, the area of the second rectangle is also given in meters, suggesting that this student may be slightly confused about units for area, a misconception that is easily corrected.

Student work in **figure 4b** seems to correctly address the general question but fails to include units on several calculations and includes incorrect units for area. As a result, this card also received two stars. Several cards with the same error pattern would suggest either a group intervention for the next class or some direct instruction to correct these patterns as they emerge. The index card recap gives teachers a simple ongoing assessment piece to guide instruction and interventions.

Data analysis using mode, median, and range

The objective of a lesson on data analysis was to enable students to find the mode, median, and range for a set of numbers. The class began with hands-on activities to generate data sets. Students reviewed graphing by organizing the data into bar charts. Then they were to compute the mode, median, and range for the various data sets. They practiced in groups, while Kavanagh circulated throughout the room, reinforcing concepts.

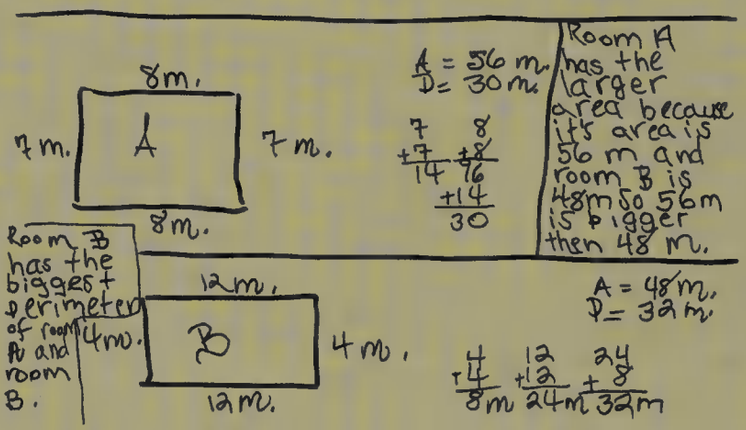
Photograph by Stefanie Burns; all rights reserved

After students complete individual recap assessments, they compare and discuss their index cards.

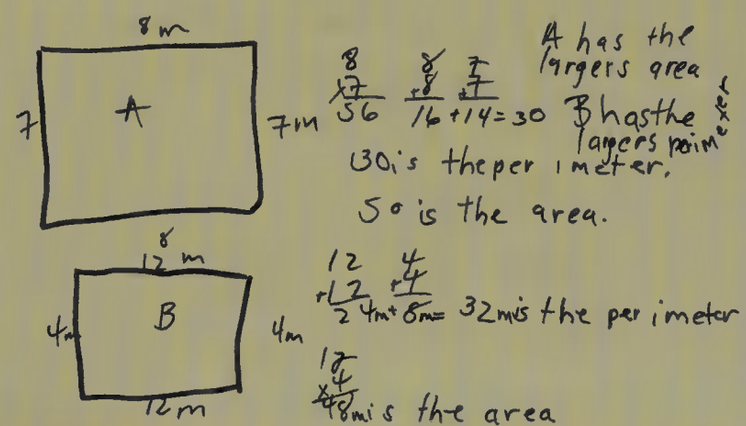
Figure 4

Several cards with the same error pattern suggest to the teacher that a group intervention is necessary.

(a) This student may be slightly confused about units for area, a misconception simple to correct.



(b) This student seems to correctly address the general question but fails to include units on several calculations and includes incorrect units for area.



As a final performance assessment, the teacher gave students five numbers arranged in random order and instructed them to compute the mode, median, and range for these five numbers. In addition, they were to explain their answers. For example, "Explain why you selected a particular number to be the mode." This performance assessment asks students to go beyond computing the correct answer to give a rationale for their answer. This approach strengthens students' ability to do well on open-ended assessments.

The card in **figure 5a** received three-star reviews. The answers of mode = 100, median = 95, and range = 14 are all correct. Fourth-grade vernacular accurately captures the idea of mode: "The mode is 100 because it showed up the most." Similarly, this student has a clear understanding of median as the middle number, once the numbers are arranged in order. Later, these fourth graders will learn how to get the median for an even number of data points, but this student is off to a good start and also understands that the range is the largest number minus the smallest.

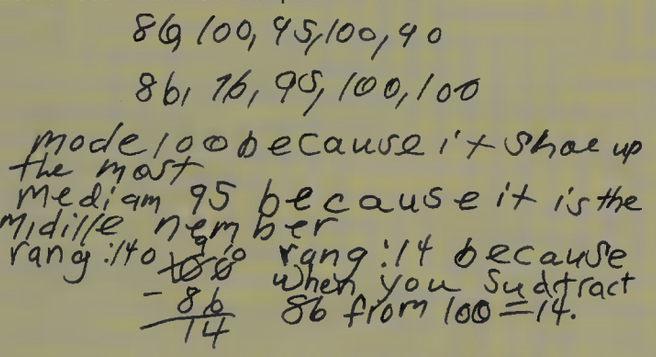
Although the student who prepared the card in **figure 5b** received three stars and got correct answers, the communication could be stronger. Notice the lack of detail in the descriptions in each of this student's explanations: "The answer to the mode is 100." "The answer to the median is 95." "The answer to the range is 14." These answers make the teacher aware that this student needs to focus on effectively communicating mathematical processes.

Peer scores of zero stars make it apparent when a student needs to review the lesson (see **fig. 5c**). The mode is correct, giving the teacher something on which to build, but Kavanagh will use this assessment to plan an intervention on median and range, check the student's understanding of mode, and work on communicating ideas.

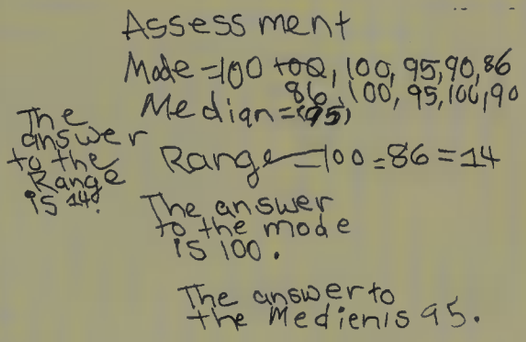
Figure 5

Students were to compute mode, median, and range for five given numbers.

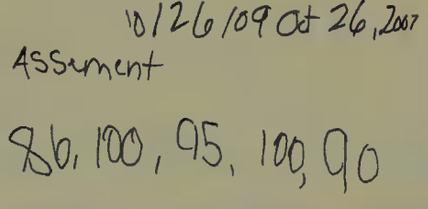
(a) This card received three-star peer reviews.


86, 100, 95, 100, 90
86, 76, 95, 100, 100
mode 100 because it shoe up the most
Mediam 95 because it is the midille member
rang: 14 because when you sudtract 86 from 100 = 14.

(b) This student needs to focus on effectively communicating mathematical processes.


Assessment
Mode = 100
Median = 95
86, 100, 95, 100, 90
Range = 100 - 86 = 14
The answer to the Range is 14.
The answer to the mode is 100.
The answer to the Medien is 95.

(c) Peer scores of zero stars on this card make it apparent to the teacher that the student needs to review the lesson.



Five Steps to Index Card Recap and Review

1. With about ten minutes of class time remaining, give each student a standard (3" × 5" or 4" × 6"), lined index card.
2. Have students write a recap of the most important points in the lesson or solve the assessment problem of the day. Make sure they put their name on the card.
3. Instruct students to pass the cards to peers in their cooperative learning group. Cooperative learning groups change biweekly to ensure that each student will receive a wide range of opinions.
4. Have peers rate the recap according to a predetermined rubric for good communication. Have the reviewers place their initials next to the rating. Record the peer evaluation values.
5. Return the reviewed index card recaps to the authors. Have a student or two who score three stars from their peers share their responses. If time permits, discuss the ratings and how the recaps can be made stronger. Periodically post examples of excellent recaps or place on the overhead along with the classroom rubric.

Fractions

The final set of examples shows how the recap cards and review can be used to document progress or emerging difficulties over time. These cards show student work involving fractions after four lessons spanning about two months in fourth grade.

The first lesson was on the meaning of various fractions, and the objective was for students to compare common fractions with different denominators. The writing prompt for the card was, "Mr. K ate half of a pizza, and a student ate 3/8 of a pizza. Who ate more? Explain your answer."

The second lesson returned to this theme several weeks later, but in the second case, students had to perform the operation of addition and then compare the sum to the number 1. First, students were to add 4/8 and 3/6 and tell whether the answer was less than, equal to, or greater than 1. Next, they were to add 1/2 and 5/8 and tell whether the answer was less than, equal to, or greater than 1. They were also to explain their answers.

In the following class, students continued to learn how to add and subtract fractions. Students drew squares representing 1 and used these squares to characterize addition and subtraction of fractions. They were to find 2/3 + 1/2. Next, they were to find 3/4 – 1/3. They were to illustrate each operation with a set of squares.

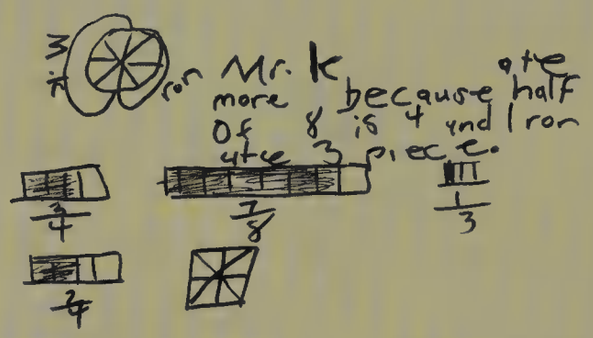
The fourth lesson featured an addition problem involving fractions, where the least common denominator was more difficult to find: 5/12 + 11/15.

Figure 6 shows four recap cards for the four lessons by a student who generally received three stars from her classmates for all four cards. She correctly concludes that Mr. K ate more pizza because 1/2 of a pizza (with eight slices) is four slices and that is more than the three slices that the student had (see **fig. 6a**). On the second card (see **fig. 6b**), she correctly concludes that 4/8 and 3/6 each are equivalent to 1/2, so they will add to 1, whereas 1/2 plus 5/8 is greater than 1 because 5/8 is greater than 1/2. The third card (see **fig. 6c**) shows the common denominator of 6 in the diagrams, 2/3 equivalent to 4/6, and 1/2 equivalent to 3/6. The student adds 4/6 + 3/6 to get 7/6 and then writes the answer as a mixed number, 1 1/6. This student also subtracts 3/4 – 1/3 correctly and shows the appropriate diagrams. Finally, 5/12 + 11/15 is done flawlessly (see **fig. 6d**), including reducing 1 9/60 to 1 3/20. All indications are that this student has strong, consistent understanding of how to add and subtract fractions.

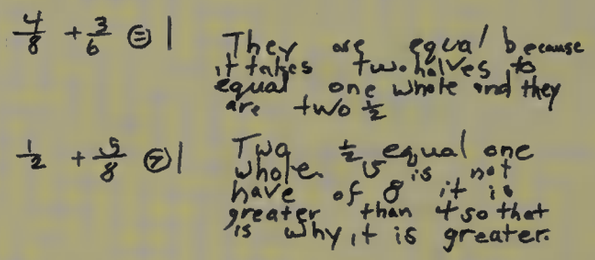
Figure 6

Students were to compare common fractions having different denominators.

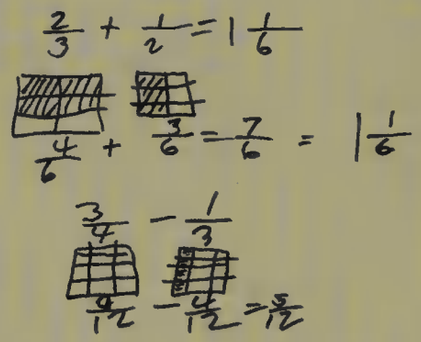
(a) This student typically received three-star assessments from her classmates.



(b) She correctly concludes that 4/8 and 3/6 each are equivalent to 1/2, so they will add to 1, whereas 1/2 plus 5/8 is greater than 1 because 5/8 is greater than 1/2.


4/8 + 3/6 = 1
They are equal because it takes two halves to equal one whole and they are two 1/2
1/2 + 5/8 > 1
Two 1/2 equal one whole 5/8 is not have of 8 it is greater than 4 so that is why it is greater.

(c) She shows the common denominator of six in the diagrams, adds the fractions, and then writes the answer as a mixed number. She also subtracts fractions correctly and shows the appropriate diagrams.



(d) All indications are that this student has strong, consistent understandings of how to add and subtract fractions.

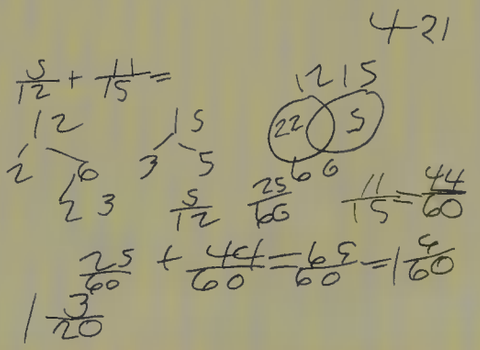
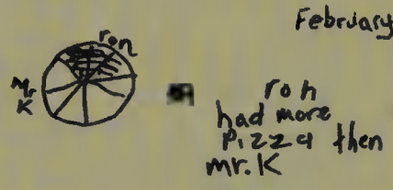


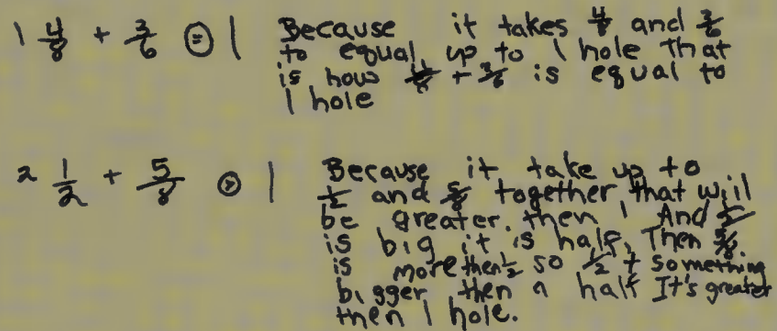
Figure 7

Peers gave the student work zeros on the first card.

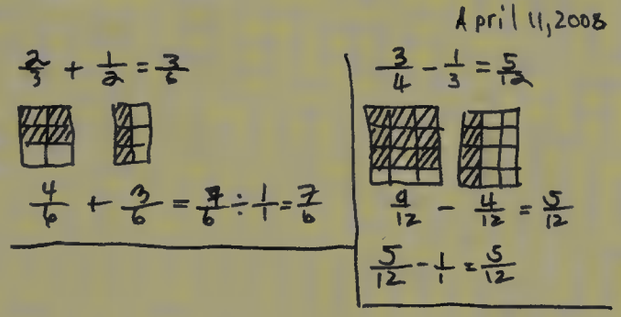
(a) The diagram appears to be correct, but the student draws incorrect conclusions.



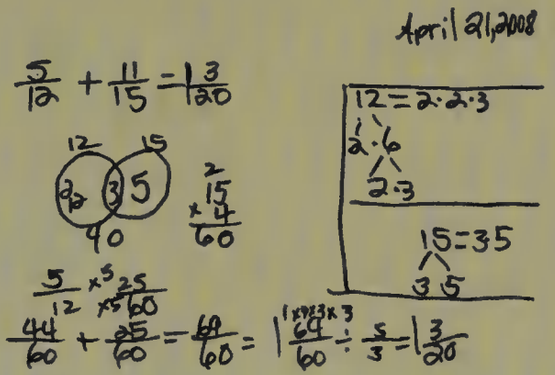
(b) With correct answers and solid logic, the student received three stars on her second card.


1 4/8 + 3/6 = 1
Because it takes 4/8 and 3/6 to equal up to 1 hole that is how 4/8 + 3/6 is equal to 1 hole
a 1/2 + 5/8 = 1
Because it take up to 1/2 and 5/8 together that will be greater then 1 And 1/2 is big it is half, Then 5/8 is more then 1/2 so 1/2 + something bigger then a half It's greater then 1 hole.

(c) The diagrams on this card are clear, and the addition is performed correctly, but only four of five peers gave her three stars because of her failure to rewrite the fraction as a mixed number.



(d) A week later, she had learned how to perform this operation. She made a refinement on the basis of a peer review, and she retained the skill.



Photograph by Stefanie Burns; all rights reserved

Students rate one another's recap cards.

Peers gave the student work in **figure 7** zero stars on the first card. This student's answer to the pizza problem is that the student had more pizza than Mr. K. The diagram appears to be correct, but the student draws incorrect conclusions (see **fig. 7a**). This student gets three stars for her second card (see **fig. 7b**). Her answers are correct, and her logic is solid. On her third card (see **fig. 7c**), four out of five peers gave her three stars. The diagrams are clear, and the addition is performed correctly. However, the student's failure to rewrite the fraction 7/6 as a mixed number gave one reviewer justification for a score of two stars and provided an opportunity for the student to seek out how to rewrite the fraction as a mixed number. The fourth card (see **fig. 7d**) for this student indicates that she learned how to perform this operation and was able to continue to do so a week or two later. She made a refinement on the basis of a peer review, and she retained the skill.

Figure 8 chronicles three cards from a student who is struggling with the material. Peer ratings consistently below three stars alert Kavanagh to arrange for regular interventions with the child, who is, however, making progress with the material. The first card (see **fig. 8a**) shows diagrams of 3/8 and 1/2 of the slices shaded. The student did not answer the question but made a start. The second card (see **fig. 8b**) reveals that the student recognized 4/8 and 3/6 as each being equal to 1/2, which led to an answer of 1. The student did not answer the second question, either. On the third card (see **fig. 8c**), the student started well, drew accurate diagrams, and performed addition and subtraction correctly. The first fraction is not rewritten as a mixed number. The second answer of 5/12 is correct, but the student goes on to compute, revealing a lack of clarity as to when the problem is finished. Peers gave this student the appropriate score of two stars. The responses

tell the teacher that he must follow through and analyze with the student the meaning of the progression 5/12, 2/12, or 3/12. On the last line, the student subtracts correctly and gets 5/12, subtracts again with 2/12, and then writes the answer as 3/14. This suggests follow-up on the meaning of subtraction and reaching closure on a problem. The misconception to continue after the problem is complete needs to be carefully addressed.

This student's responses reveal a consistent pattern. In each case, the student began well. In each case, the student faded and lost a bit of focus as the problem wrapped up. This is worthy of follow-up to help this student build strategies to strengthen the ending for each problem. Also, while the student may be discouraged in receiving mostly two-star ratings, each solution that he presented contained material on which to build. This student can honestly be encouraged for these good starts.

Classroom Results

The results of using recap cards and peer review proved excellent. One hundred percent of Kavanagh's students passed New Jersey's fourth-grade mathematics assessment, including 73 percent at the advanced proficient level. In November of that academic year, more than one-third of the class had been unable to complete forty 0–12 multiplication facts in five minutes, including several students who scored fewer than ten correct facts in five minutes. By March, all but one student were able to complete one hundred multiplication facts in less than two minutes; the other student could complete the task in less than thirty additional seconds. The index card recap and review is just one of the activities that brought success to this class. In general, the class reflects many of the NCTM standards.

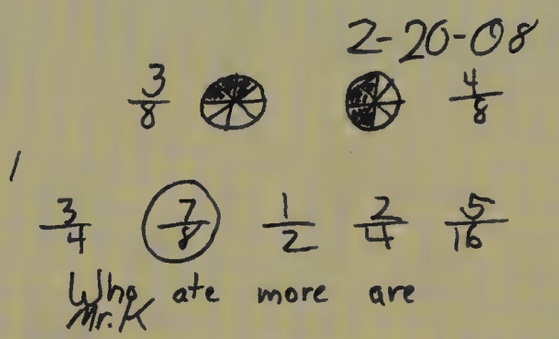
Wrap-Up

The index card recap and review is a solid, culminating activity for the end of each class—and much more. It allows students to collaborate, review peers, and learn from one another's reviews. The activity gives teachers a record of a performance assessment and insight into students' strengths and weaknesses. Trends found in the cards can guide instruction, the differentiation of instruction, and special interventions. Finally, this activity helps students focus on strengthening their mathematics communication skills, while raising appreciation for effective communication strategies among their peers.

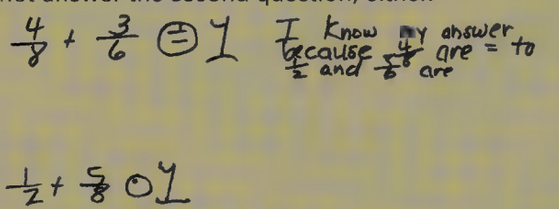
Figure 8

These three cards indicate that this student is struggling with the material.

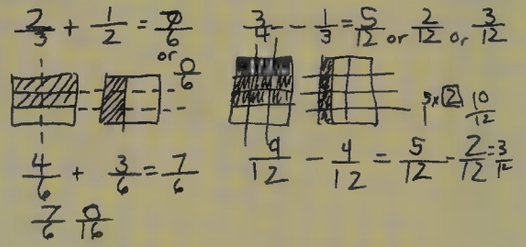
(a) Although the student did not answer the question, he attempted to do so.



(b) He did not answer the second question, either.



(c) The misconception to continue after the problem is complete needs to be carefully addressed.



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Emphasizing STANDARDS

with Classroom Activity Extensions

Even a simple exercise has potential to provide extensions for multiple standards, thereby increasing students' mathematical skills and confidence.

By David A. Coffland

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Photograph by David A. Coffland; all rights reserved

This article describes the extension of a simple activity into an exploration of multiple NCTM Standards. It includes the solution process undertaken by a combination fifth-sixth–grade class. In teaching students to attain Standards, all classroom activities should purposefully address one or more Standards. With textbooks, supplemental resource books, and especially the development of the Internet, activities abound. Not all activities are fully developed or Standards-ready, but a teacher can extend almost any activity to become a Standards-based lesson.

"Count the Triangles" Activity

The following was originally intended as a shape-recognition activity. However, the teacher chose to extend the activity to address additional Standards in greater depth. Students were to count how many triangles they could find in a diagram (see **fig. 1**). They worked individually for some time and then formed small groups to finish their counting. Their answers ranged from eighteen to thirty-two. Remarkably, no students were confident of their answer.

When the teacher asked, "What was ... most difficult ... about doing this activity?" answers included "counting all the triangles when they overlap" and "not counting the triangles more than once." Because the original activity had failed to attain closure, the teacher presented the first extension activity.

Triangle sorting

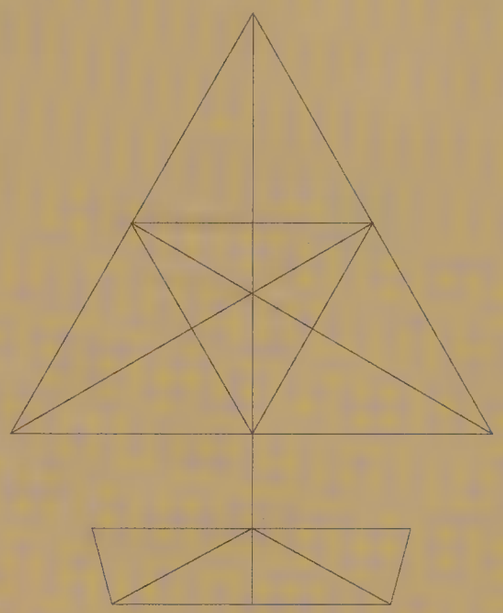
To help students organize information in a logical way, the first extension addressed the expectation noted in the Geometry Standard of specifically identifying and classifying triangle types. The teacher began by asking, "Are there different types of triangles in the figure?"

As the sixth-grade students identified and named right, equilateral, isosceles, and acute triangles, the fifth graders were introduced to these terms. The teacher's second question was, "Can we find right triangles of different sizes?" Students replied affirmatively and proceeded to find five different sizes of right triangles in the diagram. They were then instructed to determine the number of sizes for each triangle type in the figure.

The teacher's questioning strategy directed students from a random search to a more systematic one for triangles. The result was a table of different types and sizes of triangles (see **table 1**). The purpose of making and completing such a table is to identify one thorough, systematic way of searching and obtaining the solution and confirming that the answer is correct.

Figure 1

Fifth and sixth graders found the first part of the "Count the Triangles" activity to be easy, but classifying and naming the triangles proved to be more challenging.



As students began to complete the table, several issues arose almost immediately. Although the diagram has five sizes of right triangles, the students identified only two sizes of equilateral triangles. Some students stated, "It's hard to name the sizes." Some students used different columns for the two sizes of equilateral triangles. As a result, problems surfaced in communicating their ideas to other

Table 1

Triangles by Type and Size

Triangle Type	Triangle Size				
	Smallest	Small	Medium	Large	Largest
Right					
Equilateral					
Isosceles					
Scalene					
Other					

Table 2

Potential Triangles in the Diagram

"Count the Triangles" Activity—Potential Triangles

ABC	BCD	CDE	DEF	EFG	FGH	GHI	HIJ
ABD	BCE	CDF	DEG	EFH	FGI	GHJ	
ABE	BCF	CDG	DEH	EFI	FGJ	GIJ	
ABF	BCG	CDH	DEI	EFJ	FHI		
ABG	BCH	CDI	DEJ	EGH	FHJ		
ABH	BCI	CDJ	DFG	EGI	FIJ		
ABI	BCJ	CEF	DFH	EGJ			
ABJ	BDE	CEG	DFI	EHI			
ACD	BDF	CEH	DFJ	EHJ			
ACE	BDG	CEI	DGH	EIJ			
ACF	BDH	CEJ	DGI				
ACG	BDI	CFG	DGJ				
ACH	BDJ	CFH	DHI				
ACI	BEF	CFI	DHJ				
ACJ	BEG	CFJ	DIJ				
ADE	BEH	CGH					
ADF	BEI	CGI					
ADG	BEJ	CGJ					
ADH	BFG	CHI					
ADI	BFH	CHJ					
ADJ	BFI	CIJ					
AEF	BFJ						
AEG	BGH						
AEH	BGI						
AEI	BGJ						
AEJ	BHI						
AFG	BHJ						
AFH	BIJ						
AFI							
AFJ							
AGH							
AGI							
AGJ							
AHI							
AHJ							
AIJ							
KLM							
KLN							
KLO							
KLP							
KMN	LMN						
KMO	LMO						
KMP	LMP						
KNO	LNO	MNO					
KNP	LNP	MNP					
KOP	LOP	MOP	NOP				

students. After some teacher intervention, the class agreed that the two sizes of equilateral triangles would be called *large* and *small*. They made similar agreements for naming the sizes of other triangle types.

Some students found another difficulty: "There are more boxes in the table than we need." They decided to put zeroes in "extra" boxes on the chart.

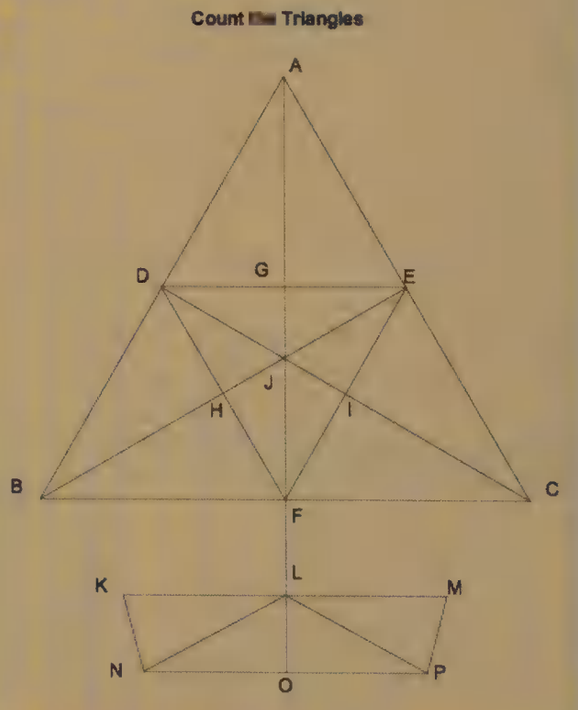
Finally, a student asked, "What do we do if there is a right triangle that is also scalene?" This more serious concern led to a discussion of how to avoid counting a triangle twice. The table was not specific enough in its labeling of triangle types. Once again, students' answers varied greatly. The class concluded that using the table was better than nothing, but it still was inadequate for making them confident of their answers.

Communication conventions

The use of proper terminology for identifying triangles is part of a larger system of communicating mathematically. Students found the attempted classification system of **table 1** lacking; they needed to name each individual triangle in order to communicate effectively.

Figure 2

Students had to agree on definitions before they could label all the intersections.



Students identified the requirement for a system for naming every triangle while excluding redundant names. They first considered the largest triangle in **figure 1** and labeled its vertices labeled *A*, *B*, and *C*. Students quickly found six possibilities for naming this triangle: *ABC*, *ACB*, *BCA*, *BAC*, *CBA*, and *CAB*. One student argued that only six names are possible because each of the letters *A*, *B*, and *C* appear twice in the first position and there are "only two directions to go around the triangle."

Because all the names identified the same triangle, students needed a system to eliminate the redundant names. The entire class accepted a communication agreement, a naming convention. They would put the letters from each triangle name into alphabetical order. They called the triangle in the example *ABC* and eliminated other possible combinations as redundant.

They also determined as a class how to label each intersection. The definition of an intersection became important; some students viewed intersections only as places where lines cross, that is, as an **X** pattern. Once the teacher pointed out that some triangles have vertices where lines meet but do not cross, students agreed that intersections include any point where two or more lines come together. Under this definition, each intersection was labeled with a letter from the alphabet (see **fig. 2**).

Once more, the class members counted the triangles in the figure, but their answers were still inconsistent. Some students suggested that they could check each student's list to make certain that they had found all the triangles; but another student pointed out that if everyone missed the same triangle, the combined list would also miss it. As a result, students decided that they needed to find a better method.

Reasoning methods

At this point, the activity was again extended to make it an exercise in mathematical reasoning. Students wanted a method that would guarantee finding all the triangles only once. They decided to make a list of all possible triangle names and then check to see which ones were actually in the diagram. Students began to write names of possible triangles. After a few minutes, one student announced that he would make a list of all the possible names that started with *A*. That set off a flurry of activity, as other students assigned themselves lists beginning with other letters. Some lists were much longer than others.

Now the teacher asked, "Can any triangle have a vertex in both the top and bottom section of the diagram?" After some debate, students decided that because the top and bottom portions of the figure are connected by a single line, the two sections of the figure could be examined separately. This meant that possible triangle names such as *ABK*, *ABL*, *ABP*, and so on did not have to be written because they included points from both sections of the diagram. **Table 2** shows the complete list of potential triangles. The two sections of the list represent the possibilities in the top and bottom portions of the diagram.

Table 3

Potential Triangles with Nontriangles Crossed Out

"Count the Triangles" Activity—Potential Triangles with Nontriangles Crossed Out

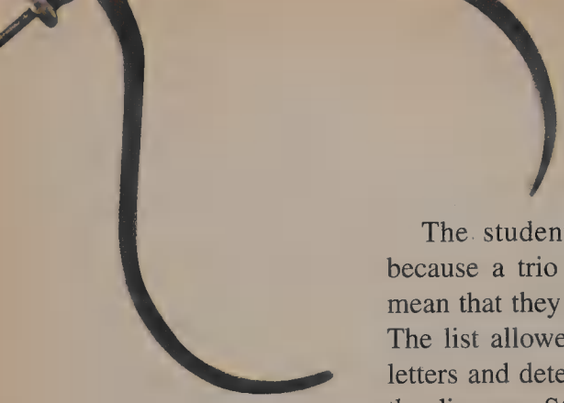
Count the triangles and eliminate duplicates.

Triangle Possibilities in the Main Portion of the Tree

ABC	BCD	CDE	DEF	EFG	~~FGH~~	~~GHI~~	~~HIJ~~
~~ABD~~	BCE	CDF	~~DEG~~	EFH	~~FGI~~	~~GHJ~~	
ABE	~~BCF~~	~~CDG~~	DEH	~~EFI~~	~~FGJ~~	~~GIJ~~	
ABF	~~BCG~~	~~CDH~~	DEI	EFJ	~~FHI~~		
~~ABG~~	~~BCH~~	~~CDI~~	DEJ	~~EGH~~	FHJ		
~~ABH~~	~~BCI~~	~~CDJ~~	DFG	~~EGI~~	FIJ		
~~ABI~~	BCJ	CEF	~~DFH~~	EGJ			
ABJ	BDE	~~CEG~~	DFI	~~EHI~~			
ACD	BDF	~~CEH~~	DFJ	~~EHJ~~			
~~ACE~~	~~BDG~~	CEI	~~DGH~~	EIJ			
ACF	BDH	CEJ	~~DGI~~				
~~ACG~~	~~BDI~~	~~CFG~~	DGJ				
~~ACH~~	BDJ	~~CFH~~	~~DHI~~				
~~ACI~~	BEF	CFI	DHJ				
ACJ	~~BEG~~	CFJ	~~DIJ~~				
ADE	~~BEH~~	~~CGH~~					
ADF	~~BEI~~	~~CGI~~					
ADG	~~BEJ~~	~~CGJ~~					
~~ADH~~	~~BFG~~	~~CHI~~					
~~ADI~~	BFH	~~CHJ~~					
ADJ	~~BFI~~	~~CIJ~~					
AEF	BFJ						
AEG	~~BGH~~						
~~AEH~~	~~BGI~~						
~~AEI~~	~~BGJ~~						
AEJ	~~BHI~~						
~~AFG~~	~~BHJ~~						
~~AFH~~	~~BIJ~~						
~~AFI~~							
~~AFJ~~							
~~AGH~~							
~~AGI~~							
~~AGJ~~							
~~AHI~~							
~~AHJ~~							
~~AIJ~~							

Triangle Possibilities in the Base

~~KLM~~			
KLN			
~~KLO~~			
~~KLP~~			
~~KMN~~	~~LMN~~		
~~KMO~~	~~LMO~~		
~~KMP~~	LMP		
~~KNO~~	LNO	~~MNO~~	
~~KNP~~	LNP	~~MNP~~	
~~KOP~~	LOP	~~MOP~~	~~NOP~~



The students recognized, however, that just because a trio of letters can be written does not mean that they represent a triangle in the diagram. The list allowed students to examine each trio of letters and determine if it represented a triangle in the diagram. Students began to identify trios, such as *ABD*, that did *not* represent a triangle. They crossed out such trios on the list. **Table 3** shows the list after all possibilities had been checked.

After counting all letter trios not crossed out, the student groups still had three different answers: fifty, fifty-one, and fifty-two. One group read its list of triangles, names beginning with *A*. When they disagreed, the entire class rechecked the diagram to determine if the triangle really existed in the figure. Then another group read its list of triangle names starting with *B*. Once again, points of disagreement were tested until everyone was convinced. They repeated this process until they had checked all the possible triangle names. The class members recalculated their answers and came to agreement that the diagram has fifty-two triangles.

Students were much more confident of this answer than they had been of their first several attempts. For most students, the reasoning strategy of listing all the possibilities and eliminating those that did not work was more valid and reliable than simply counting observable triangles and hoping that they had counted them all. The reasoning method helped one student conclude, "It's simple to check each triangle. We just have to be careful."

Students could now prove, to both themselves and their classmates, that they had identified every unique triangle in the diagram. Instead of randomly guessing or simply counting every triangle that they happened to identify, they had "reasoned out" a method to find the correct answer.

Bringing it all together

As a final activity to connect the Geometry, Communication, and Reasoning Standards, students created a table to classify all named triangles by type (see **table 4**). To complete this table, they sorted the triangles according to geometric classification and named them according to their communication convention. Last, they counted how many equilateral, isosceles, and right triangles were in the diagram.

Table 4

Triangles by Type

"Count the Triangles" Activity—Naming Triangles

Use this table to record triangle names from the "Count the Triangles" activity.

Equilateral	Isosceles	Right	Other

Conclusion

What I learned in school that day is that even simplistic activities have the potential to become multiday explorations through the Standards. An activity that began as a simple exercise in geometric visualization grew to include the more formal topics of triangle classification, naming triangles, and communication conventions.

At the same time that students and teacher were exploring geometry, they were engaged in a parallel exploration of the Reasoning and Proof Standard. Progressing from the lack of confidence in their original answers, class members identified the benefits and drawbacks of a proposed classification system, learned the necessity of one-to-one correspondence in naming triangles, and developed an exhaustive proof of their final answer.

What made this exploration possible was the teacher's consideration of the mathematics needed to solve the problem. The teacher's thinking process led to the identification of concepts and skills that the students could explore. Extending the activity to address these additional topics gave students a journey through multiple Standards. ▲

The Barn Quilt Problem

Problem

This month our problem-solving trek takes us to see barn quilts in Iowa's Sac County. This county alone has at least sixty barn quilts. As we travel from barn to barn, one quilt catches our attention. The squares on the Liske Barn quilt provide a simple yet interesting pattern: One very large square is composed of 25 different-colored small squares, which are all the same size. The squares can also be arranged to create various numbers of total squares.

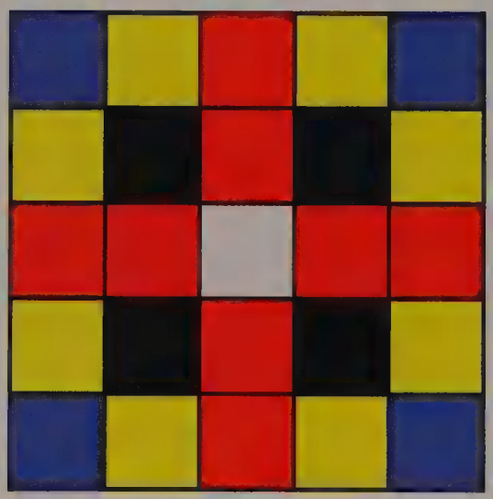


Illustration by Toni Grote www.artisttonigrote.blogspot.com; all rights reserved

The 4 squares in each corner are made up of blue, yellow, and black squares. Between these 4 corner squares are 4 red rectangles that are each made of 2 squares. In the center is a single white square. Combining the squares in this way gives us 13 square pieces of various sizes (1 white, 8 red, and 4 that are made up of yellow, blue, and black).

Look at the 3×3 black square in the center of the quilt, which is actually made up of 4 black squares, 4 red squares, and the white center square. This 3×3 square is surrounded by 16 smaller squares that are red, yellow, or blue, giving us 17 total squares. How many other quilt designs can you see or construct using different arrangements of all 25 smaller squares in this barn quilt? Can you create exactly 10, 16, or 20 total squares?

Extensions

Which other numbers between 1 and 25 could we create by combining these squares in different ways? Which values are impossible to create in this manner? Explain how you know this.

Imagine snuggling under a warm, handcrafted quilt on a chilly winter afternoon. Does anything sound cozier than that? The beautiful colors and patterns of quilts display both art and mathematics. This month, explore the Barn Quilt problem with your students and then share your insights into the mathematics of quilting.

Classroom Setup

Begin by examining the various quilt patterns on the many different barns in Sac County, Iowa (see

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Where's the Math?

This month's problem has connections to many concepts in mathematics and art. Designs using squares play a large part in making quilts. The number of different quilts that can be made from the same squares or parts of squares is amazing. Visualizing these patterns is an important mathematical skill. Some quilt designs have symmetry, either with respect to a point or a line, and that feature makes the design of the quilt attractive and visually appealing. For example, a quilt may exhibit rotational symmetry: Every half turn or quarter turn about a certain point results in the same visual pattern. A quilt could also have line symmetry: A line acts as a mirror between the two sides of the quilt design. Like harmony in music, patterns appeal to the senses; just as harmony in music sounds "right," these visual patterns look "right."

Counting and systematic counting are powerful skills in solving mathematical problems like this month's problem. Asking how many squares are in this quilt pattern forces students to organize their thinking and develop systematic ways of combining the various square sizes to account for all possible quantities of squares. Such systematic counting often produces valuable patterns and important insights.

The perfect square numbers (those numbers that are squares of the natural numbers) play an important role in this problem. Square numbers often appear in other seemingly unrelated problems; recognizing them in many contexts is a valuable skill for students. In algebra, for example, identifying perfect squares becomes essential in solving certain quadratic equations. A working knowledge of square numbers can lead to better understanding of other common patterns and applications.

Quilts, as anyone who makes them can attest, are very mathematical and allow for rich interdisciplinary investigations. Simple shapes like squares lead to an almost limitless variety in quilt patterns. Power in mathematics often comes from the ability to make connections among many representations. The strength of this month's problem lies in its powerful connections between shape, design, art, and numbers.

www.barnquilts.com). During this initial exploration, you may want to discuss the geometric shapes that appear within each of the quilts as well as the symmetry that exists.

This problem lends itself very nicely to the use of flat, square tiles. Students can arrange 25 tiles into the Liske Barn quilt pattern and then manipulate the tiles to form squares of various sizes. They can then separate and recombine the tiles as they explore possible solutions to this problem. Student results may include many individual single, 1×1-unit squares along with some larger pieces such as 2×2, 3×3, or 4×4 squares.

As they create various numbers of squares, students should record their results on dot or graph paper and describe their arrangements. A 5×5 number grid with each number from 1 to 25 written within a square provides an excellent way to document. Students can color the numbered squares according to whether that particular number could or could not be found. For example, if the color red is used to indicate total squares that can be found and blue indicates those that cannot be found, then students can begin by coloring the numbers 1, 13, 17, and 25 in red because it is possible to find each of these total number of squares. As students find other possibilities, they can color code the corresponding numbered squares in red and the remaining impossible values in blue.

As students work on this problem, note the problem-solving strategies that they use. Guess and check, looking for patterns, and starting with a simpler version of the problem are three strategies that they may find helpful. Remember that the problem-solving process for finding the answer is sometimes more important than the answer itself. Give students a later opportunity to review their problem-solving strategies. As they do, keep their focus on the different approaches they used rather than on their answers.

Share Your Student Work

Please try this problem in your classroom. We are interested in how your students responded to the problem, what problem-solving strategies they used, and how they explained or justified their reasoning. Include information about how you posed the problem, samples of student work, and photographs. E-mail your thoughts, reflections, scanned student work, and photographs to Joseph Georgeson at jgeorgeson@usmk12.org; or you may send your class experiences and photographs to Joseph Georgeson, the University School of Milwaukee, 2100 West Fairy Chasm Road, Milwaukee, WI 53217. Send your results by **February 15, 2009**, and include your name, grade level, and school name with your submission. Selected submissions will be published and acknowledged in a subsequent issue of *Teaching Children Mathematics*.

(Solutions to a previous problem begin on the next page.)

Additional Resources

Did you know that NCTM has published a collection of some past "Problem Solvers" columns? Visit **nctm.org/catalog** for more on this and other NCTM resources, including professional development offerings, other publications, and online resources.

Sakshaug, Lynae E., Melfried Olson, and Judith Olson. *Children Are Mathematical Problem Solvers*. Reston, VA: NCTM, 2002.

Solutions to the Rock-Paper-Scissors Problem

The Rock-Paper-Scissors problem is a fun way for students to develop their understanding of *probability*, the mathematical study of chance. Probability is one of the most frequent uses of mathematics in everyday life. The problem appeared in the December 2007/January 2008 "Problem Solvers" section and was stated as follows:

Problem

Vanessa and her sister, Mikaela, both want to eat the last slice of pepperoni pizza. They decide to play Rock-Paper-Scissors to decide who gets the last slice. Is this fair? Prove it. If the game isn't fair, change it to make it a fair game.

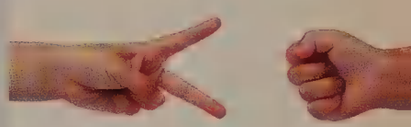
Alex wants to play Rock-Paper-Scissors with his little sister, Susanna. He decides to make it easier for Susanna to understand the game. Instead of the usual rules, he decides that he will get one point if both hands match and Susanna will get one point if both hands are different. Whoever has the most points after 10 games will be the winner. Who do you think will win? Is this game fair? Prove it. If the game isn't fair, change the scoring to make it a fair game.

Variations

For younger children, complete only the first part of the problem. Have students explain their work using pictures, tables, or graphs in addition to words.

Extensions

Challenge students to create another variation of the Rock-Paper-Scissors game. Encourage them to try using a different set of rules or more than three symbols. They must prove the game is still fair. Students can be encouraged to research how the game is played in other cultures to get further ideas.



Feng Yu/iStockphoto.com

Kelsey Kerney-Porter at Doig Intermediate School in Garden Grove, California, introduced this problem to her sixth-grade class. As a lead-in, Kerney-Porter asked students to recall their previous work with probability, which included coin tosses and spinners. She wanted students to have in mind the idea that the probability of an event occurring is related to the possible outcomes. She reminded them that they could use tables and tree diagrams to create a list of every possible outcome.

She then introduced the Rock-Paper-Scissors problem and allowed students to play the game for two minutes both to generate some excitement and also to ensure that everyone knew the game. Then students were given a handout on which the three parts of the problem were printed, with space to show their work. They worked in pairs on each part of the problem over two forty-two-minute class periods.

Investigating the first scenario, the traditional rock-paper-scissors game between Vanessa and Mikaela, Ramona used a visual approach to find each possible outcome and represented each

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Figure 1

Ramona used a visual approach to the first problem and concluded that the game is fair.

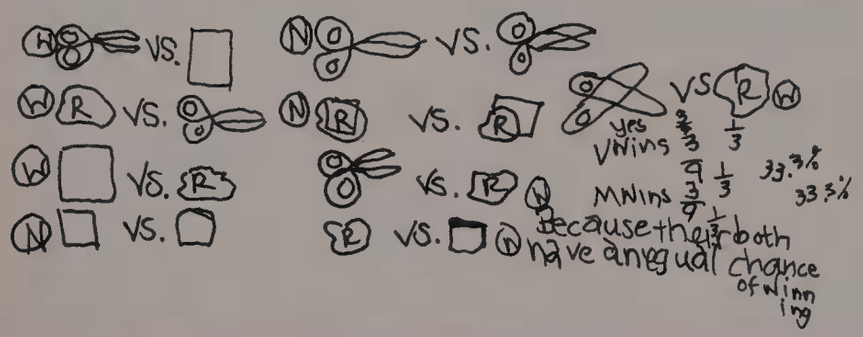


Figure 2

Martin looked at the table he made for the second problem to determine that it is not a fair game.

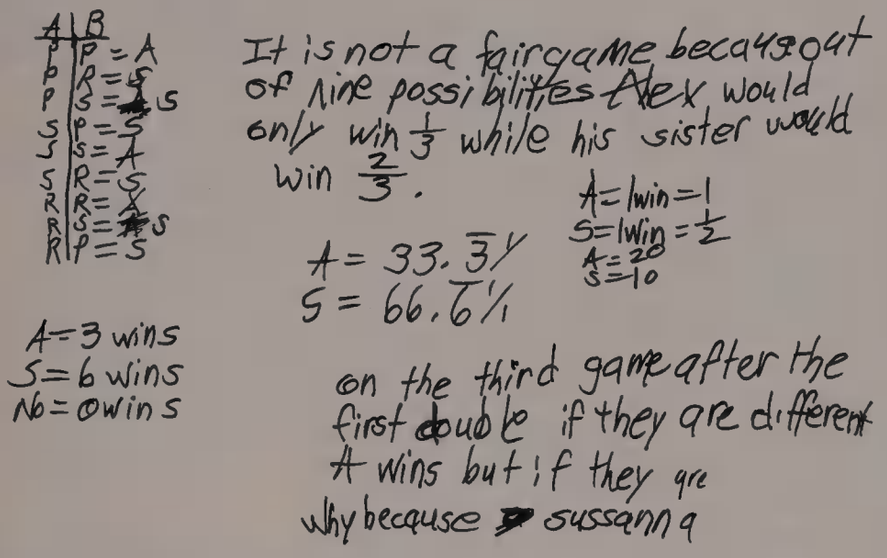
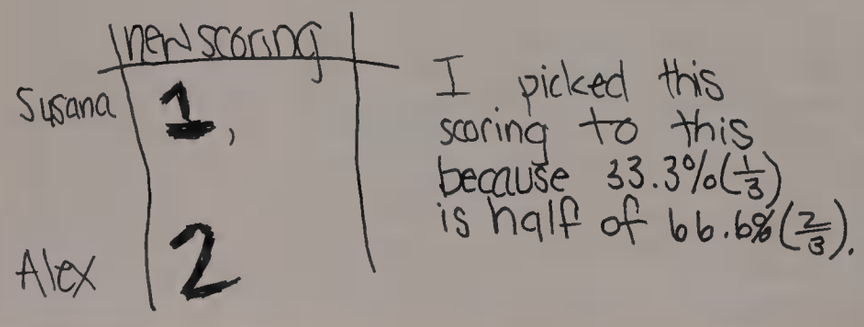

A B
P P = A
P R = S
P S = A S
S P = S
S S = A
S R = S
R R = A
R S = A S
R P = S
It is not a fair game because out of nine possibilities Alex would only win 1/3 while his sister would win 2/3.
A = 33.3%
S = 66.6%
A = 1 win = 1
S = 1 win = 1/2
A = 2
S = 1 0
A = 3 wins
S = 6 wins
No = 0 wins
on the third game after the first double if they are different A wins but if they are
Why because Sussanna

Figure 3

Students realized, as did Veronica with the second game, that solutions for making games fair depend on the combination of possible winning outcomes and points per win.



possible outcome pictorially (see **fig. 1**). Students had some sense that the game was fair but were confused when they found that each player's chances of winning were one-third or 33 1/3 percent. Kerney-Porter realized that students had learned to think that a fair two-player game meant that each person has a one-half, or 50 percent, chance of winning. She asked students to carefully examine their outcome tables. Several students noted that if players show the same object (rock-rock, paper-paper, or scissors-scissors), they tie. Ties accounted for three of the nine possible outcomes. Thus, the sum of all probabilities was 1/3 + 1/3 + 1/3, which gives 1, or 100 percent. After looking at her data, Ramona concluded that the game is fair "because they both have an equal chance of winning" (see **fig. 1**). In this way, students' understanding of "fair" became more mathematically refined.

Students continued to work in pairs on the second problem involving Alex and Susanna, who played rock-paper-scissors with slightly simplified rules. Most students predicted that Susanna would have a better chance of winning but could not immediately give a rationale for this opinion. They were encouraged to play a few rounds and then take time to make a list of possible outcomes. Looking at his table, Martin claimed, "It is not a fair game because, out of nine possibilities, Alex would only win one-third, [and] his sister would win two-thirds" (see **fig. 2**). Discussing how to make the game fair by changing the points forced students to think further about the concept of "fair" in a mathematical sense.

Students realized that they could change the points to accommodate an unequal probability of winning. Two solutions were most common. Some students, such as Veronica, said to give Susanna one point for each win and Alex two points for each win (see **fig. 3**). This way, even though Alex won half as often, he would get the same possible points. Other students, including Martin, decided to give Susanna just one-half point per win because she had twice as many possible winning outcomes. Each of these solutions recognized that fairness was dependent on the combination of possible winning outcomes and points per win.

Students approached the extension (the third problem in the teacher-created handout; see **fig. 4**) with increased enthusiasm because it was not as obvious whether the game involving three players was fair. Kerney-Porter first had students predict the game winner: Adelle, Brian, or Carlos. The majority of her class predicted Carlos would win because

Figure 4

The teacher's handout included a third problem she had created to extend and challenge her students.

Three people, Adelle, Brian, and Carlos, are playing a different version of RPS. They all show their hand at the same time, but the scoring is a little bit different. If all of the hands show the same, Adelle receives 2 points. If all of the hands are different, Brian gets 3 points. And if 2 of the hands are the same, then Carlos receives one point.

- Who do you think will win?
- Is this a fair game? Prove it.
- If the game isn't fair, change the scoring to make it fair.

it seemed more likely that two hands would match than that either all the hands would be the same or all would be different. Quite a few students thought Brian would win, reasoning that all hands being different seemed likely and noting that Brian receives three points each time he wins.

With the predictions recorded, students were eager to work out the problem. Some groups played the game a few times to get a sense for it. Others jumped right in, figuring out how to list all the possible combinations of rock, paper, and scissors that could occur. Juan's paper shows the use of a tree diagram and gives the possible wins for each player (see **fig. 5**). As students had predicted, Carlos has the most possible winning combinations with 18/27. Brian has a 6/27 chance, but students found that because he receives three points per win, he actually has the same chance of winning as Carlos in terms of points. Adelle has just three ways to win; with two points per win, her chances of being the point leader are slim. Juan explains how to make the game fair by multiplying the possible wins for each player by the points they receive per win. He found that Adele needs to get six points per win in order for it to be a fair game. In **figure 6**, Ramona explains her solution to this problem: Make equivalent fractions for each player after multiplying possible wins by points per win so that each fraction is two-thirds. Although her conversion to percentages is somewhat misguided, her understanding of fairness is strong.

Kerney-Porter's students were able to stretch their understanding of probability and fairness through their work on the Rock-Paper-Scissors problem. They moved from seeing fairness in simple 50/50 terms to understanding how to mathematically adjust the scoring of a game with unequal outcomes so that each player's chances of winning are equal in terms of points. ▲

Figure 5

Juan's tree diagram and solution for the three-player problem shows the possible wins for each player.

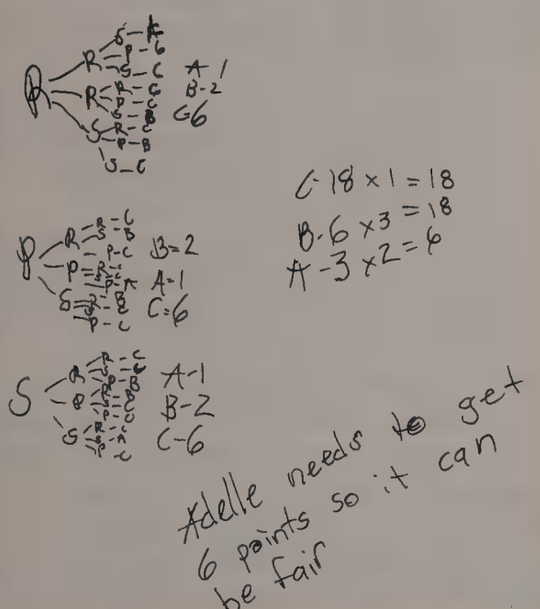
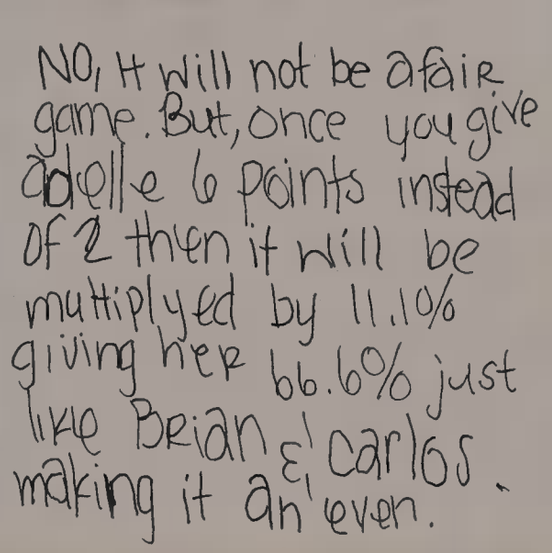


Figure 6

Ramona's conversions have errors, but her solution to make the three-player problem fair is to make equivalent fractions of two-thirds for each player after multiplying possible wins by points per win.


NO, It will not be a fair game. But, once you give adelle 6 points instead of 2 then it will be multiplyed by 11.1% giving her 66.6% just like Brian & carlos. making it an even.

Give the Gift of Books

Books

For Students

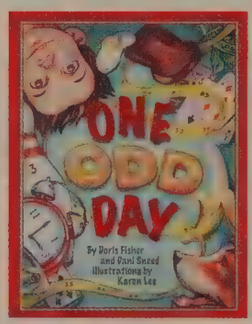
Math Matters: Mac & Cheese, Pleeeeze! *Eleanor May, 2008. 32 pp., $5.95 paper. Ages 6–8. ISBN 978-1-57565-260-3. The Kane Press; (212) 268-1435; www.kanepress.com.*

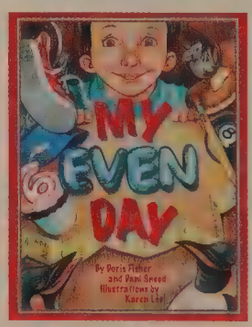
The setting for this children's literature book is an elementary school "lunch food election." Students must campaign for a food they would like to eat at Spring Fest. They create posters and go about school trying to convince others to vote for their choice.

This book is acceptable for the stated age range. The mathematics, reading, and social content is appropriate. Throughout the book, thought bubbles help readers keep track of the number of votes for each food. Strategies are noted at the bottom of some pages to help students develop shortcuts and organize the mathematics for the more challenging addition number sentences. Characters in the book connect with friends across grade levels. The curricular connections from this book could even extend to a discussion about the democratic process and could set up debate opportunities in which students defend a position they would not have chosen on their own.

The book has a couple of obvious weaknesses: Students are subject to peer pressure and unfortunately buckle to that pressure to vote according to the wishes of others in their group. At times, young students have difficulty keeping track of the many characters and the food choices they campaign for.—*Carol Buckley, Gettysburg Area School District, PA 17325.*

One Odd Day, *2006. ISBN 978-0-9768823-3-6. Sylvan Dell Publishing; (877) 958-2600; www.SylvanDellPublishing.com.* **My Even Day,** *Doris Fisher and Dani Sneed, 2007. 32 pp., $15.95 ea. cloth. Ages 4–8. ISBN 978-0-9777423-3-2.*





The authors of these engaging picture books integrate the concepts of odd and even numbers into the story of the life of a young boy who awakens to find that his world doesn't seem quite right. As the boy tumbles from his bed in *One Odd Day*, he finds that his surroundings appear to be void of all even numbers. He discovers that his clock displays only odd numbers and his dog seems to have a fifth leg. The book's pages are filled with similar examples until the next day, when the boy wakes up to find that his odd-number day appears to be over, but his mother now has two heads. It is no surprise that *My Even Day* begins where *One Odd Day* ended. The story is similar to the first except that this day revolves around examples of even numbers and then concludes with the boy waking up to find that half his head has been shaved, leading the reader to believe that the next book will focus on fractions.

These books appeal to primary students and could support a classroom investigation focused on the concept of odd and even numbers within an oddly interesting context. The rhythmic text is enticing, and the vividly creative illustrations could keep students busy for hours as they search for different objects related to odd and even numbers. Each book also contains a teaching section with

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a variety of activities, definitions, and important teaching points.—*Heidi J. Higgins, Missouri State University, Springfield, MO 65897.*

Read-It! Readers Series, *Marcie Aboff, 2008. 32 pp. ea., $19.93 ea. cloth. Grades K–2.* **The Guessing Game,** *ISBN 978-1-4048-4209-0;* **The Pool Party,** *ISBN 978-1-4048-4211-3;* **Shells Alive,** *ISBN 978-1-4048-4210-6;* **Too Many Tomatoes,** *ISBN 978-1-4046-4208-3. Picture Window Books; (888) 517-8977; www.picturewindowbooks.com.*

These picture books are part of a series of books targeting students in grades K–2. The six levels are designed to encourage the development of problem-solving skills to help children expand their thinking. The top of the back page of each book lists the concept being taught and the book's complexity level, making it easy for teachers to find an appropriate skill and skill level. These books are also very child friendly.

The Guessing Game addresses customary weight measurement. The main action takes place in a classroom, where students are guessing different objects' weights. One student changes this scene by bringing some objects from home and putting them into paper bags. Students guess what is in each bag by knowing the weight of each object and some clues.

The Pool Party is about a child's desire to have a birthday party. Pictures show temperature changes and let readers make predictions about whether the main character will have his birthday pool party. When the boy gets a fever, a body thermometer is introduced as another type of thermometer.

Focusing on two-digit addition with regrouping, *Shells Alive* tells of two children visiting their grandmother at her beach house. In *Too Many Tomatoes*, Rachel helps her grandma's neighbor become friends with others in their neighborhood and give away the bumper crop of tomatoes the two of them grew, illustrating two-digit subtraction with regrouping.

One problem is with the concept of regrouping. The two books that use addition and subtraction with regrouping do not describe how to do these operations. A simple number sentence is written, such as $47 - 19 = 28$. Students who are having trouble understanding how to rename may not understand how the difference or the sum is obtained. The books show good examples of when renaming is needed as well as writing the number sentence in word form above the numeric form. They show when to use addition and subtraction but not how to add and subtract.

All the books feature an activity at the end to further explore the mathematics concept featured. The books include glossaries of terms as well as suggestions for other books to read. Also featured is a Web site that includes a list of sites to visit that teach the concept from the book. The publisher's staff members have researched these links.

I recommend these books for use in K–2 classrooms. Third graders who need more skill development will also benefit from these books.—*Sarah Pullie, Latta Elementary School, Latta, SC 29565.*

Shape Capers, *Cathryn Falwell, 2007. 32 pp., $16.99 cloth. ISBN 978-0-06-123700-3. HarperCollins Children's Publishers; (800) 242-7737; www.harpercollinschildrens.com.*

Shape Capers is a delightful and colorful book that encourages young readers to explore shapes. The book teaches four basic shapes: circle, square, triangle, and rectangle as well as the semicircle. The theme focuses on several children looking for shapes in their everyday world and then using the shapes to create new and interesting constructions. Simple, clear illustrations of the shapes vary in size so that little ones can conceptualize the similarities and still see the same shapes. Basic vocabulary allows this to become a favorite read-along book. Although the book was written for very young children and meant as a quick read, it would have benefitted from the addition of a few more shapes commonly used with school-aged children, such as a hexagon or rhombus. I read this to my second graders, who enjoyed the imaginative qualities of the book. They commented on the colorful illustrations and the "cool" creations that were made with the shapes. I would recommend this book for the preschool or kindergarten classroom as an introduction to basic geometric concepts and as a lead-in to a lesson where students could design their own shape capers.—*Karen Lock, Tozer Primary School, Windsor, CO 80550.*

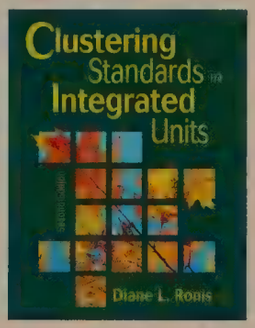
Sir Isaac Newton: Brilliant Mathematician and Scientist, *Natalie M. Rosinsky, 2008. 112 pp., $23.95 cloth. Grades 5–7. ISBN 978-0-7565-2209-4. Compass Point Books; (877) 371-1536; www.compassbooks.com.*

Rosinsky's biography portrays Newton as a talented scientific genius with major insecurities who faced obstacles from many sources. Upper elementary and middle school students have at their fingertips an understandable, interesting summary of Newton's many accomplishments and the added bonus of little-known facts and scientific explanations.

The author's straightforward, entertaining writing style is one of the book's strengths. As events in Newton's life unfold, readers can empathize with the youngster driven from his family at a young age, the young man attempting to find himself personally and professionally, and the scientist searching for the recognition due him. Newton reflected, "I seem to have been a boy playing on the sea shore … finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me" (p. 92). I highly recommend this Signature Lives book.—*Germaine L. Taggart, Fort Hays State University, Hays, KS 67601.*

For Teachers

Clustering Standards in Integrated Units, *Diane L. Ronis, 2008. 168 pp., $30.95 paper. ISBN 978-1-4129-5557-7. Corwin Press; (800) 233-9936; www.CorwinPress.com.*



The purpose of this book is to aid teachers and administrators in planning interdisciplinary units. The author believes that not only students make meaning through integrated units but also that teachers develop a better understanding of content. Each chapter is organized around building interdisciplinary units, including tools needed, planning, assessment, and implementation. On the basis of the author's belief that educators face multiple challenges in today's schools—including "... an increasing percentage of students being reared in families with less stable and responsible parenting" (p. viii)—the book's underlying theme is that all children can learn; by teaching in integrated units, we increase student success rates. Students do not learn concepts in isolation.

The book's intended audience is teachers and administrators, including assistant superintendents of elementary and secondary education. Any person responsible for staff and curriculum development may find this book useful. Each chapter explains the rationale for planning and devising units with examples for different grade levels. Sample rubrics as well as units are included in the book. The author's use of specific examples is one strength of the book. The appendices include national standards for various subject areas and blank planning templates. The appendices make the book easy to follow, especially if this work must be completed immediately.

I would not recommend this text for beginning teachers, who can often be overwhelmed learning curriculum. However, experienced teachers can benefit as they delve more deeply into the content that they are responsible for teaching.—*Natalie Armstrong, Math Specialist, West Haverstraw Elementary School, North Rockland Central School District, West Haverstraw, NY 10993.*

The Differentiated Math Classroom, *Miki Murray and Jenny Jorgensen, 2007. 216 pp., $23 paper. ISBN 978-0-325-00996-4. Heinemann; (800) 225-5800; www.heinemann.com.*

This book presents the case for differentiated instruction; summarizes research on teaching and learning mathematics; describes issues facing today's mathematics teachers; and offers many tools to support implementation, including detailed lessons and a complete unit on place value. Most important, it outlines a program to guide K–8 teachers in creating their own differentiated classrooms.

Exemplifying differentiated instruction, the place-value unit includes descriptions of the planning process and the unit's components, beginning with important place-value concepts identified by NCTM. The plan describes open-ended activities with challenges that build up to the big content ideas. These activities include student comparisons of roman numerals and base-ten number systems, a problem that leads children to invent a base-three number system, and an extensive project to create a place-value number system for a different planet. Explanations of how the unit was differentiated for readiness and accessibility are included. Photographs of student work make the unit come alive.

One of the book's major strengths is its inclusion of charts and guidelines, such as learning style charts, criteria for good problems, scoring rubrics, and steps for tiering lessons—all easily referenced in the index.

This book provides so much information that teachers without the support of a learning community may possibly find it daunting. However, it is an excellent resource for professional development. I plan to use it as a guide for a differentiated instruction workshop in my school.—*Sharon White, Little Red School House and Elisabeth Irwin High School, New York City, NY 10014.*

The Dyscalculia Tool Kit, *Ronit Bird, 2007. 160 pp. + CD ROM, $27.95 paper. ISBN 978-1412947657. Ages 7–14. Sage Publications; (805) 499-072; www.sagepub.com.*

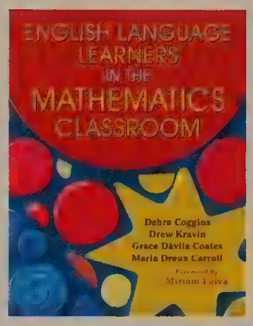
The author draws from extensive experience in working with dyslexic, dyspraxic, and dyscalculic students for this collection of teaching activities and games to address students with mathematical difficulties. The activities are appropriate for students struggling with understanding mathematics in general and numeracy in particular. The intended audience is primary-grades teachers who do not have a specialized background in mathematics or special education. An extended audience could include parents, teaching assistants, and teachers of older students.

Overall, the book is clearly written, well organized, and easy to read and follow. The text is accompanied by an easy-to-use CD that includes the entire text of the book so that teachers can tailor the activities to particular needs. Game directions are clear and understandable. Some terminology may be unfamiliar because of British colloquialisms.

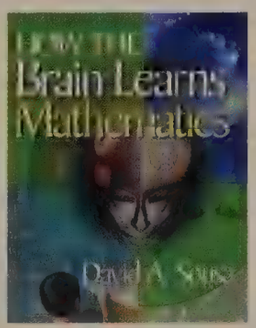
I found the materials useful with our Response to Instruction (RTI) team and recommend the materials to anyone seeking support for this instructional area. Indicators of dyscalculia, brief descriptions of difficulties, and "how to help" sections guided our teachers in their observations of students struggling with mathematics. We selected and implemented activities from each section. The complementary addition models using number lines were particularly effective with students in our program. We found the activities compatible with our mathematics curriculum.—*Maria Dufek, Jay Jeffers Elementary School, Las Vegas, NV 89115.*

English Language Learners in the Mathematics Classroom, *Debra Coggins, Drew Kravin, Grace Dávila Coates, and Maria Dreux Carroll, 2007. 144 pp., $28.95 paper. ISBN 978-1412937590. Corwin Press; (800) 818-7243; www.CorwinPress.com.*

This is a book written primarily for elementary school teachers with English Language Learners (ELLs) in their classrooms. The underlying teaching practices, however, would be beneficial in all classrooms, junior high and high school included, with all types of learners. The purpose of the book, which it readily delivers, is to give teachers specific research-based tools and strategies to use with their students so that all are successful in mathematics. This book is beneficial for the classroom teacher because it introduces the strategy and then shows a practical application of it through a classroom lesson. Seeing how to apply the strategy to a lesson makes this an immediately useful tool for the classroom teacher. For example, one chapter discusses how to develop mathematical language. It takes the reader through a lesson where students must classify shapes based on attributes of their choosing. After the students have used their own terms to describe attributes (corners, sides), the teacher then introduces the formal terms (vertices, edges). Using this method in my own classroom proved much more effective, and the students were much more engaged in the activity than in previous years when I simply told them what the terms meant. I would definitely recommend this book to any mathematics teacher. The strategies have already proved useful in my own classroom, and I will continue to implement them and watch my students' knowledge grow!—*Jennifer Tysseland, John Quincy Adams Elementary School, Dallas, TX 75217.*



How the Brain Learns Mathematics, *David A. Sousa, 2008. 264 pp., $38.95 paper. ISBN 978-1-4129-5306-1. Corwin Press; (800) 233-9936; www.CorwinPress.com.*



"Human beings are born with some remarkable capabilities. One is language. Another innate talent is number sense." If humans are born with number sense, why does mathematics continue to be such a challenge?

How the Brain Learns Mathematics attempts to meld cutting-edge research with practice. These translations from research to instructional practice are clearly indicated throughout each chapter. The book will allow its readers to answer questions such as the following: How much arithmetic can young children learn without direct instruction? How does our language affect our ability to learn counting? How does the brain deal with abstract mathematics concepts?

Chapter organization is logical and includes "Developing Number Sense," "Learning to Calculate," "Reviewing the Elements of Learning," "Teaching Mathematics to the Preschool and Kindergarten Brain," "Teaching Mathematics to the Preadolescent Brain," "Teaching Mathematics to the Adolescent Brain," "Recognizing and Addressing Mathematics Difficulties," and "Putting It All Together: Planning Lessons in Prekindergarten–12 Mathematics." Each chapter highlights recent discoveries in neuroscience and includes charts and figures inserted throughout the readings to reinforce the topic at hand.

The only weakness is an outdated recommendation (p. 192) for helping struggling students with word problems. Carpenter, Fennema, and Franke's work (1994), as well as NCTM's "Summary of Changes in Content and Emphasis in K–4 Mathematics," suggest the use of "key" words to determine which operation receives decreased attention.

Regardless of the grade level ones teaches, the book allows educators—and parents—to determine instructional approaches and materials that are compatible with what cognitive neuroscience has shown researchers about how the brain learns mathematics.—*Jane Strawhecker, University of Nebraska at Kearney, NE 68849.*

Math Dictionary: The Easy, Simple, Fun Guide to Help Math Phobics Become Math Lovers, *Eula Ewing Monroe, 2006. 216 pp., $19.95 cloth. ISBN 1-59078-413-8. Boyds Mills Press; (877) 512-8366; www.boydsmillspress.com.*

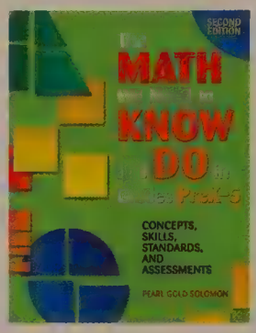
What are googols, integers, and number patterns? You can find these words among over five hundred others in this mathematical dictionary. The introduction states, "Students, parents, and teachers will find [the book] helpful, as will anyone who is looking for clear and simple definitions of basic terms needed in the study of mathematics."

Each definition has examples to help clarify the terms. Along with the definitions, added features include a "Did You Know?" section where interesting information is given about a word on the page.

One weakness of this dictionary is that each word is printed in red to separate it from the definitions, making terms harder to find. Printing the terms in bold print and the examples in red would have been a better format.

Although my students understood most of the definitions, several were unclear. For instance, the explanation for *equation* was confusing. The dictionary states that an equation "is a mathematical sentence that gives two names for the same number." The children were not sure what two names meant. This book would not be satisfactory for those who want more technical definitions, but it is useful for students who want a basic dictionary.—*Susan Snyder, Christ Our King–Stella Maris School, Mt. Pleasant, SC 29492.*

The Math We Need to Know and DO in Grades PreK–5, *Pearl Gold Solomon, 2006. 232 pp., $35.95 paper. ISBN 1-4129-1720-4. Corwin Press; (800) 818-7243; www.CorwinPress.com.*



This book is designed as a resource for teachers to use as they plan for and assess students' understanding of mathematics concepts in prekindergarten through grade 5. In chapter one, the author describes processes and dispositions needed for learning mathematics. Conceptual and procedural knowledge of content is organized into six major branches of mathematics. The author states how students use these processes to "know" math content. Chapter two comprises a table with 206 very specific embedded concepts, each explained and connected to a brief performance indicator for assessment. Suggested grade-level timing is provided for each concept. The majority of pages are devoted to chapter three, which has an example for scaffolding students' understanding for each one of the 206 concepts. These scaffolds vary in length from several lines to a bit more than one page. Illustrations and descriptions inform the teacher about background information for the concept. Teachers could find this resource useful if they seek an extremely detailed list of mathematics concepts and a timeline for when students might be expected to learn these concepts. Many teachers may find, as I did, that the concepts are in close alignment with their own state mathematics standards. The examples given for scaffolding are often routine and logical. A few illustrations in this section seem a bit skewed in terms of appearance, but this is unlikely to be much of a problem because the information is for the teacher. A short resource section provides limited information about using rubrics and written assessments.—*Wayne Gable, Austin ISD, Austin, TX 78748.*

Et Cetera

Amusement Park Math: A Ten-Minute Daily Math Adventure, *Michael Cain, 2007. 195 pp., $19.95. ISBN 978-1-59363-291-5. Grades 4–6, Prufrock Press; (800) 998-2208; www.prufrock.com.*

This teacher-friendly resource and skills-practice text provides 120 problem-solving challenges in a fun, corny story sequence about a bankrupt amusement park. For additional algorithmic practice, each page provides four expressions to solve. Rather than a typical scope and sequence, the problem solving follows the story line, targeting the upper end of the recommended audience. For instance, an early challenge asks students to calculate a debt by reading a chart that includes thousands and tens of thousands of dollars. A later but easier challenge asks students to determine how many sodas are left in a machine after a particular number are purchased from it.

Some challenges allow astute students to explore real-life economics: One town assesses penalties against people who owe back taxes. Other challenges are simply fun: As Zambul the Strongman discusses his career with a friendly farmer, students are given the arithmetic criteria to sort his twenty-two years' experience into categories of circus performer, sideshow ham, and wrestler. Teachers who want to refine the work and provide a closer companion to their curricula can easily modify the word problem challenges to match students' abilities without detracting from the story line.

The numbers in many of the problem-solving situations are well planned. Many selections use compatible numbers for the givens in the problem so that students can focus on their solution strategy and avoid being frustrated by difficult number-fact combinations needed for the solution. Skills practice in the four-problem set as well as in the ongoing story problems includes basic operations, fractions, and decimals. As is unfortunately common in textbook story problems, many of the situations presented are inauthentic and exist only in the text to provide the setup for a particular skill practice. Nevertheless, following the travel and travail of General Thumb's Theater of Thrills through sometimes unrealistic scenarios is certainly more

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enticing than one or two unrelated problems in a standard mathematics textbook.

Although recommended for upper elementary readers, the situation problems range from a level of 2.4 to 9.6 on a common reading scale. This could present a problem in a target grade level with diverse language arts ability. However, the fun of following the adventures of General Thumb's troupe may well engage even reluctant mathematicians and struggling readers. The answer key contains very few errors. I would recommend *Amusement Park Math* to teachers in the target grades of 4–6.—*Kare Marshall, Heath Elementary School, Mohawk Trail Regional School District, Shelburne Falls, MA 01370.*

Big Ideas for Growing Mathematicians, *Ann Kajander, 2007. 160 pp., $19.95 paper. ISBN 978-1-56976-212-7. Ages 10–14. Zephyr Press; (800) 888-4741; www.zephyrpress.com.*

Are you trying to get your students involved in learning math? Then you need to get this book! It includes twenty activities that "focus more on the understanding of important ideas, rather than procedural fluency."

I like how this book is organized. At the beginning of the book are several charts that show the mathematical topics, the process skills, and the prerequisite knowledge of all the lessons in the book. Each lesson lists these same items for that particular lesson and also the lesson's age appropriateness, overall "big idea," and definitions of terms used in the lesson. Each lesson also lists the lesson objectives, materials needed, preparations, procedures, suggestions, and assessment.

Although this is a good book for classroom use, I probably would use some of the lessons more for extending a skill or concept rather than for a "regular" lesson. Some of the lessons did not seem worth the effort for what students would learn from them. These lessons might be good to put in a math corner or for individual or small-group work. If you are looking for an activity-based lesson, then this book will be a big help and useful resource. —*Susan Snyder, Christ Our King–Stella Maris School, Mount Pleasant, SC 29464.* ▲

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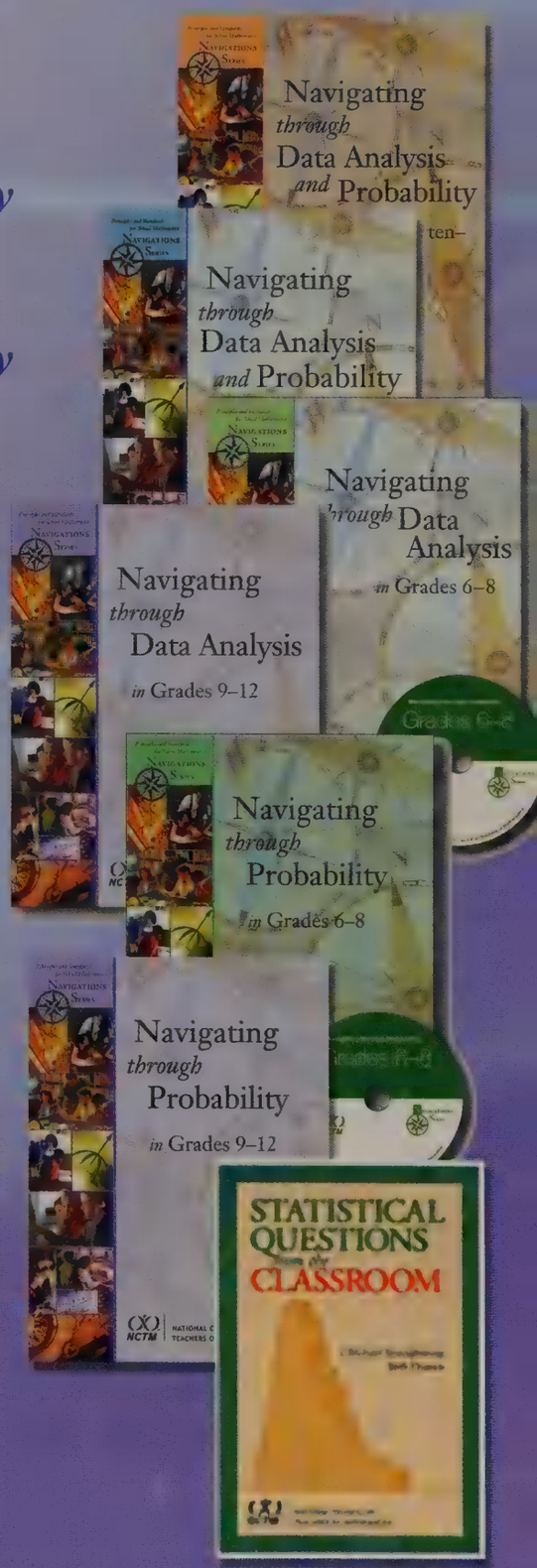
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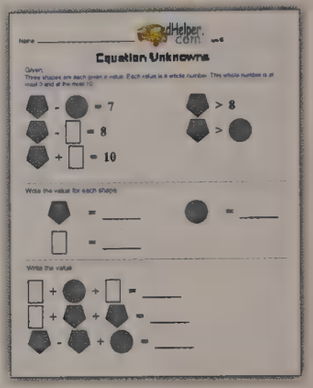

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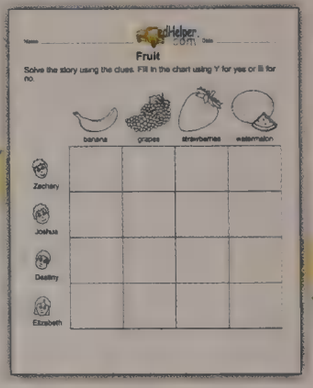

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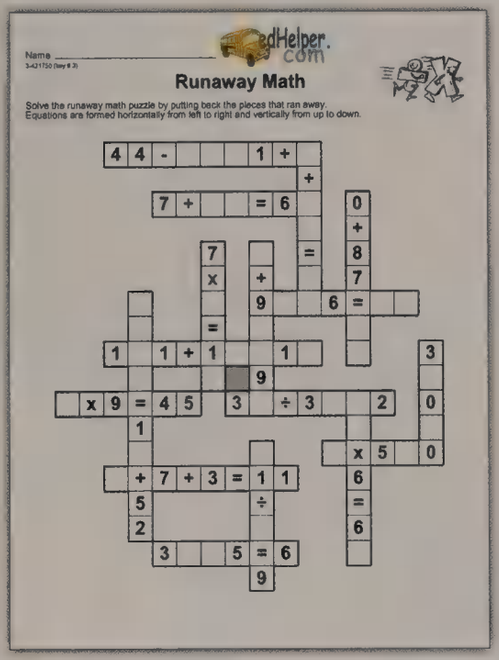
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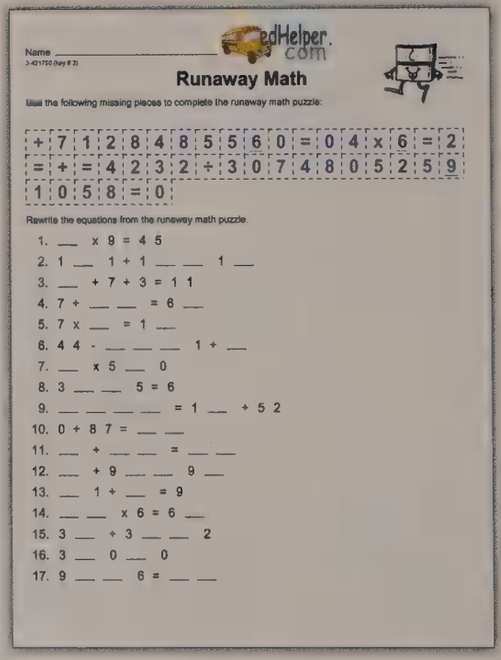

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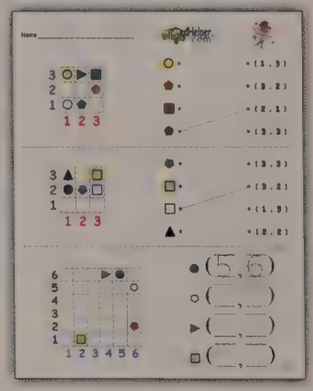

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TEACHING CHILDREN Mathematics

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Ten Is the Magic Number!

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TEACHING CHILDREN
Mathematics

Volume 15, Number 6,
February 2009

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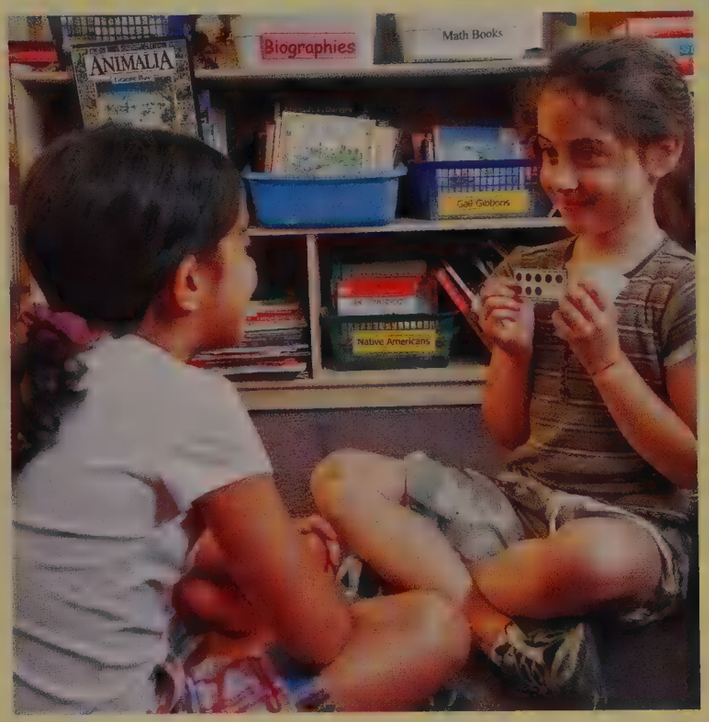

Author Lindsay Barker's second graders discovered that "Ten Is the Magic Number!" after NCTM's *Curriculum Focal Points* dramatically changed how she develops her students' composition of ten. Compared to Barker's previous classes, these students acquired greater number sense, a more positive view of mathematics, and more efficient computational skills.

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Information is available from the Headquarters Office regarding the three **other official journals**, the *Mathematics Teacher*, *Mathematics Teaching in the Middle School*, and the *Journal for Research in Mathematics Education*. Dues support the development, coordination, and delivery of the Council's services. Dues for individual membership are $78 (U.S.), which includes $33 for each *Teaching Children Mathematics* subscription, $16 for a nine-issue *NCTM News Bulletin* subscription, and *ON-Math*. Each additional school journal (*Mathematics Teacher* and *Mathematics Teaching in the Middle School*) subscription is $33. Each additional subscription to the *Journal for Research in Mathematics Education* is $60. **Foreign subscribers** add $18 (U.S.) postage for the first journal and $4 (U.S.) postage for each additional journal. Special rates for students, institutions, bulk subscribers, and emeritus members are available from the Headquarters Office.

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CALL FOR MANUSCRIPTS FOR THE OCTOBER 2010 FOCUS ISSUE

Teaching Mathematics in a Flat World

Author Thomas Friedman's "flat world" phrase points to our new century's globally connected nature, exponential technical advances, and instantaneous communication across the planet. Former U.S. Secretary of Education Richard Riley keenly observed, "We are preparing students for jobs that do not exist, using technologies that have not been invented, in order to solve problems we cannot even imagine at this time."

Teaching Tomorrow's Citizens Today

As we hurtle through this rapidly advancing flat world, what does it mean to "know," "do," and "teach" math? Attempting to prepare students for a dynamically changing environment, what mathematical knowledge and essential understanding do elementary teachers need today to help them better educate the workforce and leaders of tomorrow?

The *Teaching Children Mathematics* Editorial Panel is seeking articles to address themes related to the mathematical knowledge of these teachers. We strongly encourage submissions that reflect the value of teacher knowledge directly within the classroom and its influence on student learning and achievement. Moreover, we welcome articles related to preparing preservice teachers as well as practicing teachers. The following topics are to guide—not limit—authors in addressing one or more such issues.

Teachers' Essential Mathematical Knowledge

- Identify the skills, concepts, and dispositions elementary teachers must know to promote both the procedural fluency and the conceptual understanding of their students.
- Provide evidence regarding content areas or specific topics that enable teachers to become more confident and competent in teaching mathematics.
- Describe the *essential* mathematical understandings necessary for elementary teachers and how these may differ from typical mathematics requirements.
- Explain how these essential concepts allow teachers to empower *all* their students with the mathematical skills and innovative problem solving needed in today's flat world.
- Demonstrate the connection between teachers' essential knowledge and the elementary curriculum in state or national standards or within NCTM's *Curriculum Focal Points.*

How to Best Learn

- Provide examples of innovative courses or professional development programs that have been successful in increasing teacher content knowledge and understanding.
- Describe effective, collaborative efforts—among school districts, institutes of higher education, and other agencies—for preparing elementary teachers to teach mathematics.
- Explain obstacles to increasing teachers' mathematical knowledge and how they have been effectively addressed.
- Clarify the importance of "mathematical habits of mind" and pedagogy in relation to learning and teaching mathematics in a flat world.
- Describe technology tools and other resources that support teachers in their work and in their acquisition of essential knowledge.

Impacting the Classroom

- Explicate the effects of an increase in a teacher's mathematical knowledge on the teacher's curriculum selection, teaching practices, and assessment choices.
- Discuss the connections between teachers' mathematical knowledge and student learning and achievement in mathematics.
- Describe specific classroom examples or experiences that demonstrate the importance of teacher knowledge in learning a particular mathematical concept or idea.
- Explore beliefs that teachers hold with regard to the value and type of mathematical knowledge needed in their classrooms.
- Explore the roles of math coaches and elementary math specialists in advancing the understanding of mathematics for both teachers and students.
- Examine the role of teacher collaboration in aiding teachers to reflect on and improve their mathematical knowledge and classroom practice.
- Discuss the level of mathematical understanding required for teachers to successfully address the needs of diverse learners and differentiated instruction.

Manuscripts should not exceed 2500 words; include figures and photographs at the end. Submit completed manuscripts to *Teaching Children Mathematics* by accessing **tcm.msubmit.net** by July 31, 2009. On the cover page, please state clearly that the manuscript is being submitted for the October 2010 *TCM* Focus issue. Author identification should appear only on the cover page. For manuscript preparation guidelines, visit my.nctm.org/eresources/submission_tcm.asp. ▲

William O. Lacefield III

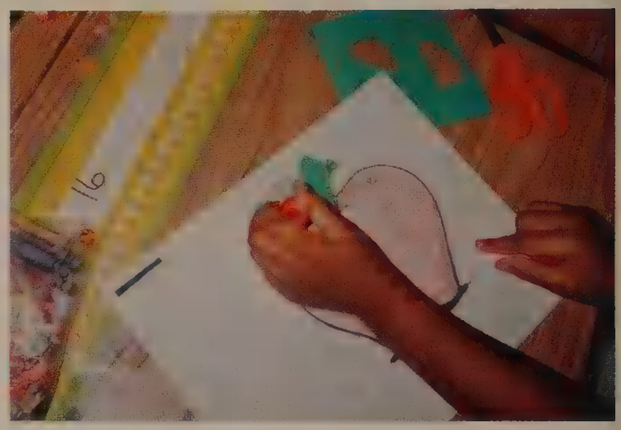
The Power of Representation:

Graphs and Glyphs in Data Analysis Lessons for Young Learners

Skillful teachers capitalize on youngsters' proclivity for exploration by implementing lessons that allow for collecting, organizing, analyzing, and discussing data—including constructing graphs and glyphs (picture graphs) with concrete materials. Such opportunities can strengthen children's abilities to develop inferences, make predictions, and recognize patterns. In the past, data analysis was often neglected in early childhood mathematics curriculum. However, educators now recognize that children's natural inquisitiveness about their experiences and about the world in which they live can stimulate them to raise a variety of questions that lead to data analysis: What kind? How much? Which of these? How many?

Inventive, resourceful early childhood teachers appreciate such questions as valuable starting points for exposing young learners to the importance of gathering and investigating data. Young children take pleasure in creating and responding to questions about their families, classmates, pets, and experiences:

- How many people are in your immediate family?
- What types of pets do your fellow students have?
- How do your classmates travel to school—on foot, in a car, or on a school bus?



Photograph by William O. Lacefield III;

A student colors the apple red on her glyph because she lives in a house and the stem green because she rides to school in a car; the five leaves represent the five people living in her household.

- What are most children's favorite kinds of fruit?
- What after-school pursuits do students enjoy—arts and crafts, sports, watching television, playing games, or other activities?

Children's verbal and written responses to questions are forms of representation, an NCTM Process Standard. Other rich forms of representation include graphs and glyphs.

Graphs

A graph summarizes data in a concise and pictorial form. Many graphs used in early childhood settings are bar graphs of one form or another. Beginning in prekindergarten and continuing through the elementary grades, teachers can find creative ways to integrate graphing into many of the lessons that they teach, always being mindful of children's development and the age-appropriateness of activities. Young learners' initial graphing experiences should involve constructing graphs with concrete materials, that is, the actual objects being examined, not pictures

William O. Lacefield III, lacefield_wo@mercer.edu, is an associate professor of mathematics education at Tift College of Mercer University in Atlanta, Georgia. The former elementary school teacher's research interests include mathematics for young learners and comparative mathematics education.

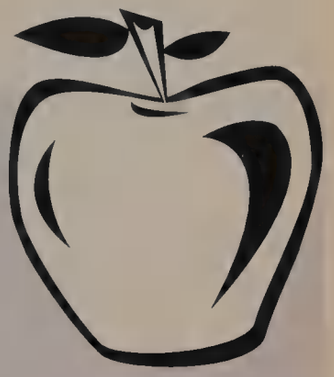
Edited by **Andrew M. Tyminski**, *atyminski@purdue.edu, an assistant professor of mathematics education at Purdue University in West Layfayette, Indiana; and* **Signe E. Kastberg**, *skastber@iupui.edu, an assistant professor of mathematics education at Indiana University Purdue University in Indianapolis.* *"Early Childhood Corner" addresses the need of early childhood teachers to support young children's emerging mathematics understandings and skills in a context that conforms with current knowledge about the way that children in prekindergarten and kindergarten learn mathematics. Readers are encouraged to send submissions to this department by accessing* **tcm.msubmit.net**. *Manuscripts should be double-spaced and must not exceed 2000 words.*

or abstract representations of the objects. In early experiences with graphing, students might be asked to compare only two objects or events. For example, ask every child to remove one shoe. Then facilitate the construction of a graph on the classroom floor. (Shower curtain liners make excellent backgrounds for large graphs and help to focus students' attention on the data being represented.) Shoes might be categorized according to type (laces versus no laces), according to color (white versus not white), or according to another chosen dichotomous characteristic (open-toed versus closed-toed, for instance).

To explore children's preferences among types of milk offered in the cafeteria, a teacher could ask class members to bring their empty milk cartons to the classroom following lunch. A floor graph could then be constructed to compare the number of students who selected plain milk with the number of those who selected chocolate milk. The milk cartons become concrete representations of data to which the students can easily relate. A similar graph could be developed on the basis of students' favorite fruits. The items placed on the floor graph would be real pieces of fruit rather than pictures of fruit.

As students become comfortable with creating and observing graphs that compare two events or objects, teachers can add more categories. With the shoe graphing activity, for instance, students might categorize shoes into three groups: tennis shoes, dress shoes, and sandals. Additional concrete graphs could be designed using students themselves as concrete representations. The teacher could assist in the formation of "human bar graphs" on the basis of children's gender, eye color, hair color, type of clothing, or other characteristics of the teacher's or students' choosing.

Regardless of the items that are used in a concrete graph, the teacher and students must be careful to align objects so that they show one-to-one correspondence. When using a shower curtain liner as a background, for instance, the teacher might want to use colored masking tape to create a grid. Place each object in a rectangle on the grid, making it easier for children to identify the number of objects in each category. If children do not have designated places in which to set objects (shoes, milk cartons, or pieces of fruit), they may position objects very close together in one row and farther apart in another row. In such cases, students who have not developed the ability to conserve number quantities under varied configurations might indicate that a row with large spaces between objects contains more objects than a row in which objects are placed tightly together, although the opposite might be true.



Apple Glyphs

Young children can create glyphs to display data about themselves, their families, their hobbies, their pets, their favorite foods, and other aspects of their lives. The Apple Glyph represents one of many examples. Creative teachers will think of many more.

Provide students with cutouts or blackline images of apples. Students will use the apples to display information about themselves. Ask them to color the apple in the following manner to indicate where they live:

- Red if they live in a house
- Green if they live in an apartment
- Yellow if they live in a mobile home

Ask students to color the apple stem to indicate how they arrive at school:

- Black if they walk to school
- Brown if they ride the school bus
- Green if they ride in a car

Ask students to cut out leaves (perhaps from green construction paper) to glue to the apple stem. The number of leaves that they attach to the stem will indicate how many people live in the household.

After students have created their apples, the teacher displays them on a bulletin board as part of an apple tree. The teacher and students then use the apple glyphs to answer thought-provoking questions. Examples of such questions follow:

- Do more students live in a house or an apartment? How can we tell?
- How do most students arrive at school?
- How many more students ride the school bus than walk to school?
- How many students have more than five people in their household?

The glyph activity leads to opportunities for creating and interpreting graphs. Teachers provide students with small pieces of paper on which they draw pictures of themselves being transported to school. The teacher and students then create a bar graph displaying the numbers of students in each category. Such graphs could, of course, lead to further opportunities for exploring questions.

After students have been involved in a variety of activities using concrete graphs, the teacher may help students to make a transition to the concrete-pictorial stage. In this stage, students discuss and explore questions similar to those from the concrete stage, but they create graphs using pictorial representations of concrete objects. Following a class discussion of students' favorite types of ice cream, for example, each learner could draw and color an ice cream cone containing his or her favorite flavor (perhaps from a limited number of choices). When comparing hair color, each child could be given a small square of paper on which to draw his or her face and hair. To explore students' choices among after-school activities, children could draw pictures of themselves participating in sports, reading, or watching television. In each of these examples, the pictures would be used to create bar graphs.

Photograph by William O. Lacefield III; all rights reserved

To represent the number of people who live in her household, this student cuts out leaves to add to her apple glyph.

Regardless of the level at which young learners are functioning, teachers should encourage students not only to pose questions and create graphs but also to reflect on the data that are shown in the graphs. Rich questions can arise from a graphing experience. In the favorite fruit lesson, for instance, the teacher might ask any or all of the following questions:

- What fruit was selected as the favorite of most people?
- How many more students selected an orange than selected an apple?
- Why do you think fewer people selected bananas as their favorite fruit?
- Why did you select the type of fruit that you selected?
- How do you think the graph might change if we include Mrs. Brown's class in our graph-making activity?

Students could respond to such questions with pictures and words in their mathematics journals, thus meaningfully connecting data analysis and representation.

Glyphs

Developing a glyph, a picture that presents information, requires a level of sophisticated thinking as well as the ability to portray that thinking. Glyphs are powerful in that they give young learners expressive, individualized ways to represent and communicate data. To extend the fruit theme discussed previously with regard to graphing, a teacher might ask students to create an apple glyph that reveals information about themselves. Give each child a large white cutout in the shape of an apple. Ask students to color their apples red if they live in a house, green if they live in an apartment, or yellow if they live in a mobile home. Use stem color to indicate how students get to school: Black stems represent walking to school, brown stems represent riding the school bus, and green stems represent carpooling. Students can color and glue green leaves to the stems; the number of leaves represents the number of people living in the home.

Of course, the types of glyphs and data represent endless possibilities, limited only by teachers' (and learners') own imaginations. Once students have created glyphs, teachers might display them on a bulletin board or mural and use them as starting points for discussions, creating graphs, and problem solving. For example, a class of students observing apple glyphs (as described above) could use the numbers of leaves on each apple to construct a graph depicting the number of people who live in each student's home.

Glyphs also provide opportunities to practice basic skills. Teachers might ask these questions: How many more students live in houses than in apartments? What number do we get if we add the number of walkers to the number of carpoolers? Again the possibilities are limitless.

Conclusion

Graphs and glyphs provide rich opportunities for exploring questions, gathering and representing data, and reflecting on such data. Teachers of young children know that students do not typically come to school with innate abilities to articulate and refine questions, to consider various ways of collecting data, or to select appropriate manners of representation. Rather, children construct these abilities through hands-on, minds-on experiences, cooperative learning opportunities, class discussions, and supportive guidance from teachers and other adults.

Early childhood educators continue to stress the importance of data analysis in early childhood curriculum. Questions, graphs, and glyphs are wonderful tools for developing critical thinking, problem solving, and representation abilities within young learners.

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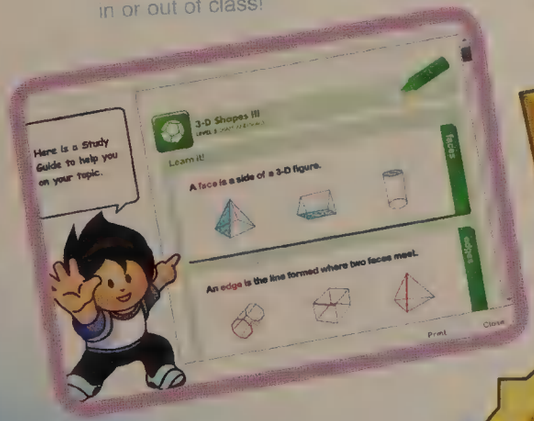
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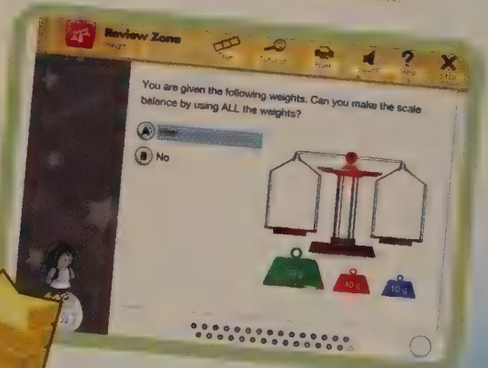

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Multiplication along the Silk Road

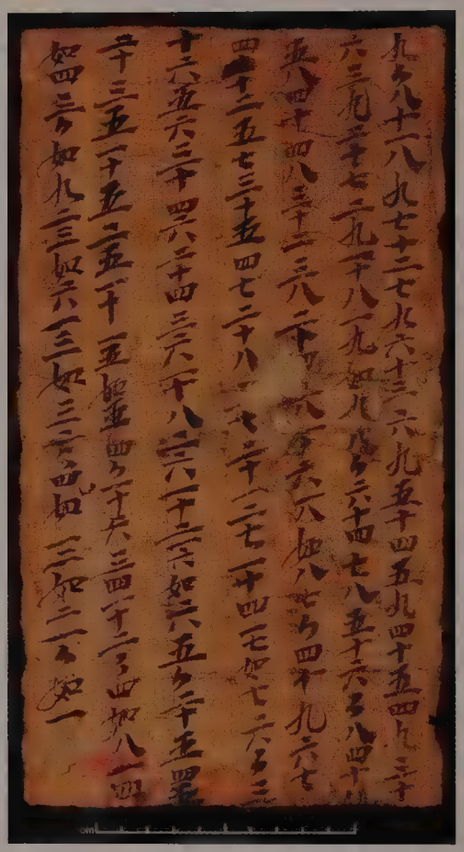
Problem

As we examine the ancient Chinese multiplication table to the right, we can identify similarities and differences between it and the modern multiplication table below. A solid understanding of multiplication facts was apparently as important during the Han Dynasty (202 BC to AD 220) as it is today. Merchants who traded along the Silk Road used multiplication to determine the value and relative worth of their various products as well as to keep track of the quantity of product in their inventory.

This month's problem focuses on basic multiplication facts. Begin by thinking of any two-digit number. The first step is to multiply the digits. Using 84 as our example gives us $8 \times 4 = 32$. Now, multiply the product's digits, resulting in $3 \times 2 = 6$. The process is complete when you obtain a single digit (0 through 9). To record the results of this example with 84, we write 84, 32, and 6 and note that this sequence has three values.

As another example, complete the multiplication process for the two-digit number 69, which results in a four-value number sequence: 69, 54, 20, 0.

The first example results in a sequence of three values, and the second example produces a sequence of four values. When this multiplication process is applied to other two-digit numbers, which numbers produce a sequence of three or four values as did the two previous examples? Which two-digit numbers result in a sequence of only two values? Which two-digit numbers produce a sequence of five or more values?



Silk Road manuscript from Dunhuang, Or8210/S.4569. © The British Library Board

Extensions

1. What happens if you begin this process with a three- or four-digit number?
2. Would the resulting sequence of values be larger or smaller? Why?

Images of this multiplication table and many other Silk Road manuscripts, paintings, and artifacts can be viewed on the British Library's International Dunhuang Project (IDP) Web site (http://idp.bl.uk). IDP is developing educational resources about Silk Road collections; see http://idp.bl.uk/pages/education_teachers.a4d.

	0	1	2	3	4	5	6	7	8	9	10	11	12
	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9	10	11	12
	0	2	4	6	8	10	12	14	16	18	20	22	24
	0	3	6	9	12	15	18	21	24	27	30	33	36
	0	4	8	12	16	20	24	28	32	36	40	44	48
	0	5	10	15	20	25	30	35	40	45	50	55	60
	0	6	12	18	24	30	36	42	48	54	60	66	72
	0	7	14	21	28	35	42	49	56	63	70	77	84
	0	8	16	24	32	40	48	56	64	72	80	88	96
	0	9	18	27	36	45	54	63	72	81	90	99	108
	0	10	20	30	40	50	60	70	80	90	100	110	120
	0	11	22	33	44	55	66	77	88	99	110	121	132
	0	12	24	36	48	60	72	84	96	108	120	132	144

The Silk Road refers to an ancient 7000-mile trading route that connected China with central Asia, northern India, and the Roman Empire. The Chinese traded their silk with the Indians in exchange for precious stones and metals. The Indians would then trade the silk with people from the Roman Empire.

The town of Dunhuang (not shown on the map) is located along the Silk Road. By the eighth century, more than a thousand cave temples existed near this town. In 1900, manuscripts, paintings, and an ancient multiplication table were discovered inside one cave. This month we use this treasure from the Orient as the starting point for our Problem Solvers exploration.

Classroom Setup

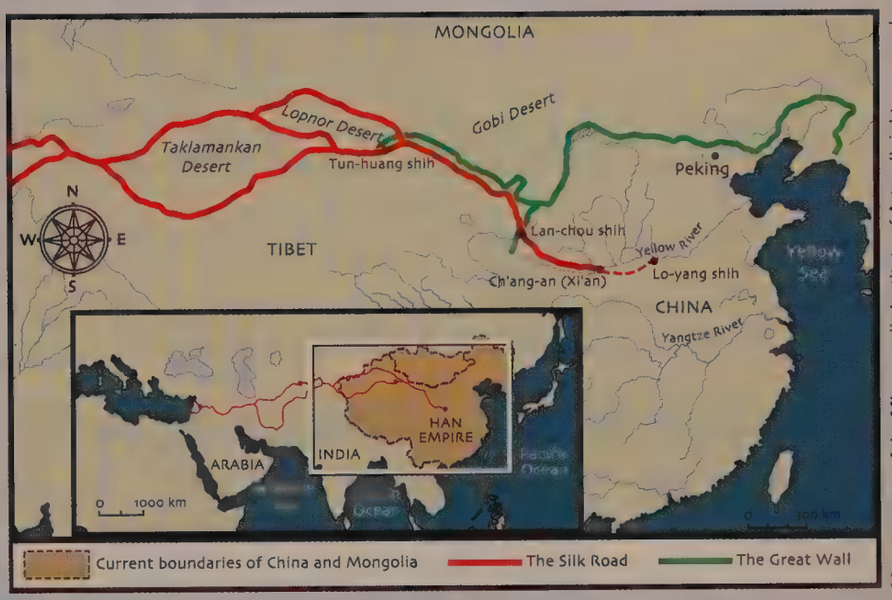
Begin by showing a map of Asia and discussing the trade route between China and the Roman Empire known as the Silk Road (see **fig. 1**). Refer to the multiplication tables in the problem. Initiate a discussion comparing and contrasting the ancient Chinese table with our modern-day one by asking questions such as, "How do these look similar?" or "How are they different?" One similarity that students may recognize is that both tables use rows and columns. Ask students if the symbols are placed on the Chinese table in a particular way or if the Chinese table has symmetry when compared with our modern table.

These are just a few of the questions that can be asked about the two tables. As you and your students examine the tables, others ideas will surface. Lead the discussion toward talking about multiplication facts and the concept of "fact power," stressing the importance of knowing our facts: Just as the merchants on the Silk Road used their multiplication facts to enhance their trade, we will use our fact power to help solve this month's problem.

Select the number 84 from the hundreds chart. Explain that by repeatedly multiplying the digits of a two-digit number, we eventually get a one-digit number and a sequence such as 84, 32, 6. Use the example of 69 to obtain the four-value sequence 69, 54, 20, 0. Note that the recorded sequence always contains the originally selected value.

Figure 1

Map of the Silk Road trade route



Map courtesy of the Minneapolis Institute of Arts; all rights reserved

Figure 2

This modified hundreds chart shows only two-digit numbers.

10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Discuss how 84 requires three steps to get to a one-digit number, but 69 takes four steps. Then ask which two-digit numbers have a sequence of three or four values as do the two examples. Challenge students to find two-digit numbers that result in a sequence of two values or, if possible, a sequence of five or more values. Make sure that students record their solutions and are able to demonstrate how they achieved them.

A modified hundreds chart (see **fig. 2**) can be used

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to color code each of the resulting answers. For example, all the numbers that result in a sequence of two values could be colored red; those resulting in a three-step sequence could be colored blue, and so on. Ask students to explore and describe any color patterns that develop within the completed hundreds chart.

If you have students who struggle with multiplication facts, use mixed-ability partners or small groups to investigate this month's problem. For students seeking additional challenge, extend the process to three- or four-digit numbers and ask students to compare these results with those from their two-digit solutions.

Share Your Students' Work

Please try this problem in your classroom. We are interested in how your students responded to the problem, what problem-solving strategies they used, and how they explained or justified their reasoning. Include information about how you posed the problem, samples of student work, and photographs. E-mail your thoughts, reflections, scanned student work, and photographs to **Sarah Bunten**, sbunten@gmail.chelsea.k12.mi.us; by **April 1, 2009**, or send them to her attention at Pierce Lake Elementary School, 275 N. Freer Rd., Chelsea, MI 48118. Include your name, grade level, and school name with your submission.

The authors thank the University of Oregon's Oscar Schaaf for ideas that appear in this problem. ▲

(Solutions to a previous problem begin on the next page.)

Where's the Math?

Multiplication is a fundamental mathematical process. This month's problem provides both a context where multiplication is the essential operation and further opportunities to solve problems and to find and explore patterns.

Why do we need to multiply? What types of questions are solved by doing multiplication?

Many younger students view multiplication as repeated addition. Multiplying 8 by 3 is really just a shortened form of adding 8 three times or 3 eight times. The fact that either way of multiplying leads to the same results is the commutative property of multiplication, an important and useful mathematical property. But other reasons exist for multiplication, and knowing them can help us solve many other types of problems.

For example, what is the area of a rectangle with a length of 3 miles and a width of 8 miles? The numerical answer is the same, 24, but the unit is now square miles, different from the unit of either of the numbers that we multiplied.

Here are two more examples of multiplication's usefulness: How many combinations could I wear if I had 3 shirts and 8 pairs of pants? The numerical answer is 24 again, but the unit is combinations of clothing. How far could you ride your bike in 3 hours if you traveled at the rate of 8 miles per hour? The numerical answer is 24 again, but the unit is miles this time. Other examples of multiplication involve scale models or changing sizes that are related to ratios.

The ancient traders along the Silk Road likely used multiplication in such varied contexts as they calculated their expenses, income, and travel details. Multiplication then and now is a powerful problem-solving tool, and knowing the kinds of problems connected to this operation leads to a deeper understanding of mathematics and where and why we use math.

The Silk Road problem involves using multiplication to find patterns. Asking students about the size of a multiplication answer, or *product*, can give them a sense of the size of numbers. Questions such as the following, in combination with basic fact practice, help students develop better, deeper understanding of multiplication:

- What is the largest value you can get when multiplying the two digits in a two-digit number?
- When will the product of the two digits be zero; when will it be another single-digit value?
- If the original number has three digits, state the largest and smallest possible products of those three digits. Can this information help determine numbers that lead to intermediate products?

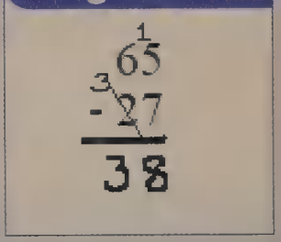
Solutions to the Creative Arithmetic Problem

Cathery Yeh accepted the February 2008 "Problem Solvers" challenge and had her students explore alternative subtraction and multiplication algorithms. This month she shares her reflections on the experience. The original problem was stated as follows:

Problem

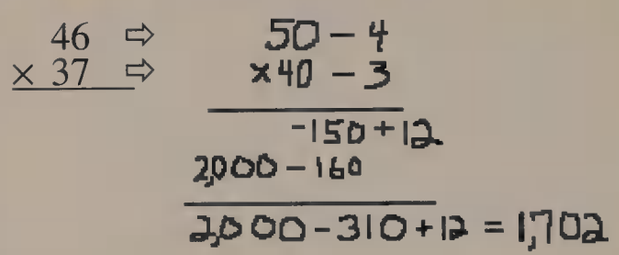
Madison was asked by her teacher, Mrs. Alvis, to take home a subtraction problem to figure out with her parents. That night, Madison's mother showed Madison a method she had learned at school when she was a child (see **fig. 1**). The next day at school, Mrs. Alvis asked Madison to explain how her method works. Madison was not too sure but knew it was the right answer. Can you help explain what Madison did and why her method works? Use and explain at least one additional strategy to solve this problem.

Figure 1



Mathias, Madison's brother, was asked by his teacher, Mr. Chavez, to solve a multiplication problem with his parents for homework. The next day, Mathias showed Mr. Chavez his method (see **fig. 2**) and was asked to explain how it works. Mathias was not too sure. Can you explain what Mathias did and why his method works? Use and explain at least one additional strategy for solving this problem.

Figure 2



Variation

Younger students can be given the subtraction problem only and asked to make sense of Madison's method. As suggested in the classroom setup section, having manipulatives such as base-ten blocks for students to use will be helpful for younger students.

Extensions

- The subtraction method used by Madison is called the equal-addition algorithm. Can you create another example of the equal-addition algorithm? Write a description of this method and how it works for any problem.
- Create a word problem that can be solved by subtracting 27 from 65 or by multiplying 46 by 37.
- Mathias's multiplication method uses equivalent expressions and the distributive property. The distributive property "distributes" multiplication to a number as you break it into equivalent parts (sometimes called *decomposing*). Figure out a different method for finding 46×37 that uses the distributive property. Explain how and why it works.
- Ask parents, grandparents, relatives, or neighbors if they know another way to do subtraction or multiplication. Have them show you an example. Bring it to class to share. How does it work?

My thirty-four fifth graders have had considerable experience with problem solving and have had regular practice with problems from the "Problem Solvers" department. I was eager to share the Creative Arithmetic problem with them because I heavily emphasize multiple approaches to problem solving and the use of alternative algorithms in my classroom. I strongly encourage my students to use

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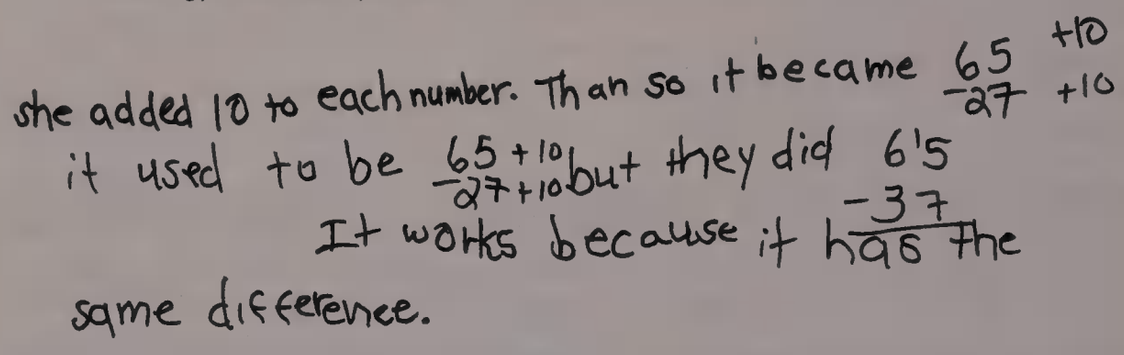
Figure 3

Hiseo clearly understands that subtraction is about finding the difference between two numbers.

Madison was asked by her teacher, Mrs. Alvis, to take home a subtraction problem to figure out with her parents. That night, Madison's mother showed Madison a method she had learned at school when she was a child (see **fig. 1**). The next day at school, Mrs. Alvis asked Madison to explain how her method works. Madison was not too sure but knew it was the right answer. Can you help explain what Madison did and why her method works? Use and explain at least one additional strategy to solve this problem.


she added 10 to each number. Than so it became 65 +10 -27 +10
it used to be 65 +10 -27 +10 but they did 65 -37
It works because it has the same difference.

any strategy they like to use as long as it makes sense to them. However, this school year had been unique for my students because I took a three-month maternity leave just after the start of the year. Before taking this leave, I had reviewed subtraction with my students and challenged them to make sense of their methods. While I was gone, students were taught multidigit multiplication and division by another teacher using an approach that relied more on learning to follow the rules shown in the textbook.

When I returned to the classroom, I was eager to see how my students would respond to the mathematical thinking of the two students portrayed in the original problem—Madison's subtraction problem (see **fig. 1**) and Mathias's multiplication problem (see **fig. 2**). Considering that two teachers had taught these concepts, would my students approach the two problems differently? I started by saying, "We often work on a problem in math by having you find multiple ways for solving one problem. Let's talk about some of the possible ways we can solve sixty-five minus twenty-seven."

Students mentioned that they could use the regrouping method, double checking the problem with addition, estimating, or adding two to sixty-five and twenty-seven so that they would not have to work with borrowing. After the sharing, I told them, "Today you are going to examine how someone solves subtraction in a different way. This student, Madison, solves subtraction using a method we haven't tried before in our class. You will need to figure out why Madison's method works."

I handed out copies of the written problems and then read the first problem to them. Students were set to work individually, in pairs, or in small groups. They seemed eager to start and had no questions for me. Most of them already knew multiple ways to subtract and were interested in understanding the equal-addition algorithm. They quickly started to discuss the problem with their partners or in their small groups. I simply walked around the room and observed.

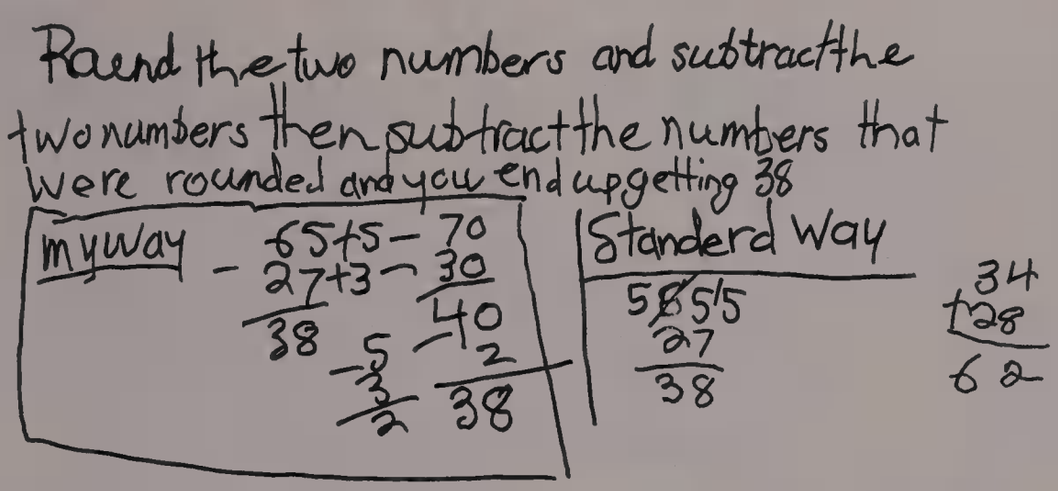
Their conversations were enlightening. Many students said that there was a correlation between Madison's equal-addition algorithm and the regrouping concept. Brenda called Madison's method "another form of borrowing where you add one to the bottom number instead of taking one away from the top number." Most students were able to easily give other examples of the equal-addition algorithm. Many saw instantly that instead of reducing the 6 tens to 5 tens, Madison made the 2 tens into 3 tens. This method works because the difference stays the same.

Hiseo was able to explain how Madison's manipulation of the numbers works in another way: "Madison added ten to the five so that sixty-five became sixty plus fifteen; so, she had to add ten more to twenty-seven. This would work because it has the same difference. The difference between the two numbers remains the same." Hiseo's explanation of Madison's method showed her clear understanding of subtraction (see **fig. 3**). As long as the same number is added to or subtracted from the

Figure 4

Pierre's work shows that he, too, has deep understanding of subtraction.

- The subtraction method used by Madison is called the equal-addition algorithm. Can you create another example of the equal-addition algorithm? Write a description of this method and how it works for any problem.

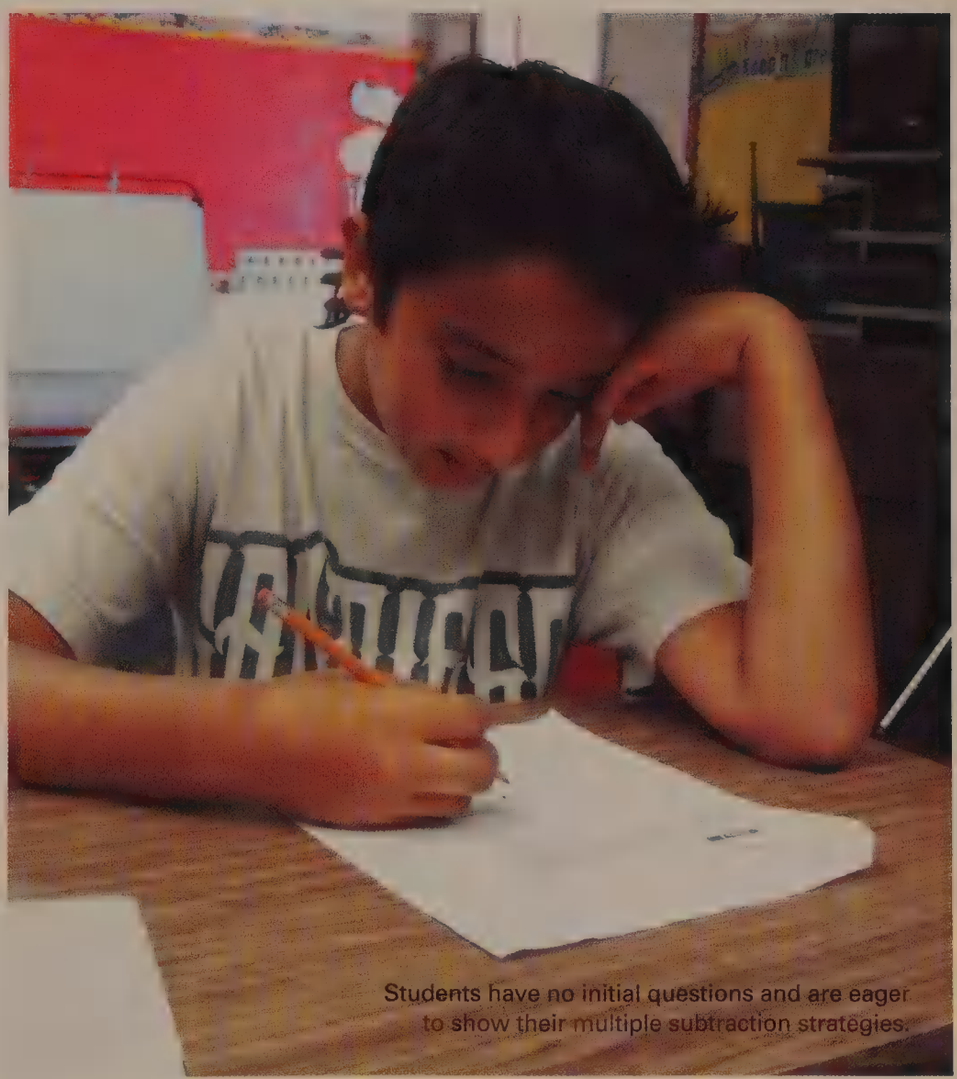


minuend and the subtrahend, the difference stays the same. Students noted that Madison's arithmetic works because the numbers are "equal" or the "same difference remains" after she manipulates her numbers. I was proud to see that they understood that subtraction is about finding the difference between two numbers.

One surprising answer came from Pierre (see **fig. 4**). He changed the sixty-five to seventy and changed the twenty-seven to thirty because those numbers are "friendlier" numbers to subtract. Then he noted that the difference between sixty-five and seventy is five and the difference between twenty-seven and thirty is three. After finding that seventy minus thirty equals forty, he then subtracted three from five "because you have to make sure the difference remains equal." His final answer showed that he understood subtraction beyond basic computational skills and had developed some deep insights.

After Pierre shared his alternative algorithm, I introduced the students to Mathias's multiplication problem. Once we read the problem together, students started working. Again I found that listening to them think out loud was interesting and informative.

All the students could easily multiply double-digit numbers but had difficulty explaining how Mathias's method works. I was not too surprised by their struggles because the students' textbook uses



Students have no initial questions and are eager to show their multiple subtraction strategies.

Photograph by Cathery Yeh; all rights reserved

double-digit multiplication. It also places emphasis on the procedural process over conceptual sense making. However, the students and I have often used the distributive property in class and call it the "break-apart" strategy.

Students had myriad reactions to Mathias's method. Some felt that the method makes no sense. Many saw it as a way of first finding an estimated answer before getting the exact answer. For example, Matthew was able to explain Mathias's procedure step by step, but he could not explain why it conceptually works (see **fig. 5a**). However, he was able to solve the same multiplication problem by using a different method and the distributive property (see **fig. 5b**).

My students' work on this problem led to a won-

Figure 5

Matthew is able to explain Mathias's procedure step by step.
(**a**) However, he cannot explain why the procedure conceptually works.

Mathias, Madison's brother, was asked by his teacher, Mr. Chavez, to solve a multiplication problem with his parents for homework. The next day, Mathias showed Mr. Chavez his method (see **fig. 2**) and was asked to explain how it works. Mathias was not too sure. Can you explain what Mathias did and why his method works? Use and explain at least one additional strategy for solving this problem.

Figure 2

4
46 ⇨ 50 − 4
× 37 ⇨ ×40 − 3
322 −150 + 12
1380 2,000 − 160
1707 2,000 − 310 + 12 = 1,702

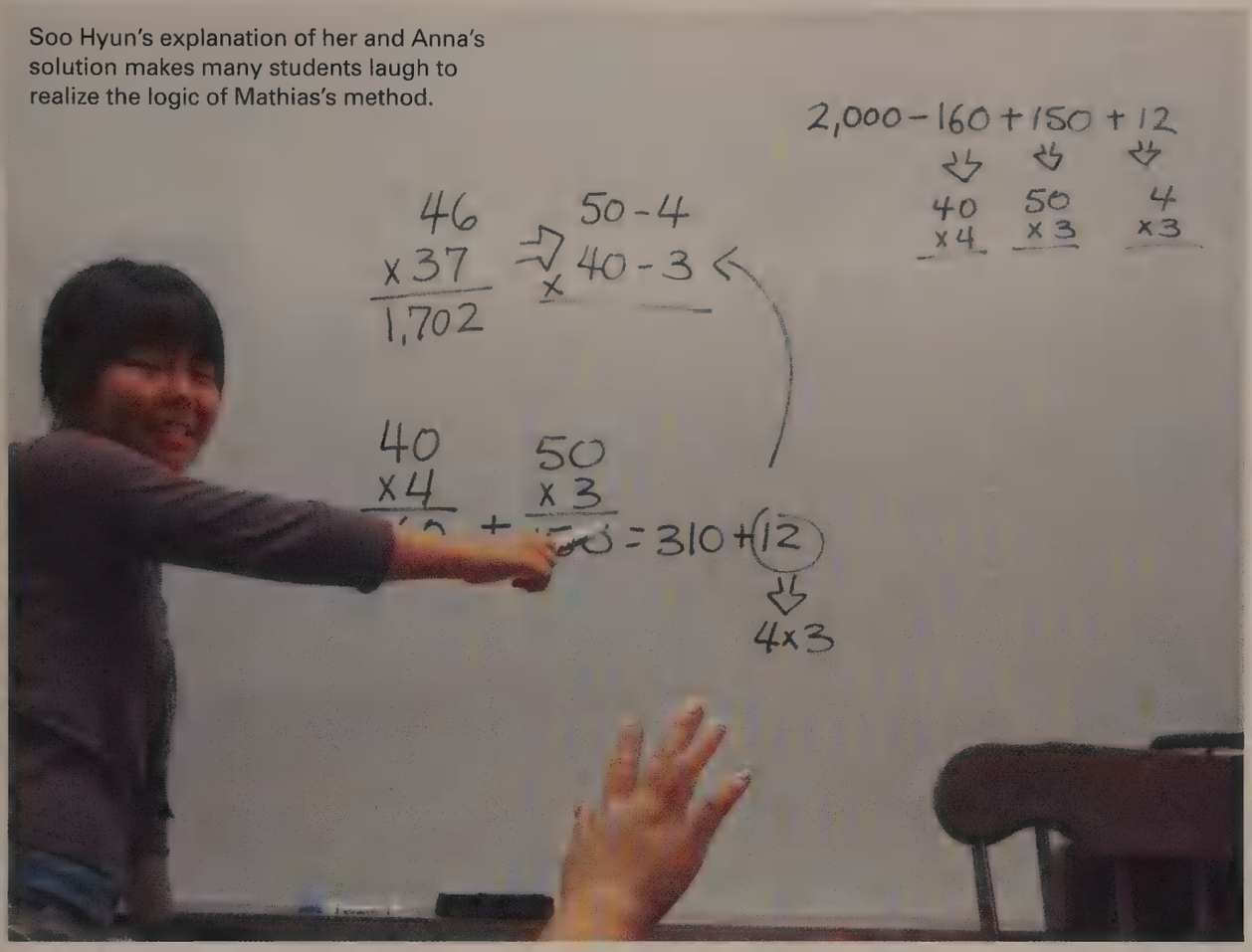
He first rounded the number and got an estimated answer. Then to make it exact, you had to subtract the product of 46×7 from the estimated answer. So multiplying the diagonal numbers and the difference from the rounded number from the exact and adding those two products would make 1,702.

(**b**) Instead, Matthew solves the same multiplication problem by using a different method and the distributive property.

- Mathias's multiplication method uses equivalent expressions and the distributive property. The distributive property "distributes" multiplication to a number as you break it into equivalent parts (sometimes called *decomposing*). Figure out a different method for finding 46 × 37 that uses the distributive property. Explain how and why it works.

4 1
46 46
× 7 × 30
322 + 1380 = 1,702

46×37 is the same as 46×7 + 46×30.



Soo Hyun's explanation of her and Anna's solution makes many students laugh to realize the logic of Mathias's method.

Photographs by Cathery Yeh; all rights reserved

derful discussion about the distributive property in multiplication. They clearly understood the problem if we broke it apart so that 46×37 becomes $(40 + 6) \times (30 + 7)$, yet they did not see Mathias's method as the same concept. However, when Soo Hyun presented the solution that she and Anna used, many students laughed because they realized that Mathias's method was so logical and was now easy for them to understand.

The Creative Arithmetic problem served as a superb exercise for the whole class, allowing students to reflect on their understanding of important mathematical concepts that we had learned throughout the year. Although we often discuss multiple ways of solving a problem, the methods that usually arise in class are familiar methods that students use in their own cultures. This problem was different: Students are not creating their own methods but analyzing another student's work.

This was a wonderful lesson for me, as well, allowing reflection on my own teaching practice. In my classroom, we often discuss the distributive property and use this popular solution method when we work on addition, subtraction, and multiplication problems. However, my students' struggle with understanding Mathias's method showed me that I need to broaden their exposure to what the distributive property really means. Soo Hyun described it perfectly in her explanation that Mathias's method "is just a mentally changed way" of seeing the exact same problem. I need to continually challenge my students to see mathematics in different ways. ▲

Together in small groups, students try to understand the equal-addition algorithms.

The Dragon Prince
L
M
N
Fairy Tales
8 + 6 = 14

Ten Is the Magic Number!

How this teacher develops composition of ten with second graders was dramatically reshaped by the 2006 release of NCTM's Curriculum Focal Points.

By Lindsay Barker

Lindsay Barker, lbarker@ric.edu, teaches second grade at the Henry Barnard Laboratory School on the Rhode Island College campus in Providence. She is interested in the development of number sense and is involved in curriculum development and evaluation at the school.

Photograph by Eugene St. Pierre; all rights reserved

Bernard, an inquisitive and friendly second grader, noticed an adult visitor in my classroom observing another child. Bernard loves to chat with adults, to share his wealth of information about the world, and to quiz grownups on his latest fascinating number puzzle. This day, Bernard had a math question he just had to share with the visitor. "What is 14,000 minus 99?" he asked with sincere wonder. We had spent the first five weeks of school studying patterns in numbers and learning addition strategies for basic facts. We had not touched on subtraction, let alone large numbers—hence Bernard's fascination.

My toes curled at the visitor's reply: "You need to line up the digits."

This is not the reply I want a curious second grader to hear to prompt his thinking. Bernard had no idea what "line up the digits" meant, and he was trying to do the problem mentally. So, what do I wish the adult visitor had said to my student? I wish she had replied to his question with one of her own.

Reshaping How I Teach

The suggestion that number sense and computation be focal points of the second-grade mathematics curriculum (NCTM 2006) resulted in positive changes in my approach to teaching these areas. Receiving extended instructional time on the composition and decomposition of ten and its applications allowed my students to become more efficient at computation and develop greater number sense and had a positive affective impact on the way they view mathematics.

I called Bernard over as soon as I could. "What do you know about 99?" I asked him.

It's almost 100," he replied. "Just like a nine is almost a ten."

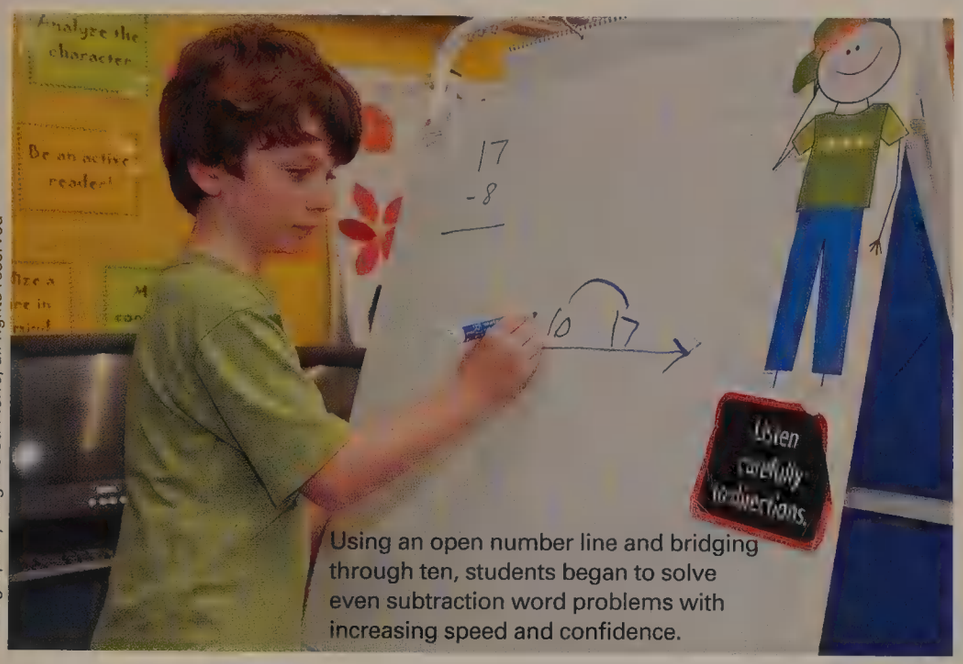
"What happens if you imagine the 99 is 100 and you take away 100 [from 14,000]?" I asked.

Bernard has good number sense. He knew that nine hundreds would be left from one thousand and quickly got to the answer of 13,900: "Oh, yeah. You need to remember the one. So, it is 13,901!" he exclaimed.

"BINGO!" I rejoiced.

The Focal Points and Curriculum Development

Bernard's is the kind of thinking that I am determined to help all of my students develop. *Curriculum Focal Points* suggests three main areas on which to focus the mathematics curriculum at each grade level. The book calls for the second-grade curriculum to focus on using "understanding of addition to develop quick recall of basic addition and related subtraction facts" (NCTM 2006). *Curriculum Focal Points* also calls for efficient methods for calculating sums and differences mentally and for fluency with efficient procedures.



Using an open number line and bridging through ten, students began to solve even subtraction word problems with increasing speed and confidence.

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The common criticism of U.S. mathematics curricula as a mile wide and an inch deep had sometimes felt close to the truth whenever I had tried to move my students along a diverse curriculum path after teaching them some strategies to solve basic addition and subtraction problems. I saw the release of *Curriculum Focal Points* as a positive step toward curriculum evaluation and development. It affirmed my thought that I should be devoting more time in my second-grade classes to developing number sense and efficient computation strategies so that all my students had the opportunity to develop the level of number sense that Bernard had.

I am in the fortunate position of working in a laboratory school, a function of which is curriculum development. As a member of the school mathematics team, one of my roles is to organize data collection about teaching mathematics at each grade level and form a scope and sequence document for the school. This document is evaluated as we determine the effectiveness of materials and how and what concepts are taught each year. The timing of this project coincided with the release of *Curriculum Focal Points*. I made a personal decision to evaluate my second-grade curriculum in light of the Focal Points and take a more in-depth approach to developing number sense and computation—in contrast to the skill-drill fashion that the media supported at the time *Curriculum Focal Points* was released. I wanted a more coherent approach with greater application and connections. My team teacher also felt strongly about developing these areas during the early childhood years and was very willing to alter the pace and depth of our scope and sequence.

After refocusing my second-grade curriculum to recognize the importance of number sense in computation and to allow more time for in-depth study of these areas, the immediate effects were an increase in mathematical dialogue and unexpected affective changes, such as students' feelings of greater efficiency and confidence with number relations. If this in-depth approach proved to provide my students with a more solid foundation for third grade, the third-grade teachers also might be willing to evaluate their curriculum in light of the Focal Points. The process could have a positive domino effect through our entire school.

Composing Ten, Factual Fluency, and Dialogue

Because our U.S. number system is based on ten, I decided that developing composition of ten would be the backbone of my curriculum. One of the key manipulatives I used is the tens frame (see **fig. 1**), which is basically two rows of five squares filled with black dots to represent the number you are showing (Van de Walle 2007).

Thompson and Van de Walle (1984) wrote about the importance of number combinations that total ten, crediting Robert Wirtz of Curriculum Development Associates with introducing them to the idea of the tens frame. I spent a week acquainting my students with tens frames as a visual reference for how ten is composed. They looked at the frames and initially noticed that ten is two rows of five. They then began to discuss each number in terms of its relationship to ten or to five. Seven became three less than ten. We flashed the tens frames in quick-fire rounds, with students identifying both the number of dots and how many more dots would be needed to make ten. In this way, my students became proficient in composing ten. They played Tens Go Fish and Turn over Ten from the Investigations in Number Data and Space series (Economopoulos et al. 1998). Each game involves students finding number pairs that add up to ten. Subsequently, students used completed tens frames to practice solving problems with an addend of ten (e.g., 10 + 7), which nicely highlighted the place-value aspect of numbers in the teens.

Our next step was to move to addition questions that involve nine as an addend. Students had found that addition with an addend of ten is a simple mental computation, so they began using the tens frame to take one from one addend and give it to the nine to compose a ten (9 + 8 became 10 + 7; see **fig. 2**). We not only modeled addition with tens frames and counters but also proved equivalence with a math balance from which we hung weights with numbers on them.

Students also became proficient at turning eights and sevens into tens. We used tens frames to model both addends, and we shifted counters from the lower addend to fill the tens frame of the higher addend. This visual aid helped students understand that removing parts from one addend and adding them to another does not change the total quantity; the balance scale beautifully represents this concept. Making tens made sense to the students, and soon they began doing it mentally.

The development of number sense in the early grades is crucial to computational fluency and understanding in later grades. I have realized the importance of providing students with the time and opportunities to compose and decompose numbers, to discover relationships between numbers, and to find alternative ways of describing them: "When children are forced to work with greater numbers before they can work fluently with numbers to ten, they become dependent upon rules and procedures that have no meaning" (Postlewait, Adams, and Shih 2003). This dependence on rules had been my educational experience; my mental mathematics skills had never developed in the way that Bernard's have. I wanted to allow all my students the time to compose and decompose numbers and develop a sense of the relationship between them.

Once students were confident with bridging to ten, it was time to extend the concept to numbers up to twenty and then beyond. We progressed quickly, beginning by modeling numbers in the teens and describing them as a tens group and some singles. Because the tens frame clearly demonstrates how far away from the next ten a number is, the relationship between numbers was a key focus of our discussions. Earlier practice at quickly determining how many more counters were needed to fill the tens frame had provided the foundation for this work. Other models I have used in the past have not been as successful in demonstrating this key relationship because the ones must be continually counted, and the distance to the next ten is not

Figure 1

The tens frame is a key manipulative for developing composition of ten.

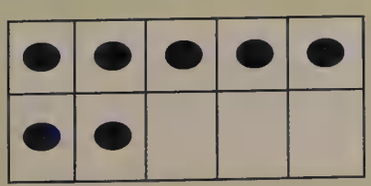
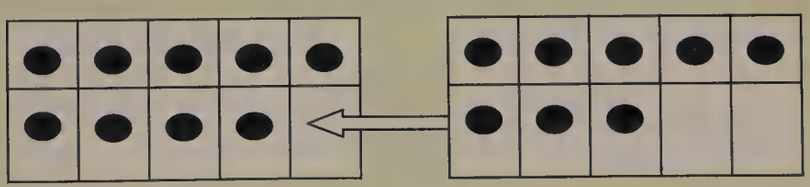


Figure 2

Students began to use the tens frame when they had an addend of nine.



visually displayed. With the tens frame, however, students excitedly connected the higher numbers to the concept of groups of ten.

Focusing on groups of ten naturally brought place-value discussions into our lessons. We had an interesting discussion about the Chinese number system, in which numbers are named by their relationship to ten. Twelve is "ten-two," and thirty-four is "three tens-four." Some researchers suggest that this naming convention is why Chinese students naturally have a better understanding of number than their counterparts in other cultures do. My students eagerly linked our tens-frame models to the Chinese number system and wondered why our naming system does not make as much sense.

Figure 3

Ten is the magic number! Students became confident in mentally composing tens and progressed to using an open number line for subtraction.

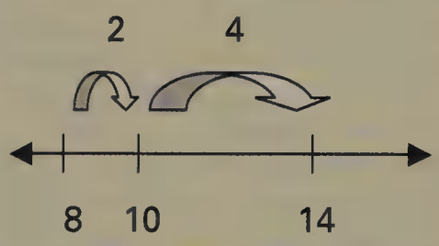
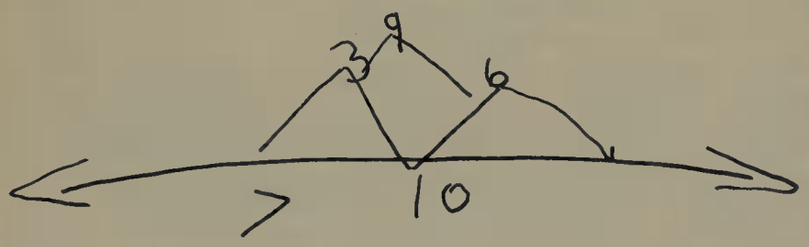


Figure 4

Using an open number line and bridging through ten, students began to solve word problems with increasing speed and confidence.

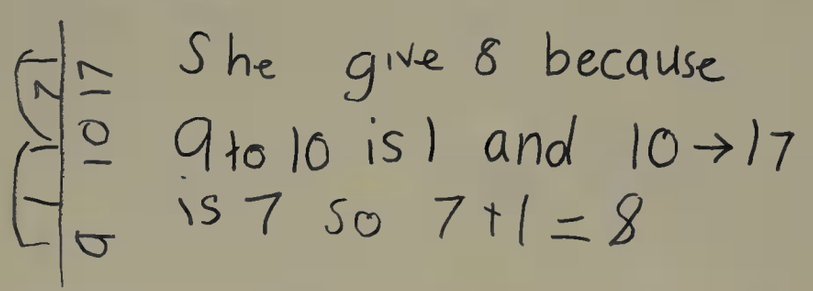
(a)

Julia was playing Cover Up with Haley. They had 16 cubes altogether. Haley covered some up, and Julia could see only 7. How many did Haley cover up?



(b)

I grew 9 pumpkins in my garden. A friend gave me some of hers, and then I had 17 altogether. How many did she give me? Show your work and explain your thinking in words.



The next day, when I asked, "Thirteen plus what makes twenty?" Lucille was eager to share her strategy: "Well, I know that thirteen is a teen. It has a group of ten and three ones. Twenty is two tens, and you have one ten already in the thirteen. We all know that three plus seven is ten, so you just need seven more to make twenty." Other students nodded in agreement.

To reinforce the composition of twenty, we played Cover Up from the Investigations series. One student showed a whole set of twenty cubes, and then covered up part of the set with a folder. The other student could see the remaining part and had to figure out what was covered. To solve the problem, students applied their knowledge of making groups of ten. The paper-and-pencil task that followed (Van de Walle 2007) encouraged them to quickly identify two (of three) numbers that have a sum of twenty. For example, of the three numbers thirteen, six, and seven, they were to find the pair that equals twenty when added.

When my students were confident in composing tens mentally, it was time to move to subtraction. In the past, I have struggled to get students proficient in subtraction. Some students never progress from relying on counting backward in ones. I wanted to build on the idea of tens because it had made sense to my students, and their mental calculations were developing well. Their success prompted me to introduce them to the open number line. (O'Laughlin described her use of this tool in *Teaching Children Mathematics* [2007]). The open number line is simply a line with no prewritten numbers on it. Students determine the start and finish number on the basis of numbers in the question they are solving and then use "friendly numbers" (in our case, multiples of ten) to jump from the start number to the finish number. Then they calculate how far they jumped.

We began our lessons by setting a context for finding a difference. Two students connected Unifix® cubes to show the length of their feet. One student's foot length was eight cubes; the other's was fourteen. To review addition strategies, we discussed combined shoe length. Again students showed their flexibility with number as they took six from the eight to give to the fourteen to make two groups of ten, and then added the final two to get a total of

twenty-two. When I asked, "What is the *difference* between the lengths of the shoes?" one student put the towers of cubes side by side and compared them. She counted the difference in ones. Another student counted down by ones.

Then Harry exclaimed, "Just use the tens strategy! Eight plus two more is ten, and four more makes six altogether!" (See **fig. 3.**) Students began to use the open number line and bridge through ten to answer questions (see **fig. 4**). My second graders were solving problems involving differences with a speed and confidence that was significantly better than in previous years when I had not taken the time to allow such an in-depth study of tens.

During one of our discussions about the open number line, Hazel wondered aloud, "It can be used for bigger numbers too, right?" I immediately wrote a double-digit problem on the white board: 27 + ? = 53. All the students began talking and asking to come to the board to solve it. I called Hazel up. "Hazel, how could knowing about tens help you [solve this problem]?"

She proceeded to put 27 on the left of her number line and 53 on the right, and then paused. After thinking for a few minutes, she added 30 close to the 27 (see **fig. 5**): "It takes three to get to thirty, then twenty to get to fifty, and another three. That's twenty-six!"

Another student added, "You can go back the other way, too. It takes three to get back to fifty, then twenty to get down to thirty, and then three more to get down to twenty-seven. That's twenty-six altogether."

Figure 5

When Hazel wondered if the open number line could be used for larger numbers, the teacher immediately challenged her with a double-digit problem, which Hazel successfully solved.

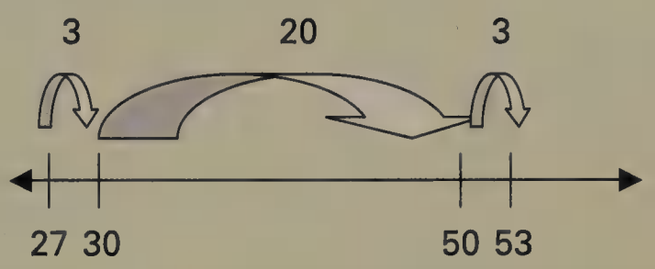


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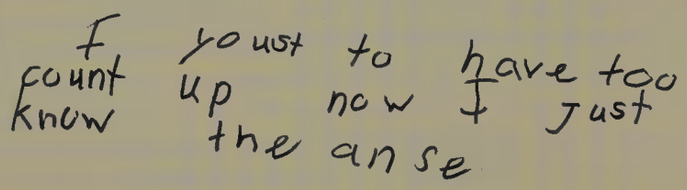
Figure 6

Stan's responses to the questions on the end-of-the-year assessment affirmed his teacher's earlier decision to study tens in depth.

19) Do you think that you have improved in math so far in second grade? Tell me about your improvement.



20) Have your feelings about math changed at all this year? Tell me about them.

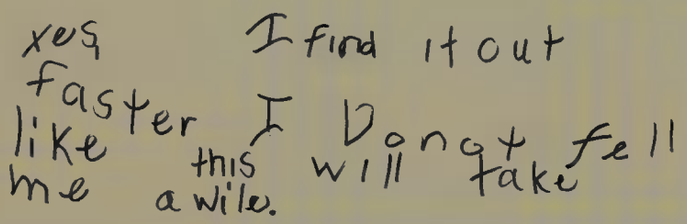
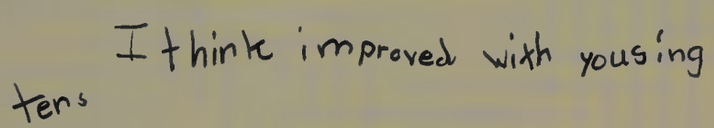


Figure 7

During the year, Victoria's confidence soared, and she recognized the change in herself.

19) Do you think that you have improved in math so far in second grade? Tell me about your improvement.



20) Have your feelings about math changed at all this year? Tell me about them.

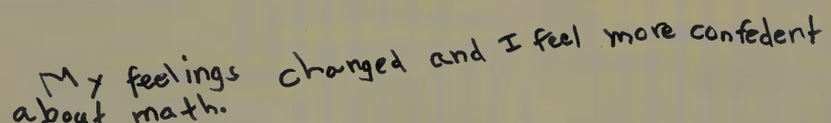
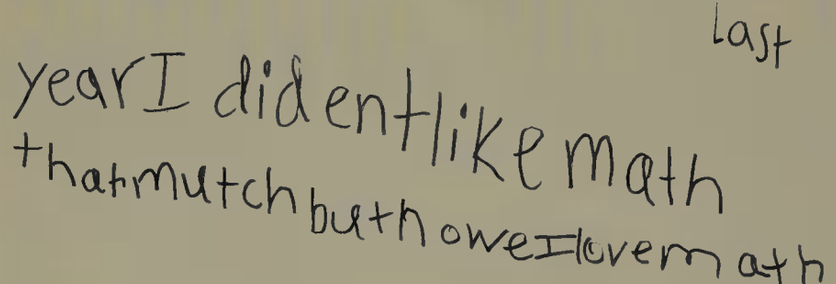


Figure 8

Lucille now loves math!

20) Have your feelings about math changed at all this year? Tell me about them.



Affective Changes

In addition to greater reasoning and factual fluency, I have noticed an increase in my students' overall confidence with mathematics. They are eager to share and excited when they apply bridging strategies to larger numbers. To find out more about the affective results of this approach, I included the following two questions at the end of students' assessments:

1. Do you think that you have improved in math so far in second grade? Tell me about your improvement.
2. Have your feelings about math changed at all this year? Tell me about them.

The responses were an affirmation of the work we had done (see **figs. 6**, **7,** and **8**). Van de Walle writes of this affective change, "Relational understanding has an affective effect as well as a cognitive effect. When ideas are well understood and make sense, the learner tends to develop a positive self concept about his or her ability to learn and understand mathematics.... There is no reason to fear or be in awe of knowledge learned rationally" (2007, p. 27).

An outcome of refocusing my curriculum that I had not anticipated, this affective change is the one that I value the most.

Teachers' Habits of Mind

Teachers must possess the correct habit of mind about mathematics. Griffin (2004) believes that teachers must see mathematics as a set of conceptual relationships between numbers and number symbols rather than as numbers that are manipulated by rules. The questions we ask, the tasks we design, and the discussions we prompt can refocus students on discovering such relationships themselves.

Emphasis on discovering relationships between numbers has been a positive change in my curriculum, but it is only a natural result of slowing the pace at which I move my students through my scope and sequence. At first, I was concerned about how I would make time to teach all that I used to teach. However, *Curriculum Focal Points* was written to encourage a critical look at curriculum revision, and studies such as TIMSS reveal that countries that outperform the United States in mathematics have a narrower curriculum focus by grade band. Schmidt (2004) identifies the three

mathematics topics for grades 1 and 2 in such high achieving countries: whole-number meaning, whole-number operations, and measurement units—the same three focus areas that NCTM suggests for second grade.

One Japanese class spent eleven days building a thorough understanding of composing ten (Murata and Fuson 2006). This prolonged study is typical of the Japanese mathematics curriculum, which spends concentrated time periods on fewer topics, and the topics build on one another. Students in China also spend time composing and decomposing ten as part of the knowledge package for related concepts, such as subtraction with regrouping (Lipin 1999).

Although I was usually further along in developing scope and sequence at this same point in the academic year, I was nevertheless confident that the depth at which my students were developing number sense and efficient computation skills would pay dividends in other concept areas and in preparing them for third grade. In fact, as I was working through my computation unit, I noted that the number-sense relationships my students were building were the perfect precursor for our next unit on place value.

Making tens made sense to students, and they soon began doing it mentally.

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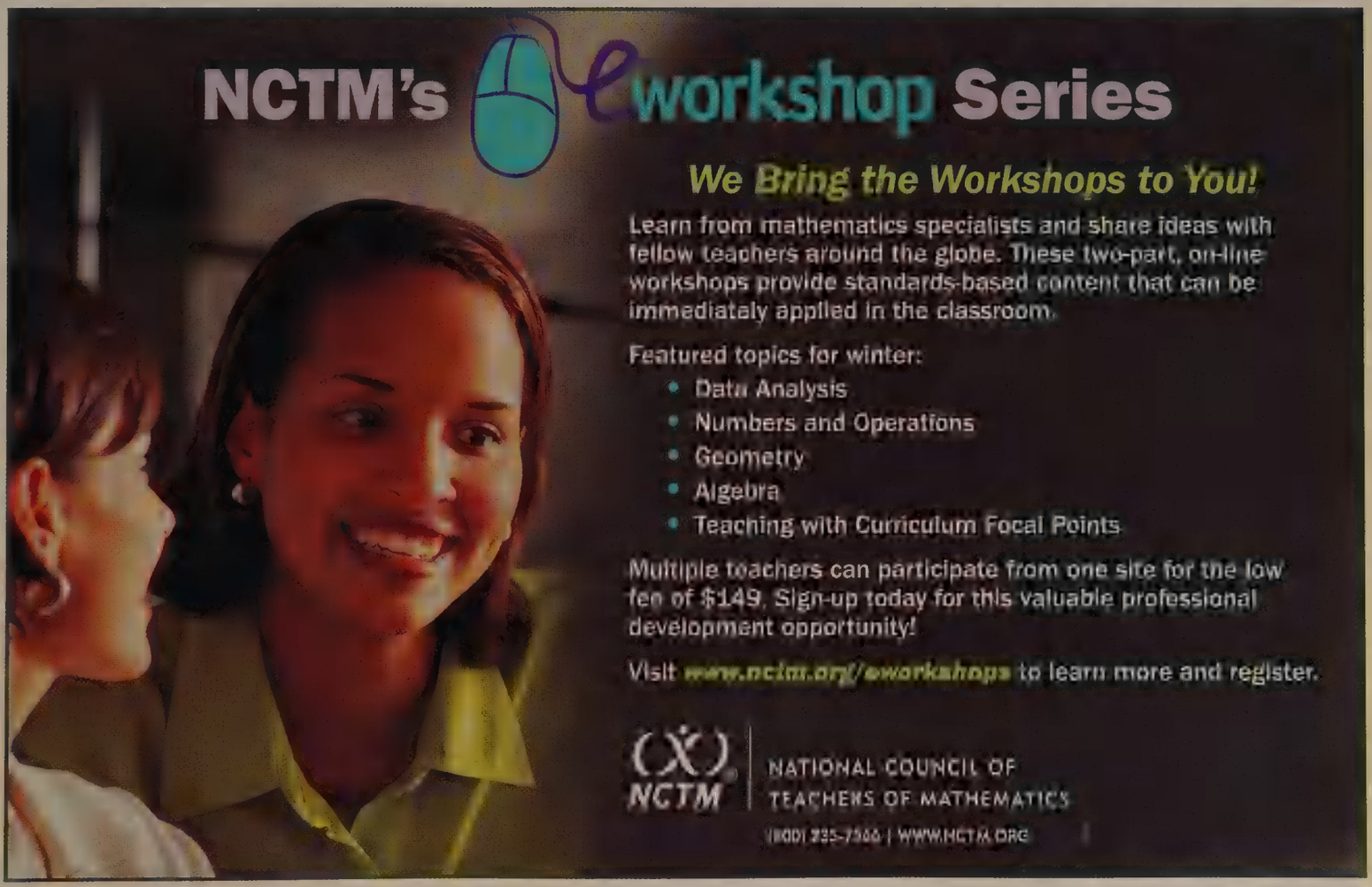
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Place Value

Our work with the concept of ten influenced our approach to place value. The first task my students had to perform was how to count a large group of beans. They quickly suggested groups of ten and then used this method to name the number of beans. They named seven groups of beans and two singles as seventy-two. Students later applied the concept of composing tens to help them find the other part of one hundred. The tens frames were used again to initially model this idea. For example, if they modeled thirty-six and wanted to find out how many more were needed to make one hundred, they bridged to forty by adding four to thirty-six, and then they applied their making-tens strategy. So, if 4 + 6 = 10, then 4 tens + 6 tens = 100. They were able to deduce that 64 is the other part of 100. This model also helped them when thinking about making change for one dollar.

My second graders were now confidently trading ten ones for a ten, ten tens for a hundred, and decomposing larger units for smaller ones. Julia announced, "Ten is just the magic number!" Many of them formed mental number lines and bridged through tens numbers to find the difference between two numbers.

My curriculum evolves each year, but publication of the Focal Points, my reading and implementing of Van de Walle's work with tens frames, and the great interest in the mathematics curricula of countries that outperform the United States have had a positive effect on how I teach mathematics and shape my students' understanding of number. The result is a more in-depth study of numbers, in particular, number relationships and composing and decomposing ten. So far, I have seen students' efficiency with computation improve, understanding of place-value relationships increase, and mathematical confidence soar. The final results of slowing down my curriculum to allow students to become more confident and efficient with numbers will be revealed in third grade. Scope and sequence will be evaluated and revised on the basis of outcomes of refocusing on the Focal Points for second grade. Further evaluation at other grades can then be a goal. As my team teacher and Postlewait, Adams, and Shih (2003) suggest, if students do not get a chance to develop such important number relationships as the composition of ten in the early grades, when will they?

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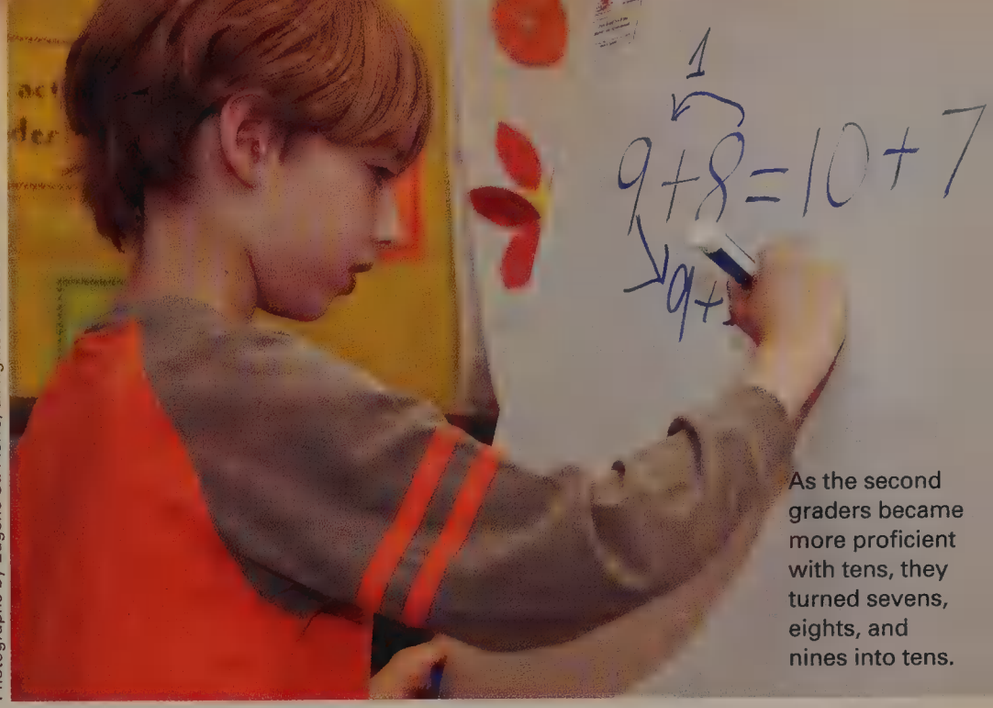

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As the second graders became more proficient with tens, they turned sevens, eights, and nines into tens.

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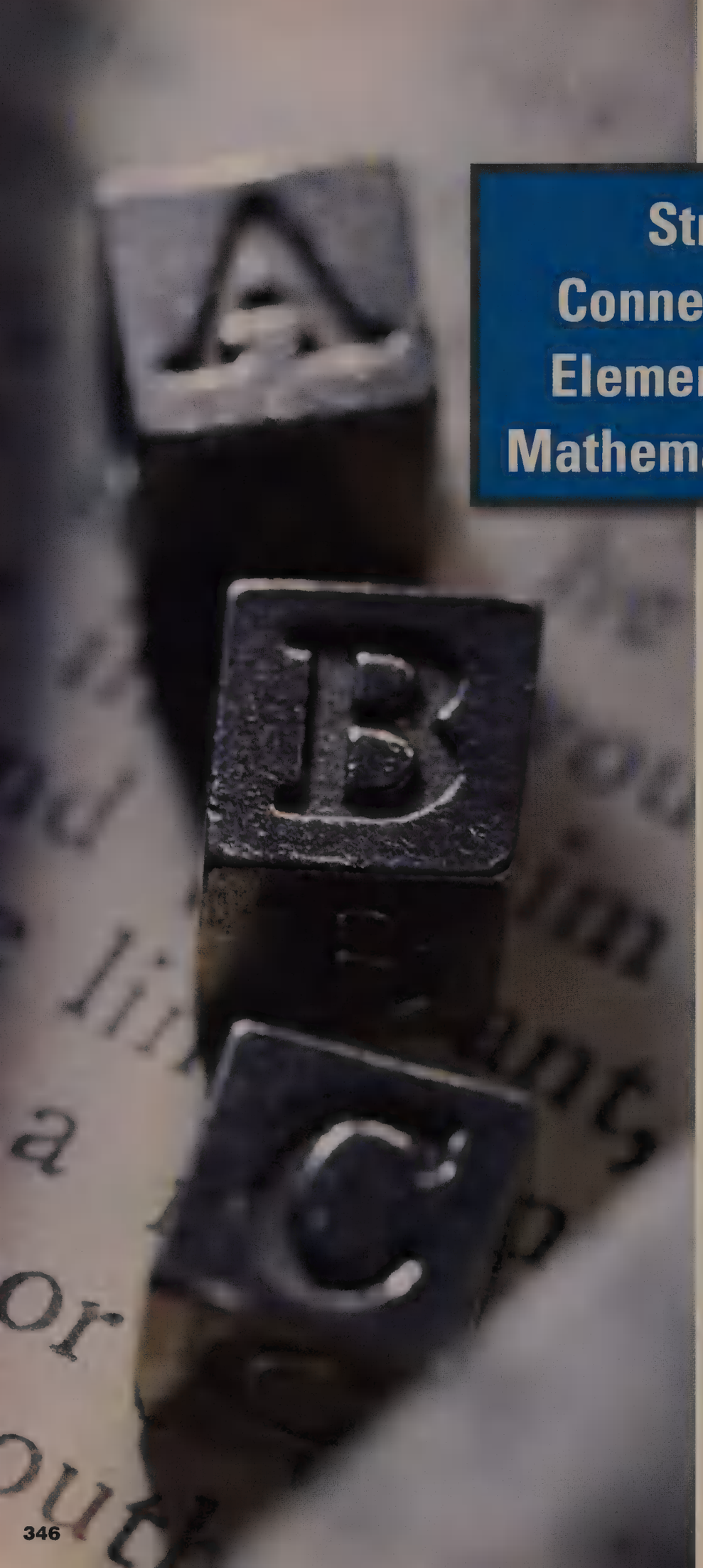

Strengthening Connections between Elementary Classroom Mathematics and Literacy

Enjoyable literacy strategies help elementary teachers reinforce students' mathematical knowledge.

Mathematics is an integral part of an elementary teacher's day; so is the important task of helping children develop literacy skills and knowledge. If teachers address isolated content areas, a school day will never have enough hours for them to teach both literacy and mathematics adequately and thoroughly. However, by creating connections between the two, teachers can help ensure that students have ample opportunity to develop both areas. In fact, many researchers stress the importance of integrating literacy strategies into mathematics (Dynak 1997; Readence, Bean, and Baldwin 2004).

Over the years, a dramatic change has taken place in the way we teach mathematics. To develop a firm mathematical knowledge base, students

By Jennifer L. Altieri

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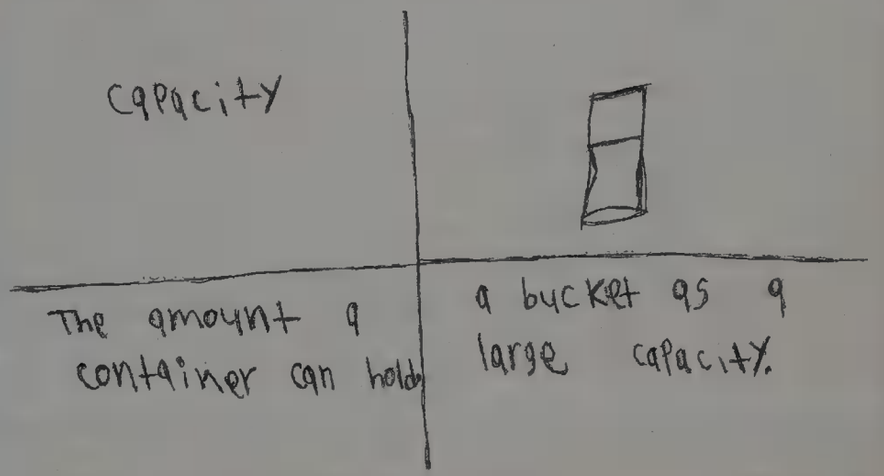
have gone from rote memorization to hands-on exploration. With reform in mathematics instruction, teachers must now reassess how they teach content areas (Draper 2002). NCTM (2000) has ten standards for pre-K–grade 12 that include Problem Solving, Reasoning and Proof, Communication, Connections, and Representation. The Communication and Connections Standards provide natural links between literacy and mathematics. Students must not only understand mathematical concepts but also demonstrate that they are able to connect their mathematical knowledge with their daily lives and communicate their knowledge to others.

This article provides teachers with a variety of enjoyable, motivational literacy strategies to help students reinforce their mathematical knowledge. (Grammar and spelling corrections were not made to the student examples discussed in this article.)

Figure 1

Carlene used the verbal and visual word association strategy to review the term *capacity*.



Word Associations

The verbal and visual word association strategy (Readence, Bean, and Baldwin 2004) is one of my favorite tools for reinforcing vocabulary development at all grade levels. This strategy provides a wonderful opportunity to help students remember terms they have learned in mathematics. Younger students may choose to complete the strategy on paper; older children may prefer to use an index card divided into four parts. First, the child writes the review word in the upper left-hand section. In a box to the right of the word, the child draws a picture to help remember the concept. The bottom left part of the page is for the definition of the mathematical term, and the child writes a sentence containing the term in the bottom right corner.

Carlene, a second grader in a local elementary school, was reviewing the term *capacity*. To help her remember the concept, the teacher had Carlene use the verbal and visual word association strategy. Completing this strategy (see **fig. 1**), Carlene demonstrated her understanding of the term *capacity* by defining it, drawing a picture, and giving an example related to the term. These activities will help Carlene to remember the term later. (Some teachers choose to have students list a nonexample for the term instead of writing the sentence in the bottom right section, but the nonexample may confuse some children.) By drawing a personally meaningful picture to remember the concept, children make connections between the word and something they have experienced. Drawing is an effective strategy for struggling readers and English Language Learners (ELLs) to learn concepts. When children use a word in a sentence, they show knowledge of the term's definition and that they can relate the term to other concepts.

When students understand how to complete the four sections, ask them to create a few additional cards at home by selecting from any new concepts that they learned that day during their mathematics activities. After reviewing the completed cards, choose some cards for games in a center: Have younger children (1) complete a word sort by looking for meaningful relationships among the definitions or (2) form a word wall bulletin board with the cards. The more often that children see, write, and read a mathematical term, the more likely they are to remember it.

Multimeaning Word Cards

Many mathematical words have different meanings in daily life. These include words like *foot*, *volume*, *scale*, and others. To many students, the difference is obvious, but such terms can prove especially difficult for ELLs. Even words that sound phonemically similar to mathematical concepts can be confusing. Twenty years ago, during my first year of teaching, I told my first graders that we were going to work with *liters* that day. When I asked if anyone had heard the word before, many hands waved in the air. But my astonishment was short-lived. When I called on Lakeisha to explain what I meant, she proudly informed the class that "a *liter* is the first person in line at the pencil sharpener." As the rest of the children nodded knowingly, I

realized the importance of clarifying mathematics concepts. Teachers must ensure that students understand that words can have multiple meanings depending on the context and that some words used in mathematics sound similar to everyday words.

An in-service teacher in my graduate literacy course related the story of a student in her classroom: To review the term *volume*, the student folded a sheet of paper in half and wrote the word at the top center of the page. She then thought of two different meanings for the word. The student drew a glass of liquid on one side of the piece of paper and wrote the word *amount*. Then she drew a stick figure on the other half and wrote "how loud someone is" (see **fig. 2**). By developing this card and talking with the teacher, the child was able to solidify her understanding of a word that can have more than one definition.

Games

Years ago, one of my preservice teachers introduced me to a popular game called I Have ____. Who Has ____? Since that time, I have used it repeatedly to reinforce sight words in the primary classroom. Each student receives a card similar to the one in **figure 3**, with the blanks completed beforehand by the teacher. I begin with a list of sight words. Then I select one of the words from the list and write it in the first blank on one card. A second word from the list goes in the second blank. Each word will occur on two different cards. For example, if I have five children reviewing colors, I might create the following five cards:

Card one: I have orange. Who has brown?
Card two: I have brown. Who has purple?
Card three: I have purple. Who has blue?
Card four: I have blue. Who has yellow?
Card five: I have yellow. Who has orange?

After creating the cards, I shuffle them and pass them out to the children, taking each child's knowledge into consideration. Students who are developing English language proficiency or are struggling readers receive cards with easier concepts or words.

One child begins the game by reading the second line of her card, "Who has _____?" The student who has that same word in the first space of his card reads next. I have had many younger children clap enthusiastically when the game ends and ask to play it again.

This game can easily be tied into the mathematics classroom at multiple levels and takes minimal time while providing a fun review of facts. Try mathematical equations in the blanks. For instance, review the multiplication tables: "I have 40. Who has 8×8?"

Use the game to review words with multiple meanings: "I have volume. Who has a tool to determine or compare weight?" and "I have scale. Who has the total of an addition problem?" The child who has "I have sum" as the first line on his or her card then continues the game.

Make this into a Guess My Rule game by stating the rule in the second blank on each card and an answer to a different rule in the first blank. For example, two cards in a Guess My Rule game might look like the following:

Card one: I have circle. Who has a shape that has four equal sides?
Card two: I have square. Who has a shape with three equal sides?

Figure 2

Using the verbal and visual word association strategy, a student solidified her understanding of a word that has more than one meaning.

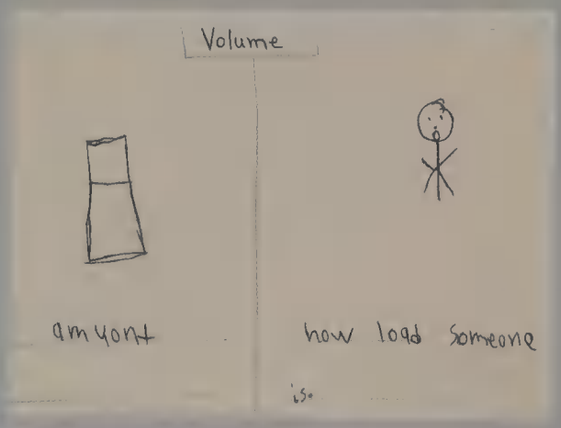


Figure 3

Games are a fun way to reinforce and review sight words in the primary classroom.

I have _____.

Who has _____?

This game is not intended to take the place of hands-on activities to develop a firm grasp of mathematical concepts. It is a motivating, quick review activity for when you have only a few minutes available in class.

Definition Concepts

Schwartz and Raphael (1985) originally developed a type of graphic organizer to help students use context to determine a word's meaning. However, this strategy can be modified to encourage a discussion of broad mathematical concepts and help ensure that students have a firm understanding of meanings beyond stating dictionary definitions.

List the word or concept in a card's center. Have the student list the category, several ways to use the word in the world around us, some examples, and a comparison. (See **figure 4** for an example for the topic *metric*.) After students create their card, they read it by starting with the term and reading clockwise, an approach that makes sense to children, especially beginning readers. In the example, the class created the following definition for *metric*:

> Metric is a way of measuring. We use metric when we visit other countries, measure large amounts of soda, or measure distance in a race. Some examples of metric include liters, grams, and kilometers. Metric can be compared to the American Standard system of measurement.

This strategy appears to work well with broad concepts such as estimating or metric. Examples can be clarified verbally, and students can talk about how the topic is used in their life. Also, the class can discuss a comparison word for a concept and then agree on the one that makes the most sense and why. When a class completes this strategy together, the discussion that creates the definition is as valuable as the final product, the definition.

Admit and Exit Slips

Many teachers I have worked with prefer to use *admit slips* or *exit slips* (Robb 2003), but some teachers use both. As part of my graduate course requirements, I often ask students to connect literacy with content areas. One graduate student, an elementary school principal, taught a lesson on perimeter to a class at her school. At the end of class, she gave each student an exit slip and asked the students to explain perimeter to her (see **fig. 5**).

Figure 4

When a class brainstorms a topic, they might develop this example.

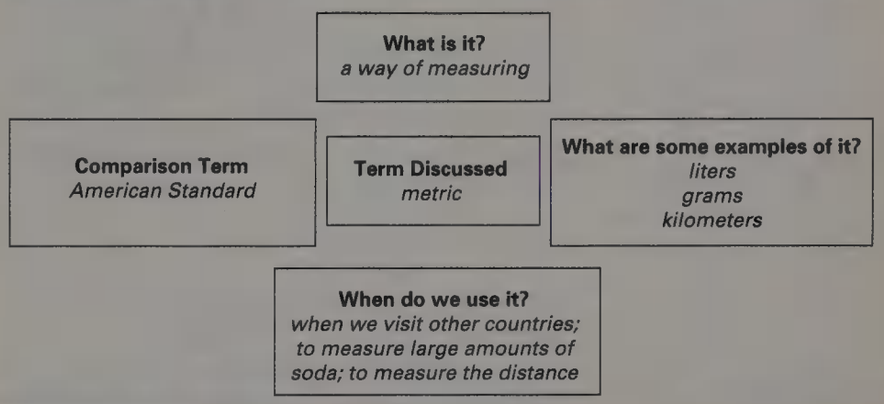
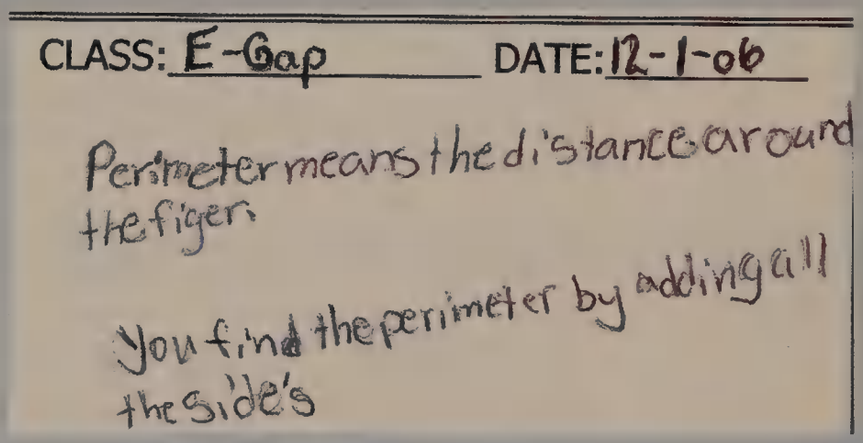


Figure 5

Tyrone's exit slip shows what he knows about the concept of perimeter.



Although Tyrone's exit slip contained several misspelled words, the content allowed Tyrone to show what he knew about the concept. The teacher could have gained even more knowledge by asking the class to include examples of when they might want to find the perimeter of an object. They could also use the slip to explain differences between two concepts.

To help students remember what was discussed in class and articulate new knowledge they gained, have them write on admit and exit slips (a) one new thing they learned that day in mathematics and (b) why it is important. Or ask students to state anything from the lesson that is still unclear. By asking this question, the teacher may develop a better understanding of which concepts need further clarification. Children too shy to announce to the entire class that they do not understand something may not mind privately sharing their difficulty with the teacher.

Teachers who prefer using the slips at the beginning of class may ask students to write one way they used mathematics at home the night before. This exercise helps students see the value of mathematics outside the classroom and make connections between mathematics and the real world. Admit and exit slips are an excellent way to have children communicate their mathematical knowledge. The activity takes minimal class time, and it helps pull the lesson together.

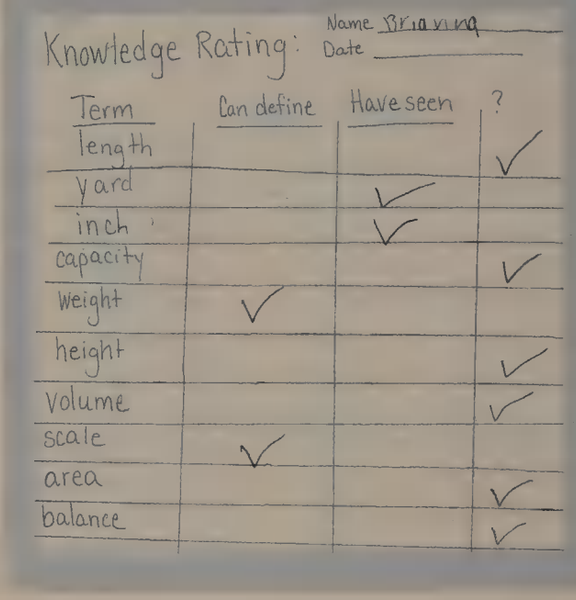
Knowledge Ratings

As an alternative to vocabulary notebooks (Blachowicz 1986) for students to rate their knowledge level of new concepts, read a list of six to eight concepts. Have students check a column denoting their level of knowledge for each concept. A primary-grades teacher, Ms. M, was discussing *measurement* with one of her students, Brianna. Ms. M selected a number of measurement terms and listed them down the left side of a paper. After reading the list of terms with Brianna, she invited the student to rate her level of knowledge and share what she knew about each term on a knowledge rating sheet (see **fig. 6**). Students must ask themselves a series of questions in order to know which column to mark. First, they must determine if they can define the concept. If that is possible, then they need to mark the first column. If not, maybe they have seen or heard the word before. If so, they check the second column. Finally, if they have never seen or heard the word and feel the concept is entirely new, then they check the last column. For each word, students must check a box in one of the three columns. Teachers may wish to use this to lead into a unit on shapes, metrics, or other mathematical concepts.

The most important thing to remember is that marking columns encourages students to think about their level of understanding. Looking at their completed knowledge rating sheet will not reveal students' misconceptions about a topic. Discussion is essential. Students might fill out the sheet on their own and then share in a large-group discussion after everyone has had time to think about the words. Teachers must then ask critical questions of their students. They may ask students who put a checkmark in the first column to share their definitions with the class. Perhaps other students can add information to clarify or improve the definition. Students can modify their sheets if they realize that they had a misconception, and this can serve as a teachable moment.

Figure 6

Rating sheets encourage students to think about their knowledge.


Knowledge Rating: Name Brianna Date

Term	Can define	Have seen	?
length			✓
yard		✓	
inch		✓	
capacity			✓
weight	✓		
height			✓
volume			✓
scale	✓		
area			✓
balance			✓

The process is key to this strategy; much more information is gained from the process (the activity and oral discussion) than from merely analyzing the product (a knowledge rating sheet). Rating one's knowledge can be a motivating activity to introduce the next topic of study, and through discussion, the teacher can help clarify misunderstandings that children have about different concepts. The teacher also develops a firm grasp of students' knowledge levels for the topic and which students need extra help or reinforcement in certain areas.

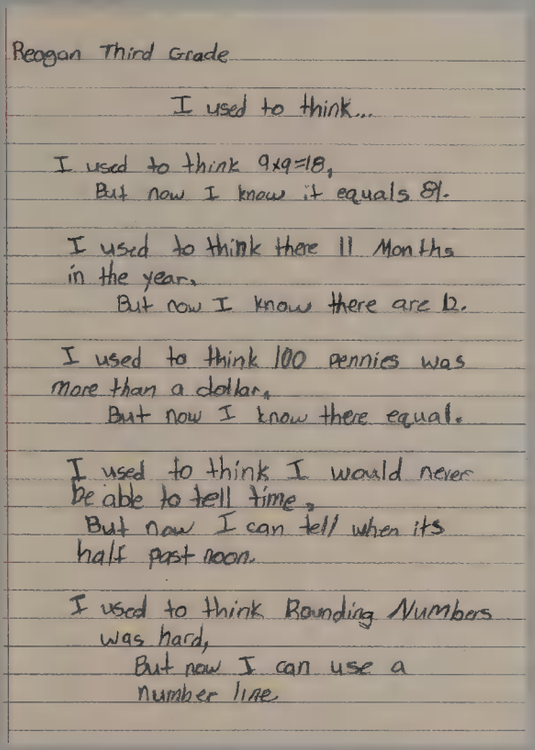
After students have an opportunity to explore a concept through hands-on and small-group activities, again allow them to rate their knowledge on a new sheet. Seeing their growth on a knowledge rating activity sheet encourages and motivates students.

Poems

Writing poetry integrates easily into mathematics (Altieri 2005). Third-grade teacher Ms. P initially talked about the process of change with a student named Reagan. Then both of them looked at a photo of Reagan when she was three months old and talked about how she had changed. Reagan pointed out, "I used to be bald, but now I have lots of hair." They continued discussing other changes related to height, food, and clothing. Then Ms. P asked Reagan to tell her what she had learned in mathematics. Finally, Ms. P invited Reagan to use a

Figure 7

Writing poetry provides another way for students to organize their thoughts about mathematics.



Reagan Third Grade

I used to think...

I used to think 9x9=18,
But now I know it equals 81.

I used to think there 11 Months in the year.
But now I know there are 12.

I used to think 100 pennies was more than a dollar.
But now I know there equal.

I used to think I would never be able to tell time.
But now I can tell when its half past noon.

I used to think Rounding Numbers was hard,
But now I can use a number line.

formulaic poem to reflect on how her mathematical thinking had changed (see **fig. 7**).

The "I used to think, but now I know" formula (Tompkins 2001) is especially beneficial to teacher and student alike because it shows how students' mathematical thinking has grown over time. This type of poetry provides yet another opportunity for children of any age to organize their thoughts about mathematics and convey the ideas to others. Children are constantly developing dispositions toward mathematics, and we want these to be positive. Allowing them to reflect on their increased understanding of mathematics can only strengthen positive attitudes toward the content area.

Strengthening student connections between mathematics and other content areas and aspects of their lives helps children improve both their literacy and their mathematical skills. Allowing them to read, write, and draw about their knowledge does just that. This article touches on seven strategies to help bridge literacy and mathematics. The examples can spark a continuing dialogue among teachers and encourage colleagues to brainstorm even more ideas for connecting literacy and mathematics in their classrooms.

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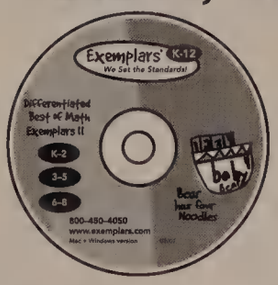

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Christine Gallego, Dora Saldamando, Grace Tapia-Beltrán, Karolyn Williams, and Thomas C. Hoopingarner

The Fine Art of Mathematics

"Math by the Month" activities are designed to engage students to think as mathematicians do. Students may work on the activities individually, in small groups, or as problems of the week for the entire class. Because no solutions are suggested, students look to themselves for mathematical justification, thereby developing the confidence to validate their work.

The theme for this collection of problems was inspired by the late Ted S. Mankowski, father of "Math by the Month" department editor Dana Islas. Ted's passion was art, particularly creating sculptures using geometric shapes. We hope to encourage students to make a connection between art and mathematics and to inspire their passion for learning through the following activities involving number sense and operations, data collection, pattern and algebra, geometry, measurement, communication, and representation. ▲

WEEKLY ACTIVITIES

THE FINE ART OF MATHEMATICS: K–2

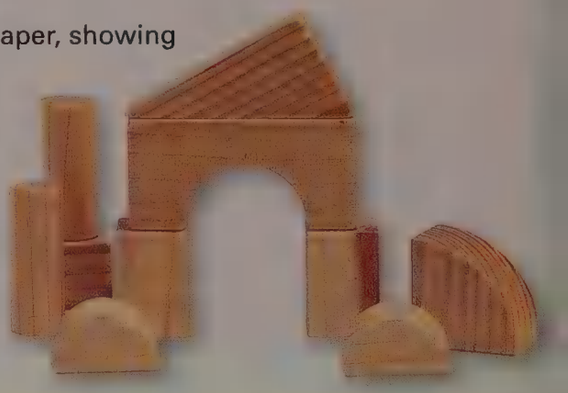
FEBRUARY 2009

2 **Use modeling clay** to form the symbols for the digits 0–9. Next, use counters to show how many each number represents. On paper, draw each of the numbers you made and the counters you used to represent each number.

9 **Design a shape collage** from a variety of die-cut geometric shapes (circles, squares, triangles, and so on) that your teacher provides. Arrange the shapes into a scene. Now glue the scene to a blank sheet of paper. Use another page to write the shape names and how many of each you used. How many shapes did you use altogether? Which shape did you use the most? How many shapes had straight lines? Curved lines?

16 **Choose any number** up to 90. Using your number, create an artistic masterpiece of Popsicle® sticks, cotton balls, buttons, stickers, beans, or any other items that your teacher provides. How many more would you have to add to your collection to make 100? How do you know? How many groups of tens and ones did you use for your masterpiece? Show your masterpiece to a classmate. Add the number of objects the two of you used altogether. Display all the class masterpieces in number order from least to greatest.

23 **Use 10 blocks** to create a structure. Now draw your structure on a sheet of paper, showing its three-dimensional attributes and noticing the various *faces* (flat surfaces), *edges* (where 2 faces meet), and *vertices* (where 3 or more edges meet). How can you show these characteristics on paper? Ask a friend to draw your structure from a different perspective. Compare your drawings; notice how they are alike and different.



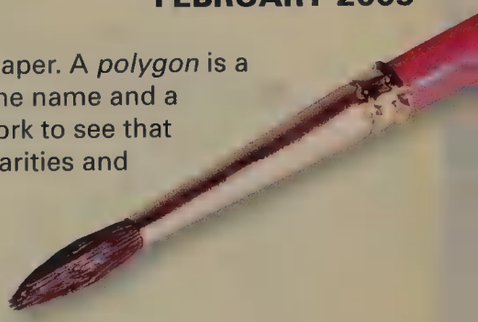
Thomas C. Hoopingarner is the mathematics curriculum and instructional specialist for Sunnyside Unified School District in Tucson, Arizona. The other team members teach at Pueblo Gardens Elementary School in Tucson Unified School District: Christine Gallego teaches first grade; Dora Saldamando teaches fourth grade; Grace Tapia-Beltrán teaches fifth grade; and Karolyn Williams teaches a third-fourth multiage class.

Edited by **Dana Islas**, *who teaches kindergarten at Pueblo Gardens Elementary School in Tucson, Arizona. Readers are encouraged to submit problems to be considered for future "Math by the Month" columns to the department editor at dana.islas@tusd1.org. Receipt of the problems will not be acknowledged; however, those selected for publication will be credited to the author. Readers are also invited to submit creative solutions and adapted problems to* **tcm@nctm.org** *for potential publication in "Reader's Exchange." Please include "Reader's Exchange" in the subject line. Because of space limitations, responses beyond the 250-word limit are subject to abridgment. Submissions are also edited for style and content.*

WEEKLY ACTIVITIES

THE FINE ART OF MATHEMATICS: 3–4

FEBRUARY 2009



Using only polygons and colored pencils or paint, draw a picture on construction paper. A *polygon* is a flat, closed shape with three or more straight sides (a triangle, a square, etc.). Write the name and a description of each type of polygon that you create. Ask a classmate to check your work to see that the polygons' names and descriptions agree. Post everyone's work. Discuss the similarities and differences between your drawings.

With paper circles of different sizes and colors, create a pleasing design and then glue it to construction paper. Count how many circles of each size you use and how many circles altogether. Write the numbers as fractions. Which colors are the largest and smallest fractions of your arrangement? Find the *difference* between the two fractions. Now add all the fractions. What do you notice about the sum's *numerator* and *denominator*?

Create a sculpture of roughly $5 \times 5 \times 10$ inches. Work with a partner. Use materials such as wood scraps, wire, aluminum foil, Popsicle sticks, and pipe cleaners. To safely transport the sculpture to where it will be displayed, it must fit snugly into a cardboard box that you will build. Draw a plan and note how to use measurement to create the box. Now, construct the box. Did you fold your box or piece it together? Record each step, including mistakes, how to correct them, anything you learned, and suggestions for others making boxes for their sculptures.

Use a straightedge or your fingernail to sharply crease 4 random folds on an 8 1/2″ × 11″ sheet of colored paper. Unfold the paper and study the polygons that your folds formed. Label each with its geometric name. Predict whether each angle is right, obtuse, or acute. Now use a protractor to measure each angle; record its measurement in degrees. Cut the polygons apart along the creases. Sort and classify your polygons by shape; number of angles; types of angles; and lines of symmetry, congruency, and similarity. Record your findings. Arrange your polygons into a pleasing abstract design. Glue the design to construction paper. Name your artwork and share it with classmates.

WEEKLY ACTIVITIES

THE FINE ART OF MATHEMATICS: 5–6

FEBRUARY 2009

You have a painting that measures 172 square inches. Will you be able to use a 10″ × 10″ frame that you found? Explain why or why not. Predict what happens if you increase the sides of the frame by 10 percent. Use a sheet of graph paper to represent a 10×10 square. Next, increase the sides of the square by 10 percent to check your prediction. What happened to the area? What do you notice as you increase the sides by 15 percent and then by 20 percent? Will you be able to find a square frame that encloses 172 square inches exactly? Explain how you know.

Create a stained glass window that is 16 square units of red, yellow, blue, and green paper. Show 2 or more ways to be certain that you used equal parts of each color. How many lines of symmetry does your window design have? Create similar designs with 9, 25, and 36 square units. Prove that your 4 window designs are similar.

Famous artist M. C. Escher used repeating patterns called *tessellations* to create art. After you study some of his work, make your own tessellation: Draw or trace a polygon onto tag board. Cut out the polygon, cut off a piece, and tape it to another side to create a new shape. On plain paper, trace the new shape multiple times to create a tessellation. Shapes cannot gap or overlap. How many did you get on a page? Choose 2 colored markers to create a pattern on your tessellation. Do all shapes tessellate? Which shapes tessellate best? From observing your pattern, state the percentage of each color in your tessellation. Compare tessellations and findings with your classmates.

Chart human body ratios by working in groups of 4 to find these measurements for each group member: height, foot length (heel to tip of longest toe), foot to knee (flat foot to knee joint), knee to hip joint, hip joint to shoulder, shoulder width, elbow to wrist, wrist to longest fingertip, longest fingertip to longest fingertip with outstretched arms. What percentage of the height is each of the other measurements? Note any patterns or relationships. For each measurement, find the average (mean) of your group's data to create a proportional drawing of a person on a large sheet of butcher paper. As a group, discuss how knowing the relationships between body measurements can be useful to an artist.

A Venue for Peer Reviews

Online conversations help teachers engage in constructive criticism and attend more carefully to aligning lesson plans with problem solving.

As part of a recent weeklong professional development workshop, we asked teachers to read an article entitled "Teacher Leadership: A Promising Paradigm for Improving Instruction in Science and Mathematics" (Pellicer and Anderson 2001). The article emphasized the idea that teachers must work with principals and other school administrators to develop a shared vision of how mathematics instruction should be carried out within a school building. It argued against the notion that principals are the sole "managers" or "instructional leaders" and teachers are simply the "workers." We asked teachers to consider the ideas in the article because we felt it was important for them to view themselves as educational decision makers who share the responsibility for designing and implementing lessons that promote mathematical problem solving and student understandings. We hoped that reflecting on such issues would encourage the teachers to take more active instructional leadership roles in their schools and districts.

By Randall E. Groth and Claudia R. Burgess

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When participants in our workshop began their discussion, one of the first sentiments expressed was that the ideas in the article on teacher leadership are out of date. Some claimed that the article is no longer relevant because "doing teacher leadership" is not an option in today's schools. Their perceptions were that with the rise of high-stakes testing, many administrators have become prescriptive about enforcing curricular mandates. The teachers felt that in light of administrators' prescriptive bent, the task of making pedagogical decisions is no longer among teachers' primary responsibilities. Such perceptions of reduced professional power caused by high-stakes testing resonate with those of other U.S. teachers (Cochran-Smith and Lytle 2006). We were disconcerted by the idea that teacher leadership and collaborative, instruction-oriented decision making could ever be considered "out of date." Therefore, we began to think seriously about how we might facilitate a shift in teachers' perceptions of themselves and other teachers as instructional leaders and curricular decision makers in collaboration with school administrators.

An Online Extension

Fortunately, the workshop did not mark the end of our contact with the teachers. As part of a collaborative project among several school districts, teachers were scheduled to extend their professional development via an online discussion that

followed the weeklong, face-to-face workshop. The online discussion was intended to help teachers think about how pedagogical ideas from the workshop might translate into the context of their own classrooms. Because teachers were separated by distance and scheduling constraints, the online discussion board served the practical purpose of facilitating pedagogical discussions that otherwise would have ended at the workshop. In addition, these online conversations were intended to—

- encourage contributions from those who are ordinarily silent in face-to-face settings (Groth 2006);
- allow teachers extended time to reflect on one another's comments (Newell et al. 2002); and
- spark productive debates about pedagogical ideas (Groth and Bergner 2007).

As a way of helping teachers perceive themselves as professionals who should be deeply involved in the instructional decision-making process, online discussions were used as a platform for peer review of lessons. Using some of the resources provided during the workshop, teachers developed the lesson plans for the peer review process (Burns 2000; Cannon et al. 2001; Cuevas and Yeatts 2001). They put the lesson plans into word processing documents that could be posted to the discussion board for the rest of the workshop participants to read. At the end of the workshop, teachers were told that they would be reviewing one another's lessons online. The group then brainstormed norms for how they should critique the lessons. They decided, for example, that they should exhibit professional behavior in critiquing one another and that they should not take criticism of a lesson personally. They also agreed that because problem solving had been one of the themes of the workshop, lessons ought to be examined for whether or not they actually exemplified the NCTM Problem Solving Process Standard (NCTM 2000).

Logistics

The online conversations took place over five weeks immediately following the face-to-face workshop. The thirty teachers involved were randomly divided into two discussion groups of fifteen teachers each. Each group was given access to only one of the discussion boards. At the beginning of each week, three of the teachers' lesson plans were posted to each discussion board; and teachers were asked to post feedback they believed would be helpful for improving, refining, or extending the lessons. Teachers' names were removed from the lessons in an attempt to set everyone at ease when posting constructive criticism. Participants were asked to make at least four discussion posts each week—two early in the week and two later in the week—allowing teachers a chance to read and reflect on others' contributions to the conversation.

To set up the online discussions, we used the software program Desire2Learn, which all teachers had used in the past as part of a preworkshop online discussion. The fact that the participants were already familiar with the software helped to avoid some of the technical issues that can impede the progress of an online conversation (Salmon 2004). Readers interested in trying similar activities with teachers may want to consider holding an initial face-to-face meeting with the teachers to help them understand how to make their first posts to a discussion board, respond to the posts of others, and download file attachments. Readers may also want to look into using Web sites that allow users to set up online discussions without charge (e.g., groups.google.com and groups.yahoo.com).

Discussion board dynamics

As the online discussions unfolded, interesting group dynamics emerged. In this section, we will describe some patterns of teacher interaction by outlining the following topics:

- The dynamics observed during initial discussions
- Our attempt to initiate a shift in interaction patterns
- Our observations of how our actions as discussion moderators influenced interaction patterns

Initial Discussions

As teachers began to read and comment on their peers' lessons, some of them indicated that the lessons helped them add to their personal repertoires of pedagogical techniques. For example, after reading a lesson on using audible patterns to help introduce the idea of numerical patterns (see **fig. 1**), one teacher commented, "Great idea about the music instruments.... I'm the band director after school, and I have percussion equipment in my room! I'll have to remember that for patterns lessons next year." (Activity sheets referenced in the figures appear on pp. 361–62.)

Teachers also acquired pedagogical ideas from the lessons via Web site links embedded in lesson plans. Two of the sites that teachers had explored during

Figure 1

Lesson on audible patterns

Pattern Lesson* for Early Grade 4

Objectives
Students will—

1. identify repeating and growing patterns
2. observe patterns and their relationships to one another;
3. identify conjectures, and make and test them with patterns; and
4. discuss, verbalize, generalize, and represent patterns and their relationships.

Materials

- Colored tiles
- Manipulatives (pattern blocks, counting cubes, etc.)
- Paper, graph paper
- Laptops or computers

Before the lesson
The teacher will do the following:

- Introduce a pattern of clap, clap, clap, snap, clap, clap, clap, snap ... and have students tell things they know about the pattern (identifying the sounds verbally; a, a, a, b pattern; square, square, square, triangle; and so forth). If repeated pattern is not mentioned, write this term on the board and discuss it.
- Provide students with manipulatives to build the pattern. They may also draw or cut out shapes to represent the corresponding letters of the alphabet. Have them share their findings with the class and discuss various methods used.
- Have students practice pattern identification with the selection of Mastermind at nlvm.usu.edu/en/nav/grade_g_2.html (use four pegs).
- Have students create a pattern of their own to share with the class or group. This can be done with any of the manipulatives. Promote discussion of the choices and manipulatives used.
- Introduce a second pattern of clap, snap, clap, snap, snap, clap, snap, snap, snap. Again, have students tell what they know about the pattern. Provide them with manipulatives or paper to create a growing pattern of their own. Share and discuss these patterns with the class.
- Assign them the task of identifying what the tenth iteration of the pattern will be if this pattern continues. Identify and discuss strategies to arrive at the solution. Discuss how their attempts to extend their patterns are called conjectures.
- Tell students that they will be working with the task of designing a two-colored square patio for their residence with a garden in the center and border tiles around it. Let them choose which color the garden will be and which will represent the border (blue and white are used here).
- Show them a picture of the three smallest patios that can be built and let them build them (see **activity sheet 1A**).
- Have students use the table to record the number of blue and white tiles for each patio (see **activity sheet 1B**).
- Explain that their task is to determine what the fifth patio would look like and think about the number of blue and white tiles that will be used.

During the lesson
The teacher will do the following:

- Allow students to use manipulatives to work through the next two patterns.
- Encourage them to think about the number of (white) border tiles with the given number of blue tiles. Reverse this and think about the given number of blue tiles with the number of (white) border tiles.
- Ask students to describe their strategies for determining the number of blue and white tiles.
- Prompt them to look for patterns with the blue and white tiles.
- Have students check their pattern or rule with all of the patterns in the chart.

After the lesson
The teacher will do the following:

- Encourage students to share their problem-solving methods.
- Allow students to justify and evaluate their results and methods. Accept all solutions.

Extensions
Students may be asked to—

- discover a rule that describes one or both of the patterns (blue and white tiles);
- graph each set of tiles and determine how they are different (provide graph paper); or
- become familiar with the term linear for the white tiles and discuss how the blue tile pattern grows faster than the white tile pattern.

**Portions of this lesson were adapted from* Navigating through Algebra in Grades 3–5 *(Cuevas and Yeatts 2001, pp. 19–20), and* About Teaching Mathematics *(Burns 2000, p. 112).*

the workshop were used in several of the lessons: the National Library of Virtual Manipulatives (nlvm.usu.edu/en/nav/vlibrary.html) and United Streaming (www.unitedstreaming.com). The former gives students online access to a variety of electronic manipulatives that represent tools usually manufactured in plastic form, and the latter site provides access to video clips aired on educational television. Teachers who reviewed lessons that used these Web sites frequently commented that they planned to use the Web sites with their own students.

Discussing lessons also helped facilitate conversations about curricular expectations across the pre-K–8 spectrum. This was largely because twenty-two teachers taught grades 3–5, and eight others taught grades 6–8. One example of a cross-grade-level discussion took place after a middle school teacher read a fourth-grade lesson on functions (see **fig. 2**):

> As a middle school teacher, I really appreciate the fact that you are using function tables in fourth grade. I have found in the past that many students haven't used function tables a lot by the time they reach middle school. It is a really good lesson, and I think students will like the real-life feel of it. How much time do you think it will take a fourth grader to complete this whole activity?

This comment prompted a brief discussion about the time frame for the lesson. It also led to another middle school teacher's observation:

> I know that we were not introduced to an actual "function table" until high school, and I can remember having such a difficult time grasping the idea even though math was my strong point. I feel that I was over-thinking them and if function tables had been used in lower levels of mathematics, I would have been much more comfortable with them!

Hence, the online peer review of lessons sparked some of the discussions across grade levels that are commonly referred to as "vertical articulation."

During the first two weeks, teachers began to post ideas for improving the lessons that were under review. For a lesson that involved getting students to make comparisons among graphs of linear relationships, one teacher suggested putting graphs on overhead transparencies and laying them on top of one another so students could more easily compare and contrast the graphs. The teacher who

Figure 2

Fourth-grade lesson on functions

Function Mathematics

Objectives (Grade 4 MD VSC)*
Students will—

1. (1A1a, b, c, d) identify, describe, extend, and create numeric patterns and functions;
2. (1C1b) identify positions on a coordinate plane;
3. (3C1a, b) determine area and perimeter;
4. (4A1a) collect, organize, and display data;
5. (4B1b) interpret line graphs; and
6. (6C1a) add whole numbers.

Materials

- Counter chips
- Paper, graph paper
- Colored pencils

Beginning
The teacher will do the following:

- Have children complete the pizza table with a partner (see **activity sheet 2A**).
- Ask children to explain how they completed the table.
- Ask, "What do you notice about all of the numbers of slices?"
- Ask, "Why do you think we have an even number of slices?"

Extension
If each pizza has 8 slices, how many pizzas will be needed for ___ students? Create another chart that shows this relationship.

Think Pair Share
Think: Have the children complete the cookie function table on their own.
Pair: Then have them explain it to a partner.
Share: Have children explain how their partner filled in the chart. Ask: "What do you notice about the numbers?" (Hint: Look at the odd and even numbers.) Why do you think this?

Extensions
Students may be asked to—

- solve: If each box of cookies contains 2 dozen cookies, how many boxes of cookies would need to be purchased for ___ students? (If needed, ask, "Can you buy just part of a box of cookies?");
- create a chart to show the number of boxes needed;
- use a colored pencil to create a line graph of the results from our pizza chart (provide graph paper);
- let x represent the number of students, and let y stand for the number of pizza slices; and
- graph the cookies per person as well. Use a second colored pencil for the cookies and remember to indicate that in your key.

Think Pair Share
Think: Ask, "How could you describe the difference in the two lines on your graph?" (Hint: Why is one line steeper than the other?)
Pair: Have students pair up with a partner.
Share: Have students share with the class the ideas they and their partner discussed.

**These objectives are in the Maryland Voluntary State Curriculum (MD VSC) for Grade 4. The entire document is available online at www.mdk12.org/instruction/curriculum/.*

wrote the lesson plan posted a note to the discussion board that she liked the change and would use it to improve the lesson. In another lesson, a teacher planned to pose a problem about the best way to construct a rectangular garden against the side of a house, using twenty-four feet of lumber. One teacher who read the lesson recommended having a variety of Popsicle® sticks available to help students act out the problem. The teacher who wrote the lesson plan adopted this suggestion. Through such interactions, teachers began to critique one another's lesson plans.

An Interaction Pattern Shift

Although we saw the beginnings of critical feedback and pedagogical reflection among teachers during the first two weeks, we were somewhat unsatisfied with the depth of the lesson plan critiques. Many posts to the discussion boards could be equated to electronic "pats on the back" for doing a good job with lesson plan design. When teachers criticized others' lessons, they simply provided feedback with small adjustments that could be made to lessons (such as using overhead transparencies or Popsicle sticks as teaching tools). Although we could see that teachers were providing suggestions for simple lesson alterations during the first two weeks of discussion, no teachers at any time posted comments that questioned the overall conceptual structures of lessons. Similarly, no teachers questioned the traditional "show and tell" nature of particular lessons although these lessons claimed to align with the NCTM Standards associated with teaching through problem solving. Given the teachers' lack of questioning and their previously expressed desire to evaluate lessons for their potential to teach through problem solving, we felt it necessary to add a twist to the final three weeks of the online conversation. This twist was intended to promote dialogues that focused on pedagogical questioning. We hoped to help teachers be more careful in examining the structure of the posted lessons and to work collaboratively to develop lessons that more closely resonated with NCTM's Standards (2000).

At the beginning of the third week of discussion, we sent the following e-mail to the teachers:

Subject: New discussions open, with a twist!

Hello. The new discussion board on peer review of lessons is now open. We added a new feature for the final discussions: Sometime during the final three weeks, you can expect to see lessons on the board that we have intentionally constructed to not represent teaching through problem solving. These lessons won't have any special identifying markers but will be mixed in with the rest.

A helpful review of what it means to teach through problem solving can be found on the NCTM Web site at standards.nctm.org/document/chapter3/prob.htm. Keeping the criteria given on this Web site in mind should help sharpen our comments on all the lessons on the discussion board. Have fun!

Changes in Interaction Patterns

During the week of discussion that took place immediately after delivery of the e-mail message, we did not post any fabricated lessons. We did, however, begin to see teachers engage in some critique of the extent to which lessons fostered problem solving. Interestingly, though, their critiques were more focused on their own teaching practices rather than the lessons posted to that week's discussion board. The following discussion board comments reflect this trend:

Teacher A: A lot of times we (in the upper elementary [grades]) go into the pictorial arena of a lesson without doing the concrete or hands-on [part] because of lack of time or an assumption that hands-on [activities] may have already been done at the lower elementary [grades]. And students often still need that concrete, hands-on [aspect] in upper elementary and middle [grades].

Teacher B: I like that you didn't tell them how to organize their data. It is important to be able to organize information on your own. I am guilty of giving the children too much guidance. If a group or a child has difficulty organizing the data, then maybe it would be beneficial to help them devise a method. This year I hope to help students gain more independence.

Such comments highlighted the positive aspects of lessons posted to the discussion board and then used those aspects as mirrors for participants to reflect on what they needed to work on personally. Initially, such conceptual self-critique, rather than critique of another individual's lesson, was prevalent.

During the fourth week of discussion, we began to see teachers make more pointed critiques of others' lessons. Their critiques were the sharpest when it came to evaluating a fabricated lesson (see **fig. 3**) posted to the discussion board. Some teachers criticized the lesson for its transmission-oriented approach to instruction:

Teacher C: The teacher is telling the students what to do and then giving them the manipulatives to practice. It is really a teacher show, guided practice, and independent practice model on the same skill.
Teacher D: Although this lesson appears to be very organized and even has the before, during, and after sub-headings, it seems very traditional in nature.... By traditional, I mean the way most of us have been encouraged to teach math in the past. It was very much teacher directed and was lacking the inquiry component of discovery.

Others focused on criticizing the idea to reward students for finishing their homework quickly at the end of the lesson:

Teacher E: I do not agree with using the race-to-the-finish approach with work completion. With the last component of the lesson, no importance was placed on the problem-solving process. Instead, finishing first was reinforced. I would never use the "get-out-of-homework" idea.... It puts a negative spin on homework and reinforces the mindset that it is a "pain."

Overall, we were encouraged that teachers perceived many of the weaknesses in the fabricated lesson and that they were becoming more comfortable with direct criticism of lesson plans needing improvement.

Conclusion

Our experience with the online peer-review activity suggests that fostering critical pedagogical conversations among teachers is a challenging task made all the more difficult by teachers' reduced sense of power in light of administrative mandates related to the current culture of standardized testing. We were heartened that our online peer-review activity gradually nudged teachers toward forming a community in which pedagogical ideas were critically evaluated and critiqued. At the same time, we acknowledge that much more could be done to build and strengthen such communities. We hope that other teacher educators will use and build on the activities and experiences we have described

Figure 3

Fabricated lesson posted to the discussion board

Grade 5 Lesson on Comparing Fractions

Objectives
Students will—

1. compare and order fractions; and
2. compare, order, and describe decimals.

Before the lesson
The teacher will do the following:

- As a warm-up exercise, ask students to do the following division problems: $3 \div 4$; $5 \div 6$; $9 \div 10$.
- After a few minutes, ask three students who did the problems correctly to go to the board to show their work.
- When the warm-up problems are done, tell students that in today's lesson, they will use the skill of dividing one number by another.
- For a whole-class discussion, ask students to give examples of times when they have compared fractions or decimals. Record their examples on a sheet under the document camera at the front of the room.

During the lesson
The teacher will do the following:

- Give each student a set of pattern blocks.
- Tell students to take out a hexagon pattern block and put a trapezoid pattern block on top of it.
- Tell students to sketch these manipulatives on their paper and write the fraction "1/2" below the picture, because a trapezoid is half a hexagon.
- Tell students to solve the division problem $1 \div 2$ and write the answer as a decimal right below the picture they have drawn.
- Have students take out another hexagon and put a green triangle on top of it. Then tell them to sketch these manipulatives on their paper and write the fraction "1/6" below the picture, because a triangle is one-sixth of a hexagon.
- Tell students to do the division problem $1 \div 6$ and write the answer as a decimal right below the picture they have drawn.
- Teach to a greater range of learning styles by giving students laptops and directing them to the National Library of Virtual Manipulatives to repeat the exercises that involve putting a triangle and a trapezoid on top of a hexagon piece. This approach will help them see fractions in a different way.
- Ask any students who finish early to find two other kinds of virtual manipulatives that could be used to represent the fractions one-half and one-sixth.

After the lesson
The teacher will do the following:

- Ask students to do more problems to extend the work they did with one-half and one-sixth with the pattern blocks. For example, ask students to get out a hexagon and a blue trapezoid and see how many times the blue trapezoid fits into the hexagon. Because the blue trapezoid fits three times, students will write the fraction "1/3," divide one by three, and write the decimal.
- Wrap up the lesson by telling students to note that bigger decimals go along with bigger pattern blocks. So, to compare fractions, students should always change fractions into decimals by dividing the top number by the bottom number. Ask them to write this rule in their mathematics writing journals.
- Using this rule about comparing fractions, have students do some fraction comparison problems with fractions that have numerators bigger than one.
- To motivate students to do the problems during class, announce that the first three people to complete the set of exercises will not have to do tonight's homework.

in this article as they work toward fostering critical conversations among the teachers they serve.

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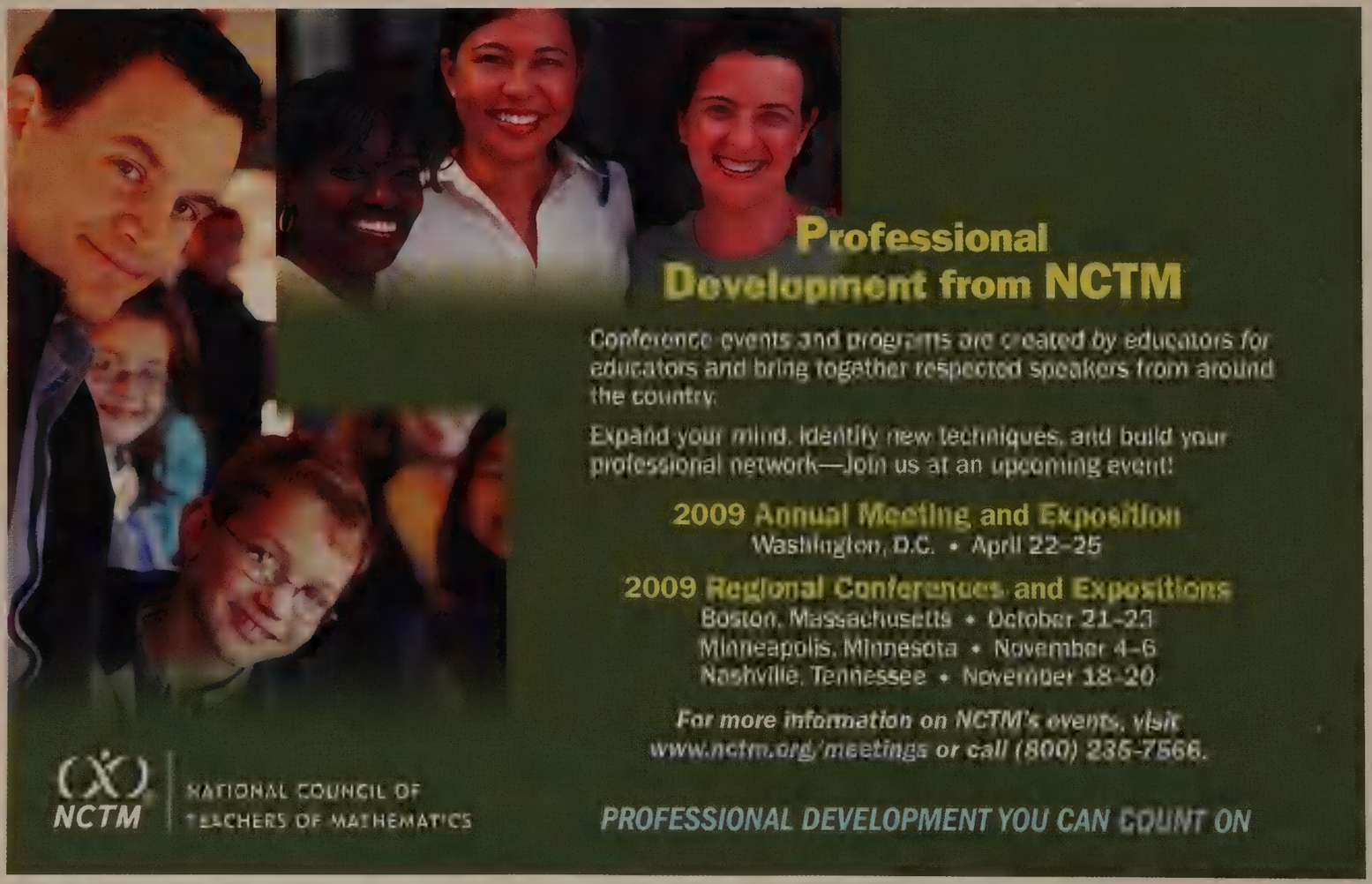
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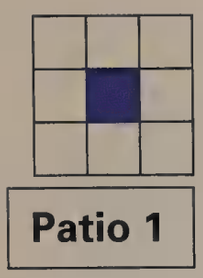
Activity Sheet 1. Online Discussions

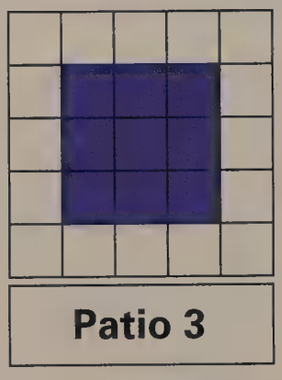
PATIO PATTERNS Name _____

A. If a patio has 49 blue tiles, how many white tiles does it have? Explain how you got your answer.



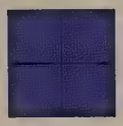
Patio 1

Patio 2

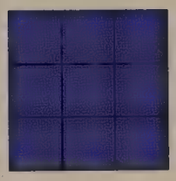


Patio 3

Blue square 1



Blue square 2



Blue square 3

B. Use the table to record the number of blue and white tiles for each patio.

Patio Number	Number of Blue Tiles	Number of White Tiles	Total Number of Blue and White Squares
1. _____			
2. _____			
3. _____			
4. _____			
5. _____			
6. _____			
7. _____			

From the February 2009 issue of TEACHING CHILDREN Mathematics

Activity Sheet 2. Online Discussions

FUNCTION MATHEMATICS Name _____

A. Pizza table. The fourth grade is planning a pizza party. Each child will be allowed to eat 2 slices of pizza. Complete the table below.

Number of Students	Number of Pizza Slices
1	2
2	4
5	
	20
25	
30	
	86
100	
	1000

B. Cookie table. The children will also be able to eat cookies at the party. Complete the table below.

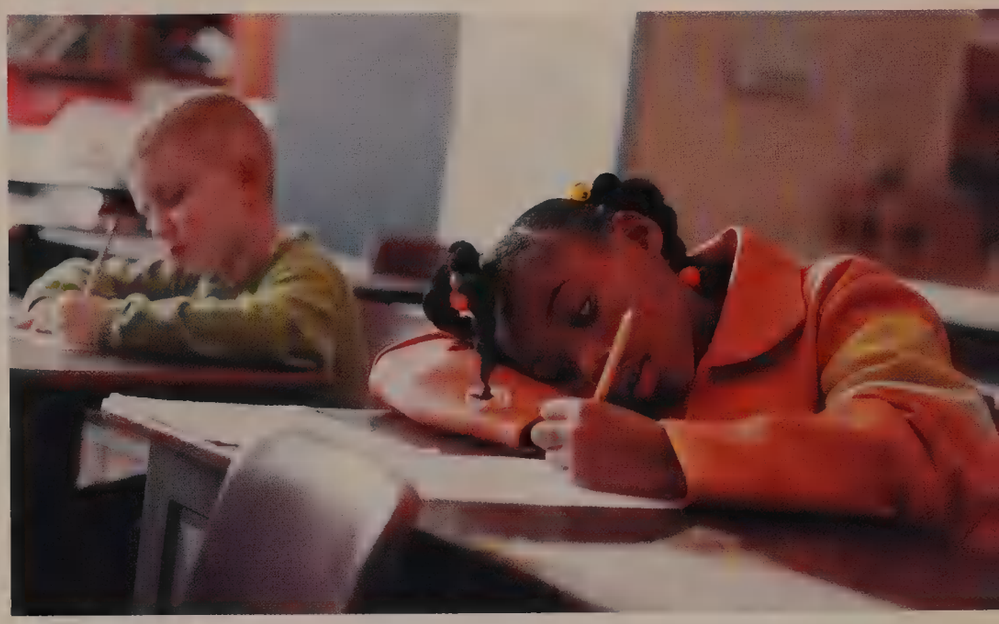
Number of Students	Number of Cookies
3	
6	18
	21
9	27
12	
	45
19	

C. Line graphs. On graph paper, graph the results from your first table. Let x represent the number of students and y represent the number of pizza slices.

Let's graph the number of cookies per person as well. Use a different colored pencil for the cookies and remember to indicate that in your key.

From the February 2009 issue of TEACHING CHILDREN Mathematics

A Generation of Progress: Learning from NAEP



Often referred to as "The Nation's Report Card," the National Assessment of Educational Progress (NAEP) was designed in the 1960s as a tool for monitoring precollege student performance in various subject areas. The original design included assessment of nine-, thirteen-, and seventeen-year-old students. The first mathematics assessment was completed in 1973 with additional mathematics assessments following at two- to four-year intervals. In contrast to assessments like the SAT, which are usually taken by college-bound students only, NAEP is given to a sampling of all students across the United States regardless of ability or aspiration. As such, it is the best available measure of mathematics achievement for the nation as a whole (Kenney and Kloosterman 2007).

With this fact in mind, we looked to NAEP data to determine how students' mathematics skills and their ability to solve various types of problems have changed over twenty-five years. Specifically, this article focuses on overall trends in NAEP results. More specifically, we focus on what the changes in performance on items released to the public suggest about increases or decreases in knowledge, particularly knowledge of basic skills, over time.

The Sample and Overall Results

One of the problems with assessing nine-, thirteen-, and seventeen-year-olds is that students have to be pulled from classes at different grade levels to get a complete sample. Age 9 students, for example, are commonly found in both grades 3 and 4. Thus, to make administration of NAEP easier, the sample was changed with the 1990 NAEP in mathematics to students in grades 4, 8, and 12. Additionally, although some of the items in NAEP were used across multiple years, new items were continually added, and the content of NAEP was adjusted to account for the changing mathematics curriculum. To ensure that the ability to compare student performance over time was not lost, however, the items that were used in 1982 continue to be administered to national samples of nine-, thirteen-, and seventeen-year-olds. The NAEP given at grades 4, 8, and 12 has become known as the Main NAEP, whereas the unchanged NAEP is now known as the Long-Term Trend (LTT) of NAEP. Because the LTT assessment is essentially identical to what it was in 1982, it is the best indicator of change in performance over time.

Because the items in the LTT assessment have not changed, one limitation is that it represents the mathematics that was taught and assessed in the 1970s. The LLT assessment was administered in mathematics in 1973, 1978, 1982, 1986, 1990,

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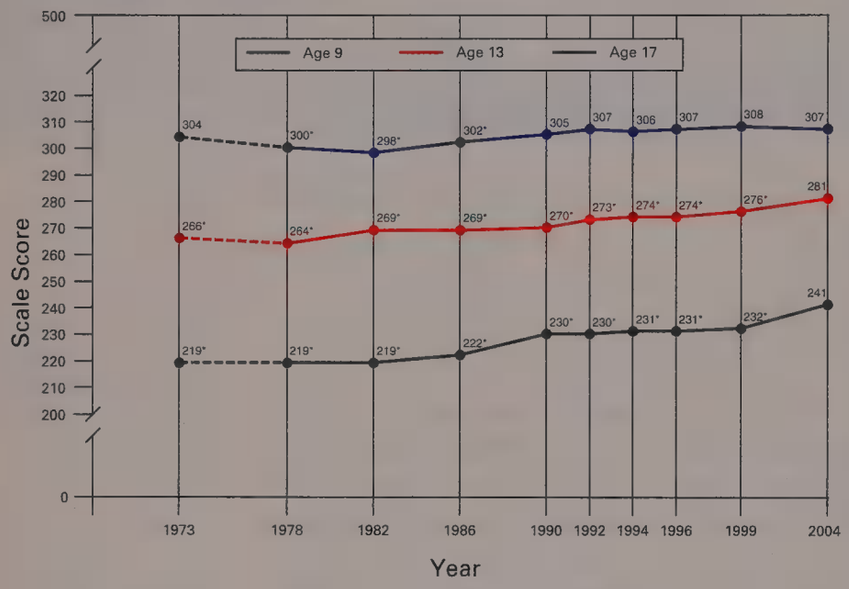
1992, 1994, 1996, 1999, and 2004. **Figure 1** shows the overall mathematics scale scores for each of those years. Scale scores can range from 200 to 500, and, as on the SAT, the scores indicate relative performance of students but are not easily tied to a "percentage correct" on the assessment.

NAEP scores are not intended to be used for comparisons across ages and have no grade-level benchmarks, but our analyses of NAEP data have enabled us to approximate the point gains that might be expected from one grade level to the next. For example, to get from the average age 9 score (241) to the average age 13 score (281) requires a gain of 40 points (or 10 points per year). Age 9 scores climbed substantially between 1986 and 1990 and again between 1999 and 2004. Using the approximation of 10 points per year, the 22-point gain between 1978 and 2004 for age 9 students means that age 9 students in 2004 have mathematics skills about two grade levels higher than their peers in 1978 (see Kloosterman and Walcott [2007] for additional explanation). Age 13 scores climbed 5 points between 1978 and 1982, remained relatively stable until 1990, and then rose 11 more points by 2004. With a difference of 26 points (or 6.5 points per year) between average age 13 and age 17 scores in 2004, to say that age 13 students have also had a substantial gain since 1978 is appropriate. Seventeen-year-olds' scores have been relatively stable throughout NAEP's history.

The overall improvements over the years for age 9 mask the fact that large gaps exist between the average scale scores of white students compared with those of black and Hispanic students. **Figure 2** displays the scale scores over time for these three groups. The disparity between the scores of black students and white students as well as between Hispanic and white students decreased to their lowest levels ever in 2004 (23- and 17-point differences, respectively). As promising as those numbers sound, the disappointing news is that when we think in terms of 10 points representing a year of gain at age 9, we see racial gaps of about two grade levels by that age. Also note that the size of the gaps has varied in the past, so it is unclear whether the current narrowing is a permanent trend. For example, disparity in the scores between white students and black students for the most part decreased from 1978 until 1994, after which the gap began to widen until the most recent administration in 2004, when it decreased by 5 points.

Figure 1

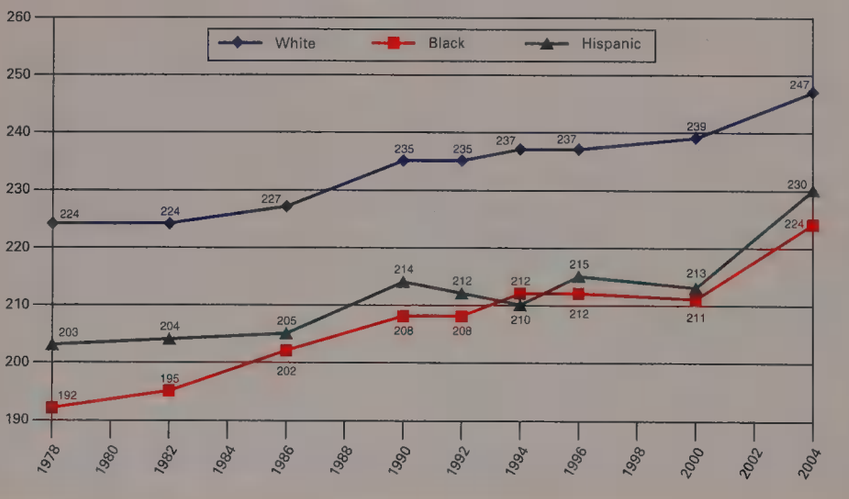
Overall mathematics scale scores on the LTT assessments in mathematics from 1973 to 2004 indicate relative performance but are not easily tied to a "percentage correct."



* Indicates a significant difference from 2004. Dotted lines are used to connect 1973 and 1978 results because the 1973 scores are extrapolations based on limited data (Perie, Moran, and Lutkus 2005, p. 17).

Figure 2

Age 9 mathematics scale scores by racial and ethnic group compared white students to black and Hispanic students over time.



Released Items

For the first time since 1982, a subset of the items used on the LTT assessment was released to the public in 2006 and is now available on the NAEP Web site (Perie, Moran, and Lutkus 2005). (Released items are available under Sample Questions. Clicking on Questions Tool leads to a menu that can be used to retrieve the 2004 mathematics

LTT assessment released items.) Although the Web site includes information regarding performance on the released items in 2004, it does not contain information regarding changes in performance over time. The remainder of this article focuses on performance from 1982 to 2004 on the subset of released items administered to nine-year-olds and compares these results to what is known about grade 4 performance on the Main NAEP. Specifically, overall performance on the LTT assessment has been increasing steadily over time for nine-year-old students, but looking at performance on individual items gives a sense of some topics on which student performance is increasing more than others. Unfortunately, the released items focus on a relatively limited set of mathematical skills; thus, analysis is constrained to conclusions that can be drawn from items that happened to have been released.

Table 1 displays the percentage of students who chose the correct answer for the released items from the age 9 LTT assessment. To better understand the data presented in **table 1**, we consider some of these items individually. To structure the discussion, we have categorized items as involving computation or use of the metric system, and we also describe additional items that have seen substantial changes in performance over the years.

Computational items

Items 3, 4, 5, 11, and 12 (see **table 1**) are the only strictly computational items released from NAEP. Although some released items were multiple choice (e.g., item 2), students were required to write out their responses on all five computational items. Performance on all these items has increased.

Table 1

Percentage Correct on Age 9 Released Items

	Item[a]	1982	1986	1990	1992	1994	1996	1999	2004
1.	$3 + \square = 7$ (Which number goes in the box?)	91	92	95	95	91	93	94	96
2.	Which is worth the most: 35 pennies, 1 quarter, or 4 dimes?	74	74	78	73	78	75	75	77
3.	35 + 42 = ?	90	96	97	96	96	96	96	96
4.	55 + 37 = ?	84	87	90	89	87	88	87	87
5.	59 + 46 + 82 + 68 = ?	46	53	54	52	51	52	52	57
6.	Which one of the following represents "nine tens": 9, 90, 900, or 910?	73	70	76	74	74	77	76	80
7.	Solve a story problem involving subtraction (see **fig. 4**).	72	71	76	76	79	76	78	77
8.	What number is 10 MORE than 95?	67	65	71	72	73	72	76	76
9.	Determine the length of the side of square, given the length of another side.	65	62	65	60	65	62	63	69
10.	What digit is in the thousands place in 45,372?	77	73	79	81	79	80	78	76
11.	$21 \times 3 = ?$	61	65	71	68	72	69	72	71
12.	$314 \times 12 = ?$	14	12	16	11	14	12	13	20
13.	Which is worth the most: 11 nickels, 6 dimes, 1 half dollar?	58	57	62	57	62	58	58	65
14.	How many kilograms does a bicycle weigh: 1.5, 5, 150, 1500?	37	25	23	24	23	23	24	40
15.	Which is the largest unit of measurement: centimeter, kilometer, meter, millimeter?	39	29	38	42	41	35	34	40
16.	Solve a two-step story problem involving money (see **fig. 5**).	34	38	43	43	48	44	44	47
17.	Find the perimeter of a rectangle, given its length and width (see **fig. 3**).	25	19	25	25	29	34	35	51

[a] Exact items, and 2004 performance data for each item, can be found online at nces.ed.gov/nationsreportcard/itmrls/startsearch.asp.

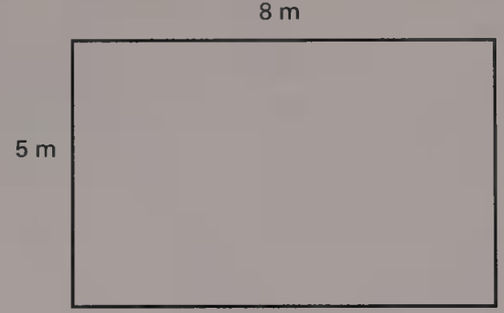
Items 3, 4, and 5 were addition problems. Performance on item 3 (35 + 42 = ?) increased 6 percentage points even though 90 percent of students correctly responded in 1982. Percentages on item 4 (55 + 37 = ?) increased from 84 percent to 87 percent over the period, and, as with item 3, performance was high to begin with. The somewhat lower performance on this item compared to item 3 is likely because item 3 does not require the student to regroup, whereas item 4 does. Item 5 is the most difficult, requiring students to add 59, 46, 82, and 68. Of the three items, item 5 had the greatest increase in performance through the years, moving from 46 percent in 1982 to 57 percent in 2004. This increase is evidence that students' skills are improving on tasks that were challenging to many students a generation ago.

Today's students know substantially more mathematics than their peers of a generation ago

The remaining two items (11 and 12) required students to compute products of two- and one-digit numbers (21 × 3) and three- and two-digit numbers (314 × 12), respectively. Performance on item 11 increased from 61 percent to 71 percent, and performance on item 12 increased from 14 percent to 20 percent. Most of the change for item 11 occurred before 1990, and most of the change for item 12 occurred between 1999 and 2004. Given that gains on both items were relatively modest, it is hard to know if this pattern is the result of changes in curriculum over time or due to the specific group of students surveyed.

Figure 3

Item 17 requires students to compute the perimeter of a rectangle when given the lengths of two adjacent sides of the rectangle.



17. What is the PERIMETER of this rectangle?

A) 13 meters B) 26 meters C) 40 meters D) 80 meters

Metric system knowledge items

Two multiple-choice items (14 and 15) in **table 1** require students to apply their knowledge of the metric system. Although both items involve knowledge of metric units, these problems are subtly different, and performance variances on the items may reflect this. Specifically, item 15 (Which is the largest unit of measurement?) requires students to recall the relationship between different prefixes used in the metric system (e.g., *kilo*, *centi*, *milli*). These prefixes have been a part of the curriculum since the 1970s. Performance on this item has been 34 percent to 42 percent correct and has no real pattern other than the relatively poor performance in 1986, when only 29 percent got the item correct. This poor performance is probably a result of the big push for metrics in the 1970s waning with the back-to-basics movement of the 1980s. The rebounding of performance on item 15 in 1990 is likely connected to renewed emphasis on metrics as part of concern about globalization in the late 1980s.

In contrast to item 15, item 14 (How many kilograms does a bicycle weigh?) requires that students have a sense of how much weight a kilogram represents. In 1982, 37 percent of students got this item correct, but, as with item 15, performance dropped to 25 percent correct in 1986. However, performance on item 14 stayed low, hovering at 23 to 24 percent correct until 2004, when it jumped to 40 percent correct, the highest performance ever. Clearly, something happened between 1999 and 2004 that helped students gain a better sense of what a kilogram represents. We suspect that the gain results from more hands-on experience with kilograms. Some of this gain may be due to emphasis on metric measurement in elementary school mathematics and science lessons, but some may also be due to students' increased exposure to metric units outside school.

Additional items

Having considered computational items and those involving knowledge of the metric system, we now look at other items where significant changes in performance took place over time. Performance on items 7, 16, and 17 increased, with item 17 rising more than any other released item, moving from 25 percent to 51 percent correct. Item 17 requires students to compute the perimeter of a rectangle when given the lengths of two adjacent sides of the rectangle (see **fig. 3**). The most common incorrect response in 2004 was 13 meters, selected by 29 percent of students. Answer A results from simply

adding the two numbers in the problem together—something students often do when they have no idea of where to start. Fourteen percent of students selected 40 meters (answer C), suggesting possible confusion between perimeter and area. Regardless, the large increase in performance on this item is likely a reflection of an increased emphasis on geometry in the elementary grades in recent years, especially since 1999.

Items 7 and 16 are related in that both are story problems. Item 7 (see **fig. 4**) is a typical one-step subtraction problem that includes the key phrase *how many more*. Students have always performed well on this item, with performance increasing from 72 percent correct in 1982 to 77 percent correct in 2004.

Item 16 is a two-step problem requiring students to add two two-digit numbers and then subtract the sum from one dollar (see **fig. 5**). Performance on this item is much lower than performance on one-step items, although it has risen from 34 percent to 47 percent since 1982. The most common incorrect answer was answer B (18¢), which 21 percent of students selected. These students likely got their answer by subtracting 37 from 55, suggesting that they may have seen the question, "How much change does she get back," as an indication to subtract but did not fully grasp the problem's context. An additional 16 percent selected answer E (92¢), which results from adding 55 and 37. In brief, this item demonstrates that although many nine-year-olds have not mastered two-step problems, the percentage of students who have is steadily increasing and thus indicates improving performance on the basic skill of solving multiple-step problems.

LTT Assessment and Main NAEP

As the introduction mentions, comparing performance on these Long-Term Trend assessment items with performance on similar items administered in the Main NAEP is also an interesting exercise. Kloosterman and others (2004) describe items that were administered at various times on the grade 4 Main NAEP throughout the 1990s that involve metric knowledge, computational skills, and computations with money. Specifically, performance on straightforward computational items was high and relatively stable. For example, performance on an unreleased item that requires adding "two three-digit numbers with regrouping" (p. 76) hovered between 88 percent and 90 percent from 1990 to 2003; but performance on other items, particularly more complex items where performance was lower to begin with, was improving. This trend is similar to the one for the released LTT assessment addition items (3, 4, and 5 in **table 1**).

Figure 4

Students have always performed well on this one-step story problem involving subtraction.

7. A school bicycle rack holds 18 bicycles. There are 12 bicycles already in the rack. How many more bicycles can be put in the rack?

ANSWER: _____

Figure 5

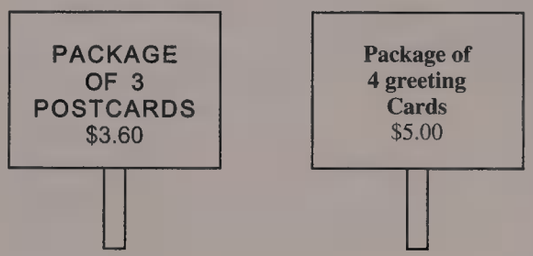
Performance on this two-step story problem involving money is much lower than performance on one-step items.

16. Carol buys a ball for 55 cents and a game for 37 cents. How much change does she get back from $1.00?

A) 8¢
B) 18¢
C) 45¢
D) 63¢
E) 92¢

Figure 6

The most challenging problem in the block has multiple steps and was answered correctly by only 17 percent of students.



16. Rico bought 10 cards, which cost $12.20 before tax. How many packages of each type did he buy?

_____ Packages of postcards

_____ Packages of greeting cards

Explain how you know your answer is correct.

Rico said that one postcard is cheaper than one greeting card. Show that Rico is correct.

Kloosterman's article also analyzed an item that was described as requiring students to "identify the change from a given amount [of money] when buying multiples of a given item" (Kloosterman et al. 2004, p. 80). Performance on this item increased from 27 to 35 percent correct between 1990 and 2003. This item and performance on it are similar to the LTT assessment's two-step problem (item 16 in **table 1**). The Main NAEP item was more complicated, as it involved getting change for purchases that cost more than a dollar, but item 16 and the Main NAEP item indicate that students are improving. Yet, more work must be done in teaching our students to tackle complicated and realistic problems.

With respect to items involving knowledge of the metric system, the Kloosterman article described an item on the Main NAEP that required students to "identify an appropriate unit of length" in metric units (p. 82). This item was administered only in 1996, 2000, and 2003; performance improved from 34 to 36 percent correct over this period. The item is similar to item 14 in **table 1**. Both questions are easier to answer if students have an intuition about metric units rather than just the rote ability to convert between them and English units. On the basis of students' similar performance on these two items, it seems reasonable to assume that about one-third of students in grades 3 and 4 have an intuitive grasp of metric units for mass and length.

NAEP and Your Students

What is the overall message of the NAEP in mathematics? We should be proud of the improvements made in mathematics over a generation.

In addition to the LTT assessment items noted in this article, the Questions Tool on the NAEP Web site (nces.ed.gov/nationsreportcard) contains 262 items released from the grade 4 Main NAEP between 1990 and 2007. NAEP items can be used as assessment and learning tools in the classroom. You could administer a subset of these items to your students to evaluate how well they are doing in relation to their peers across the country. They can then see how they solved problems in comparison to a national sample. You may select items from the site on the basis of mathematical content or difficulty level, or you might give your class the items in the same arrangement that NAEP gives them by choosing all the items in one block. Grade 4 students who take the Main NAEP are given twenty-five minutes to complete a block of items. For example, the 2007 released block M7 has seven items that are considered easy, two that are of medium difficulty, and seven that are hard.

Figure 6 shows that only 17 percent of students correctly answered item 16 from block M7, the most challenging problem in the block. The question requires an "extended constructed response" because it has multiple steps; in addition to providing an answer, students are required to justify their answer. Few students are able to provide a complete answer to this problem. It is more challenging than traditional textbook problems and therefore provides an excellent learning activity for a class discussion on problem-solving methods.

Of the 262 released items, only 11 are classified as requiring extended constructed responses; however, many multiple choice questions and those requiring short constructed responses can be used for both assessment and as tasks for discussion. For instructors who are interested in using NAEP constructed-response items for assessment or class discussion, the NCTM publication *Learning from NAEP: Professional Development Materials for Teachers of Mathematics* (Brown and Clark 2006) includes a CD with a database of student responses to released constructed-response items. This database includes thirty student responses to each of fifteen different grade 4 constructed-response items. These responses reflect the range of possible solutions as well as the common errors for each item. *Learning from NAEP* is intended for professional development workshops on how to use NAEP data to understand student thinking; however, the activities are easy to follow for teachers who want to use the items to assess their own students. In short, although the focus of this article is on gains in mathematics achievement over time, the tool for assessing those gains—NAEP—is also an excellent source of mathematics problems and solutions for the classroom.

Conclusion

So, what do the LTT assessment results reported in this article mean with respect to teaching elementary school mathematics? Similar to overall performance on the entire 68-item LTT of NAEP, performance on almost all the released items has improved significantly since 1982. Moreover, much of this improvement mirrors the Main NAEP results. Students' skills have improved or remained stable across a variety of item types. Performance on items involving only computation has been stable or has improved over time. Performance on more complicated items such as two-step story problems has improved. Finally, students are becoming more adept at elementary geometry. In short, the data are clear: Performance on relatively basic skills is stable or improving, and performance on most items involving more complex skills is improving as well.

More than anything, the analyses reported here suggest that students are learning what is being taught. Performance is improving on item types that have been emphasized in NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) and *Principles and Standards for School Mathematics* (2000). Thus, the LTT assessment and the Main NAEP results provide evidence that Standards-based instruction is having a positive effect at the elementary level. Although the extent to which students experience Standards-based curricula varies by school and classroom, if such curricula were detrimental to computational or problem-solving skills,

then performance on the NAEP would be decreasing. Instead, we see gains on the LTT assessment and the Main NAEP, both overall and on an item-by-item basis. Students know substantially more mathematics than their peers of a generation ago did; thus, we need to keep moving in the same direction.

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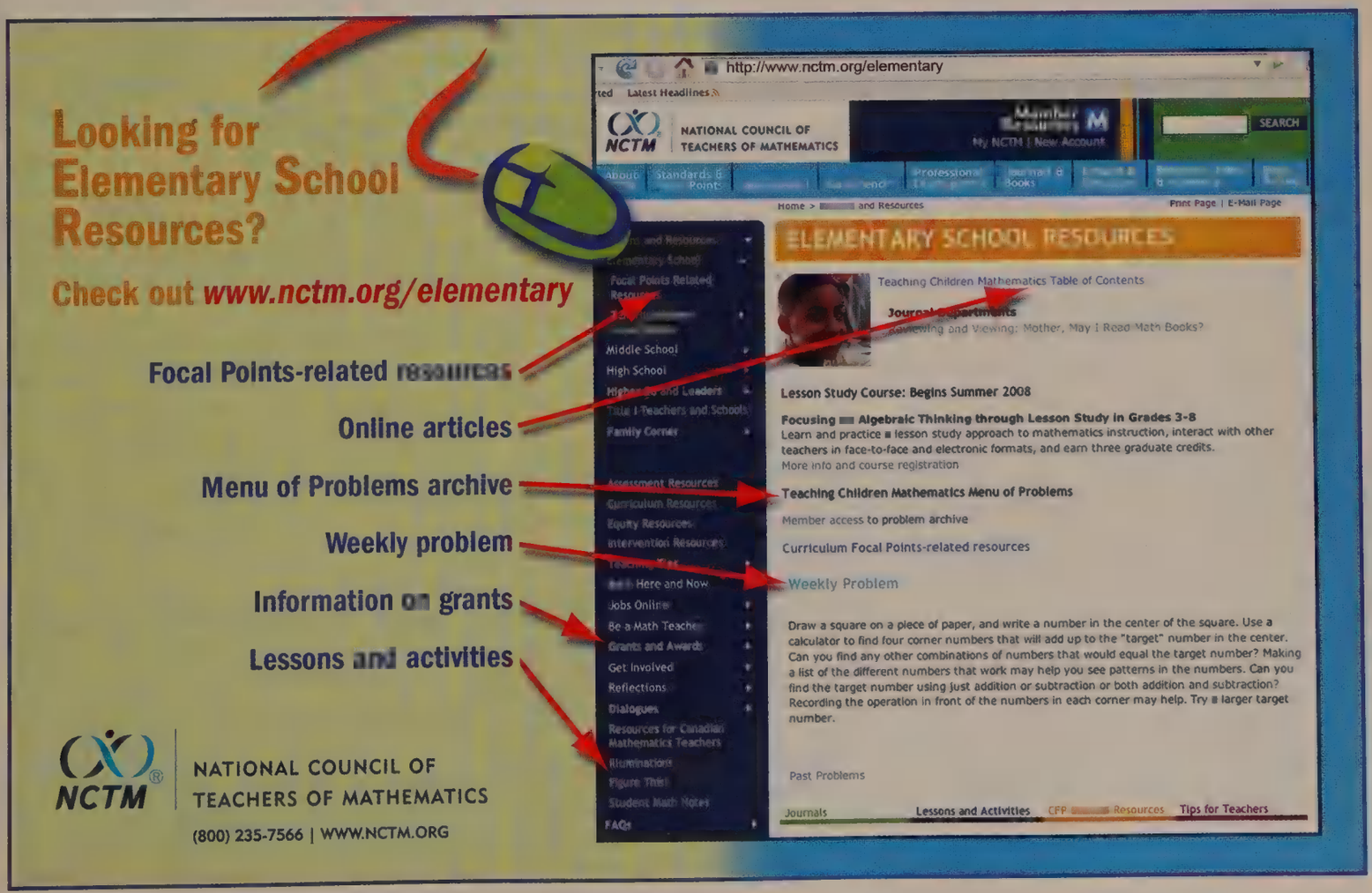
Editors' Note: Readers may also want to read NAEP-related articles that are appearing this month in Mathematics Teaching in the Middle School *and* Mathematics Teacher.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Weekly Problem
Draw a square on a piece of paper, and write a number in the center of the square. Use a calculator to find four corner numbers that will add up to the "target" number in the center. Can you find any other combinations of numbers that would equal the target number? Making a list of the different numbers that work may help you see patterns in the numbers. Can you find the target number using just addition or subtraction or both addition and subtraction? Recording the operation in front of the numbers in each corner may help. Try a larger target number.
Past Problems
Journals
Lessons and Activities
Tips for Teachers

Marvin E. Harrell and Dawn R. Slavens

Using Base-Ten Block Diagrams for Divisibility Tests

"Investigations" features children's hands-on and minds-on explorations in mathematics and presents teachers with open-ended investigations to enhance mathematics instruction. These tasks invoke problem solving and reasoning, require communication skills, and connect various mathematical concepts and principles. The ideas presented here have been tested in classroom settings.

This investigation provides students with a rich context in which they explore and discover divisibility tests using base-ten blocks and diagrams. The activities were carried out in a sixth-grade classroom and took a significant amount of instructional time, but the time was well spent. The activities are also accessible to students in grades 4 and 5.

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"Investigations" highlights classroom-tested multilesson units that develop conceptual understanding of mathematics topics. This material can be reproduced by classroom teachers for use with their own students without requesting permission from the National Council of Teachers of Mathematics. Readers are encouraged to submit manuscripts appropriate for this department by accessing **tcm.msubmit.net**. *Typed, double-spaced manuscripts should not exceed 3000 words and two reproducible pages of activities.*

The Investigation

Learning goals, rationale, and pedagogical context

The benefits of testing whether a number is divisible by another number are often overlooked. Teaching and learning divisibility tests and why they work can be of value when it comes to understanding number sense related to large numbers and place value, simplifying fractions, learning more advanced number concepts such as greatest common factor and least common multiple, and providing a context to practice mental math strategies. A quick review of many mathematics textbooks for grades 4–6 shows little, if any, discussion on how to discover or justify divisibility tests. Often such tests are simply stated, and then students apply these tests to a set of numbers.

Objectives of the investigation

Students will—

- be able to apply divisibility tests for the numbers 2, 3, 4, 5, 6, 8, and 9;
- understand the strategy of decomposing a number into the sum of two numbers and checking each for divisibility; and
- understand why various divisibility tests work.

Materials

All lessons

We recommend that students work in heterogeneous groups of four to complete the lessons within this investigation. For each lesson, each student needs two different colored pencils, a calculator, the appropriate activity sheet, and the accompanying base-ten block diagram page. Calculators are useful for checking the outcomes of the divisibility tests, but most of our students used mental solution strategies. Use base-ten blocks on the first day; you may continue to use them throughout the investigation, but students must have paper representations to shade. **Activity sheets** and accompanying **Base-Ten Blocks pages** follow this article.

Lesson 1 (1 day)

The teacher will need—

- a transparency of **activity sheet 1**, "Divisibility and the Number 2";
- a transparency of the accompanying "Base-Ten Blocks" page for the number 2; and
- 2 transparency markers, different colors.

Each student will need—

- 1 copy of the "Junior Mathematicians" letter (see **fig. 1**);
- 1 copy of **activity sheet 1**, "Divisibility and the Number 2";
- an accompanying "Base-Ten Blocks" page for the number 2;
- 2 pencils, different colors; and
- a calculator.

Lesson 2 (1 day)

Each student will need—

- 1 copy of **activity sheet 2**, "Divisibility and the Number 5";
- the accompanying "Base-Ten Blocks" page for the number 5;
- 2 pencils, different colors; and
- a calculator.

Lesson 3 (1–2 days)

Each student will need—

- 1 copy of **activity sheet 3**, "Divisibility and the Number 4";
- the accompanying "Base-Ten Blocks" page for the number 4;
- 1 copy of **activity sheet 4**, "Divisibility and the Number 8";

Figure 1

A fictitious company is looking for a team to invent a sorting machine that uses divisibility tests for numbers.

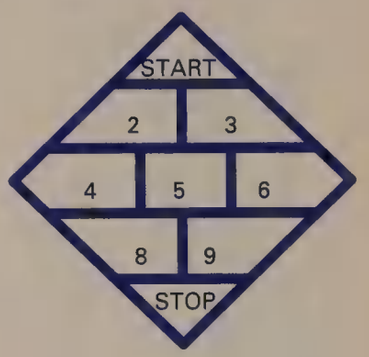
Numbers R Us Inc.
Office of the Presidents

March 13, 2008

Junior Mathematicians:

You have been assembled to work as a team of junior researchers in the Development and Research Department of our company. Your team's task is to find a solution to one of our customers' problem. This customer is Extremely Large Numbers Inc. They requested that our firm build a sorting machine that will work more efficiently than their current dividing machine, which sorts numbers into bins on the basis of whether a newly produced number is divisible by a selected number. Their current machine uses long division to do its sorting, and this has proven to be very slow as the numbers get larger and larger. As a result of this slowness, the company's production has decreased considerably, which is costing them lots of money.

Our sorting machine is to sort numbers into two bins on the basis of whether a number is divisible by a number pressed on the machine's keypad. Our customer has given us the keypad we are to use. A picture of the keypad is included in this memo for your team to think about when developing a shortcut for our scientists to program into the computer of this new sorting machine.



When ■ key is pressed, the machine is to sort the numbers into two bins, one bin for the numbers divisible by the pressed button and one bin for those not divisible by the pushed number. For example, when 2 is pressed, the machine sorts the company's numbers into two bins: One bin will contain only those numbers divisible by 2, and the other bin will contain all the numbers not divisible by 2.

Our senior researchers have provided you with some directions to consider as your team approaches this task. For you to find a shortcut, they suggest that you look for patterns in numbers that are divisible by each number. Their notes and questions will be distributed to you shortly. Your success is very important to both Numbers R Us and to our customer. Please work hard to complete this task in a timely manner.

Sincerely,

Dr. Summy Addly and Dr. Divinely Divides
Founding Presidents of Numbers R Us Inc.

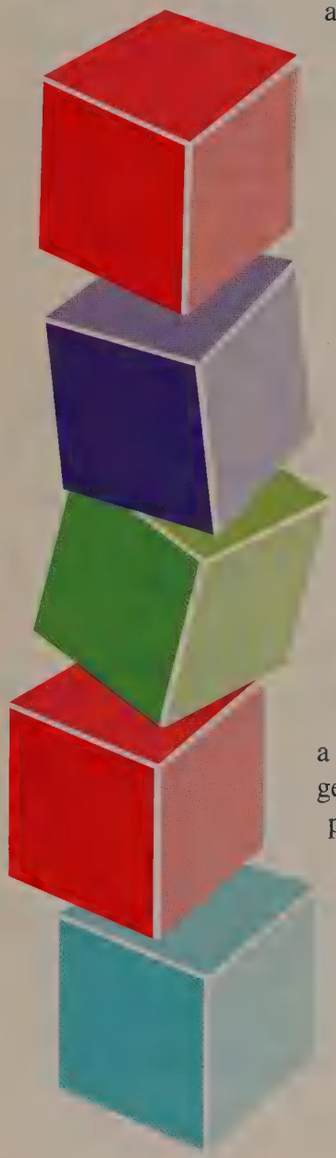
- the accompanying "Base-Ten Blocks" page for the number 8;
- 2 pencils, different colors; and
- a calculator.

Lesson 4 (1 – 2 Days)
Each student will need—

- 1 copy of **activity sheet 5**, "Divisibility and the Number 9";
- the accompanying "Base-Ten Blocks" page for the number 9;
- 1 copy of **activity sheet 6**, "Divisibility and the Number 3";
- the accompanying "Base-Ten Blocks" page for the number 3;
- 2 pencils, different colors; and
- a calculator.

Previous knowledge

Students who completed this investigation were already familiar with base-ten blocks, place value of large numbers, even and odd numbers, factors, multiples, and multidigit division. Students were also comfortable making conjectures and supporting why their conjectures appear to be correct.



Lesson 1: Why Does the Test for 2 Work?

Set the stage for this investigation by giving each class member a copy of the "Junior Mathematicians" letter (see **fig. 1**) from the presidents of Numbers R Us Inc., a fictitious company specializing in creating number machines. As the letter states, junior mathematicians (your students) are to determine divisibility tests for the numbers 2, 3, 4, 5, 6, 8, and 9 to be used to program a number-sorting machine for a customer, Extremely Large Numbers Inc. Students are told that long division is too slow for determining if large numbers are divisible by a given number. Our junior mathematicians diligently completed their tasks as workers who were paid with small pieces of candy.

To start the activity, ask your students if they know the divisibility test or rule to determine if a number is divisible by 2. In our group of students, a few responded that they thought they knew it. One student said, "If it is an even number, it is divisible by 2," and another student stated, "If it ends in 2, 4, 6, or 8, it is divisible by 2." We asked about numbers ending in 0, and the students quickly added it to the list of numbers.

Ask students, "Why do these tests work?" Our students were not sure how to explain why the different, but equivalent, statements work for a divisibility test for 2.

To facilitate the investigation, place students into groups of four and distribute **activity sheet 1**, "Divisibility and the Number 2," with the accompanying "Base-Ten Blocks" page (see **figs. 2a** and **b** for completed activity sheets displaying student work). Have students use base-ten blocks to model the first number in the chart: 16. Then use a transparency to lead your students through the shading process, alternating the two colors as you shade the groups of 2 in the largest base-ten blocks first. Students use two colors to help them see the different groups when shading the diagrams.

Next, have students use base-ten blocks to verify the number of groups of 2 that they shaded. Once students understand the shading process and how to complete the chart, have them shade the blocks for the remaining numbers and record the information in the table as they go. In this and subsequent lessons, you may wish to save time by having students divide the shading task and share their information with one another. However, having every student shade at least half of the numbers is important for helping them see how the blocks relate to the divisibility tests.

For the number 2, our students did not take long to make the connection between the shading and the quotient with remainder. If your students are unable to see this connection, ask them outright, "How does the shading relate to the quotient and remainder when you divide a given number by 2?" Our students realized that they did not need to shade all the longs because each long had five groups of 2 in it. They could multiply the number of longs by 5 to get the total number of shaded groups of 2 in the longs. This eventually led them to discover why the divisibility test for 2 works. That is, shading all the longs left no remainders when placed in groups of 2. As a result, students determined that all they needed to concern themselves with were the units and whether the units could be placed into groups of 2. They concluded that numbers with even digits in the ones place were the only numbers divisible by 2.

In a class discussion following this initial activity, ask students, "If we have a three-digit number such as 352, will your team's shortcut still work? Why, or why not?" Our students kept returning to the idea that all they needed to look at was the 2 because of its place value.

Figure 2

Students discovered the connection between the quotient and remainder when dividing by 2 by completing

(**a**) the "Divisibility and the Number 2" activity sheet and

Divisibility and the Number Two Robby

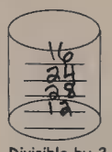
To help you get started, the senior researchers suggest that for each number in the table below your team shade, on the base-ten blocks page provided, as many groups of 2 as possible using different colored pencils. For each number, they suggest that your team record the number of different groups of 2 that can be formed. Also, record the number of squares that could not be put into a group of 2. Dividing up the numbers amongst your team members will prove to be quicker.

Number	Number of groups of two	Number of Unshaded Squares
a) 16	8	0
b) 13	6	1
c) 21	10	1
d) 24	12	0

Number	Number of groups of two	Number of Unshaded Squares
e) 19	9	1
f) 28	14	0
g) 12	6	0
h) 25	12	1

When a natural number is divided by 2 and the remainder is zero, the number is divisible by 2. Using the information in the table above, which of the numbers in the tables should be sorted into the bin that contains only the numbers divisible by 2? Why?

2 fits into the number evenly



Divisible by 2

If your team were to tell someone a "short cut" for determining if a number is divisible by 2 without dividing, what would it be? Once your team determines a short cut, consult with a senior researcher before your team continues.

The last number in a number has to be even or a zero.

Using your team's "short cut" from above, would 280 be divisible by 2? Explain.

Yes, because the last numbers a zero.

Write a 5-digit number that is divisible by 2. Why is it?

13,724 It's divisable by 2 because it ends in a even number.

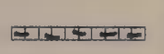
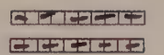
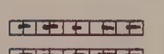
Write a 6-digit number that is not divisible by 2. Why is it not?

137,243 The last number in the number is not an even number or zero.

(**b**) the base-ten blocks page.

Divisibility and the Number Two
Base-ten Blocks Page

Name: Robby

a) 16		8 0
b) 13		6 1
c) 21		10 1
d) 24		12 0
e) 19		9 1
f) 28		14 0
g) 12		6 0
h) 25		12 1

To help students justify their response, ask them to think about the number of longs that are in a flat. In other words, how many tens are in 352? One student responded, "Thirty-five." When asked why, she stated, "Each flat has 10 longs; so, 3 flats have 30 longs. Along with the 5 longs, this gives us a total of 35 longs or 35 tens."

Next, write the following on the board: $352 = 35 \times 10 + 2$, which shows a different way to represent the number, and emphasize 352 as 35 longs and 2 units. As a class, talk about how the digits to the left of the ones place represent a multiple of 10, even for a very big number. Because each long is divisible by 2, we need look only at the digit in the ones place to see if it is divisible by 2. That is, when thinking about base-ten blocks, we have to concern ourselves with the unit blocks only, because all the other blocks can be represented as a multiple of the long block, and we know that each long block is divisible by 2.

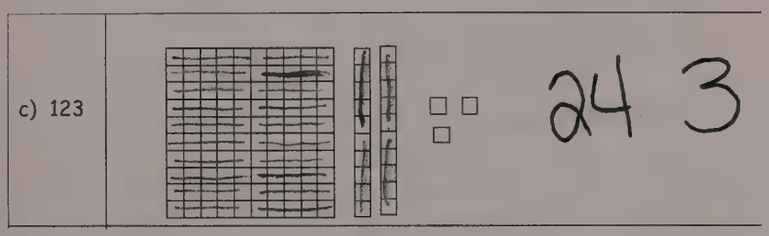
Lesson 2: Number 5 Looks a Lot Like Number 2

In this lesson, students will discover that a natural number is divisible by 5 if the digit in the ones place is either 0 or 5. Distribute **activity sheet 2**, "Divisibility and the Number 5," with the accompanying "Base-Ten Blocks" page. Because a long contains two groups of 5, this activity is very similar to that for the number 2. That is, any block larger than a long can be thought of as a multiple of a long, and because each long is divisible by 5, any block larger than a long is divisible by 5. As our students shaded groups of 5 on the "Base-ten Blocks" page,

Figure 3

This student's work shows groups of 5 and an alternative to shading

(a) the number 123 and



(b) an alternative to shading the number 175.

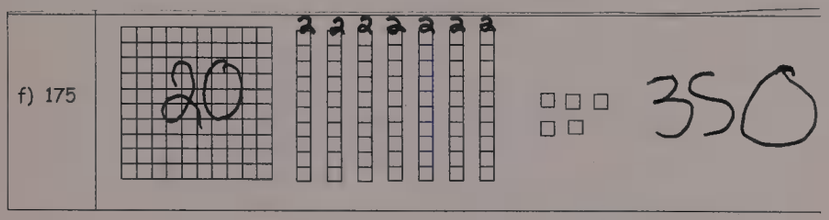


Figure 4

Student work connects base-ten blocks and divisibility by 4.

What does your team notice about the coloring of the flats? How does it differ from the longs and units?

the flats don't have leftovers and the longs and units do

they noticed that each flat had 20 groups of 5 and each long contained 2 groups of 5. Several of the students asked if it is necessary to continue shading the groups of 5 or whether they could simply write 20 for each flat and 2 for each ten (see **fig. 3**). Of course, this was exactly the outcome we desired.

The class discussion that followed this activity mirrored that of the discussion after the lesson 1 activity for groups of 2. Our conversation reinforced the idea that within any natural number, the digits to the left of the ones place represent a multiple of 10, and as a natural number, because 10 is divisible by 5, these digits always represent a number that is divisible by 5. So, when thinking about base-ten blocks, the only time a natural number is divisible by 5 is when the unit blocks can be placed into groups of 5 with none left over or when there are no unit blocks. To help students with these ideas, ask them, "If you are allowed to use only longs and units to represent the number 123 with base-ten blocks, what is the greatest number of longs you could use?" One of our students answered, "Twelve."

Next, ask your students, "Is the number represented by 12 longs divisible by 5?" Several of our students replied yes. One student explained that "since a long has 2 groups of 5, 12 longs have 2 × 12, or 24, groups of 5." To help your students make a generalization about this important idea, ask them whether every set of longs represents a number that is divisible by 5. Several of our students answered yes, explaining that "each of the longs within a group of longs contains two groups of 5, so the group of longs will contain twice as many groups of 5 as there are longs." As a class, we summarized this discovery: "Any number represented by a group of longs is a multiple of 10, and since each 10 contains 2 groups of 5, any multiple of 10 is divisible by 5." To sum up the importance of this discovery, we returned to the number 123. Because we know that the number represented by the digits in and to the left of the ones place, 120, is a multiple of 10 and is therefore divisible by 5, we need only determine whether or not groups of 5 can be formed from the 3 unit blocks with no unit blocks left over. In this case, 3 cannot be placed into a group of 5 with no unshaded squares leftover, and so 123 is *not* divisible by 5. This discussion was grounded in what students had experienced from shading groups of 5 within the base-ten block representations of numbers, and it led them to understand why a natural number is divisible by 5 only when the ones digit is 0 or 5.

To prepare students for investigating a divisibility test for the number 4 in the next lesson, ask them, "How many flats or hundreds are in 4349?" One of our students responded, "43." When asked to explain why, the student stated, "There are 10 flats in a cube, and we have 4 cubes; so we have 40 + 3 flats."

Lesson 3, Part 1: Number 4

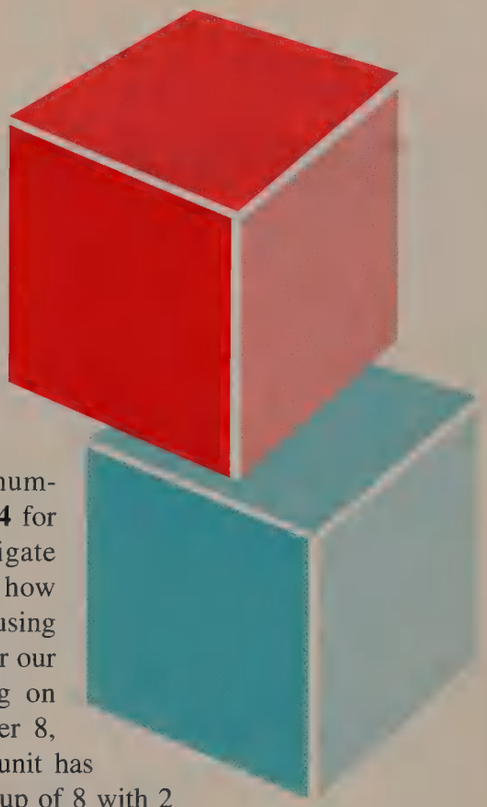
The key is in the flat

"Divisibility and the Number 4" (see **activity sheet 3**) and the accompanying "Base-Ten Blocks" page will focus students' attention on the differences among the shading of the various types of base-ten blocks. Distribute these pages to the students in their respective groups and have them complete the activity. As students shade groups of 4, they will realize that each flat contains complete groups of 4 with no leftovers, but each long contains leftovers when forming groups of 4 (see **fig. 4**). Once students discover that a flat is the smallest base-ten block that can be shaded into groups of 4 with no

squares left over, they are able to use this discovery to develop the divisibility test for 4; that is, a natural number is divisible by 4 if the number represented by the two right-most digits (tens and ones) is divisible by 4. For example, to determine if 512 is divisible by 4, one need only check whether12 is divisible by 4.

Next, have your students check their divisibility test for 4 with four-digit numbers by using a modified-1000 block, which consists of 10 flats (see **fig. 5**). This requires them to decompose a natural number into the sum of a multiple of 100 and a two-digit number. That is, in any natural number, the digits in and to the left of the tens place represent a multiple of the flat, and since the flat is divisible by 4, this portion of any number is divisible by 4. As a result of this fact, our students realized that in order to determine whether a natural number is divisible by 4, all they need to do is determine if the number represented by the longs and units is divisible by 4.

To help solidify your students' understanding of this divisibility test, ask them to find the smallest and largest four-digit numbers that are divisible by 4. Our students determined these four-digit numbers by decomposing natural numbers into sums of two numbers, one that was known to be divisible by 4 and another that had to be checked to determine if it was divisible by 4. Specifically, several students stated that "the smallest four-digit number divisible by 4 is 1000 because the modified cube was divisible by 4. The largest number is 9996 because 9996 = 9900 + 96, and both are divisible by 4." Our students' approach showed that they had gained an understanding that if two numbers are divisible by a number, then their sum is divisible by the number. Furthermore, if only one of two numbers is divisible by a number, then their sum is not divisible by the number. Much of our students' work and conversations demonstrated the use of both decomposition of numbers and mental math strategies when determining divisibility.



Lesson 3, Part 2: Number 8

Flattening the cube

Unlike the activities for the numbers 2, 4, and 5, **activity sheet 4** for the number 8 does not investigate individual numbers but rather how groups of 8 can be formed by using each type of base-ten block. After our student groups started working on the activity sheet for the number 8, they quickly determined that a unit has no groups of 8, a long has 1 group of 8 with 2 unshaded squares, and a flat has 12 groups of 8 with 4 unshaded squares. Knowing that a modified-1000 block consists of 10 flats and that each flat contains 4 unshaded squares, several of our students shared comments similar to the following: "The number of unshaded squares in a modified-1000 block is 40 because each flat has 4 unshaded squares, and there are 10 flats in a 1000 block, and this is the same as 5 groups of 8. So, a modified-1000 block is divisible by 8" (see **fig. 6**). As a result, one must consider only whether the flats, longs, and units as a whole can be shaded into groups of 8, because the larger blocks can be thought of as multiples of the 1000 block. From this discovery, our students realized that the divisibility test for 8 requires one to consider only the three right-most digits as a whole number and determine if this three-digit number is divisible by 8. In other words, if the three right-most digits can be divided by 8 as a whole number

Figure 5

Working with the modified-1000 block on Activity sheet 4 requires students to decompose a natural number into the sum of a multiple of 100 and a two-digit number.

Think about shading groups of 4 within the modified-1000 block below. Will there be any unshaded squares? Explain. (Hint: Does your team really need to do all the shading? Instead, think about how many flats are in a modified 1000 block and how shading this 1000 block relates to the shading of a flat.)

No because 100 is divisable by 4 and 1,000 is only 100x10.

Figure 6

The color of the shading shows modified 1000-block groups of 8.

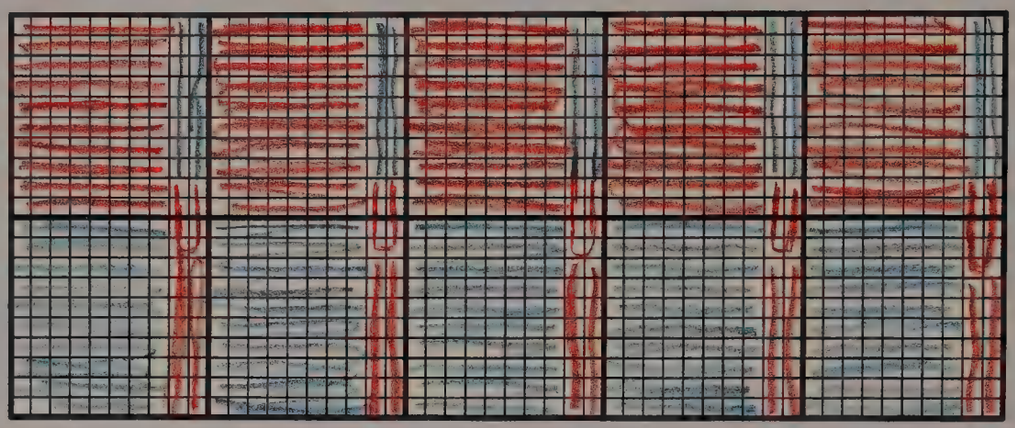
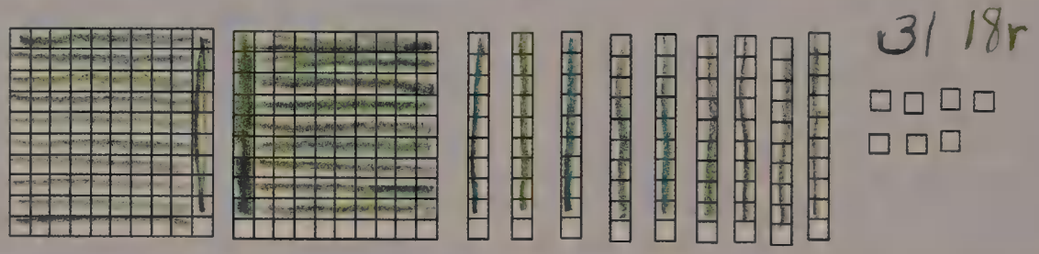


Figure 7

This shading shows groups of 9 for the number 297.



with no remainder, then the original number is divisible by 8. For example, to determine if 5160 is divisible by 8, one need only check whether 160 is divisible by 8.

Lesson 4, Part 1: Number 9

It is all in the leftovers

The approach used to investigate divisibility tests for the numbers 9 and 3 differs from those used in earlier parts of the investigation. Distribute **activity sheet 5** and the accompanying "Base Ten Blocks" page and have students shade groups of 9 within each base-ten block representation. *Be sure to emphasize that students are not to combine the leftover unshaded squares as they did in the previous lessons.* **Figure 7** shows one student's shaded blocks for the number 297. As students completed their shading, they recorded their information in columns two through six of the table on **activity sheet 5** and only then combined the unshaded squares into additional groups of 9 and completed the last three columns of the table.

Activity sheet 5 has key questions to guide students to the critical discovery that the total number of unshaded squares for each type of base-ten block corresponds to the digits in each of the place values within the number under investigation. For example, the number 297 has 2 total unshaded squares from the flats (hundreds place), 9 unshaded squares from the longs (tens place) and 7 unshaded units. Many of our students quickly recognized this fact and that the 18 unshaded squares could be combined into 2 additional groups of 9 with no remainders. Because the number of unshaded squares for each place value equals the digit with that place value, the students began considering whether the digits of the given number could be used to form groups of 9. For example, some students said "the 2 and 7 make a 9, and then there is one other 9, so 297 is divisible by 9." Other students realized that the sum of the digits corresponds to the total number of unshaded squares; they then determined whether this sum was divisible by 9. Our students discovered that "if the sum of the digits of a natural number is divisible by 9, then the natural number is divisible by 9."

Lesson 4, Part 2: Numbers 3 and 9

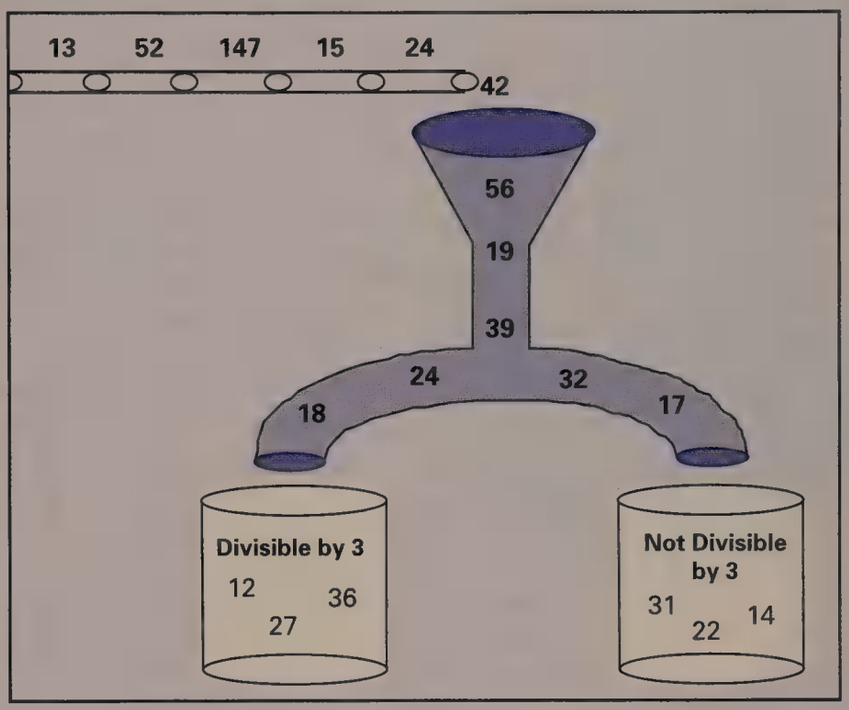
Fraternal twins

This activity is designed to guide students to the discovery that the divisibility test for the number 3 is similar to that for the number 9. Distribute **activity sheet 6** and the accompanying "Base-Ten Blocks" page for the number 3. After starting on the activity sheets, students were quick to realize that when shading groups of 3 within the different types of base-ten blocks, one unshaded square always remained, just as with the 9s. This discovery, as would be expected, quickly led them to realize that "a natural number is divisible by 3 if the sum of its digits is a multiple of 3" or "if the sum of the digits in the number is divisible by 3."

During a class discussion ask, "If a number is divisible by 9, does it have to be divisible by 3?" Our students answered with a resounding yes. When asked why, one student responded, "Because the sum of the digits would be divisible by 3 since 9 is a multiple of 3." Follow up by asking, "If a number is divisible by 3, is it divisible by 9?" This question brought mixed replies until one student mentioned the number 6, which is divisible by 3 but not by 9.

Figure 8

A sorting machine that represents a function can have only one of two outputs.



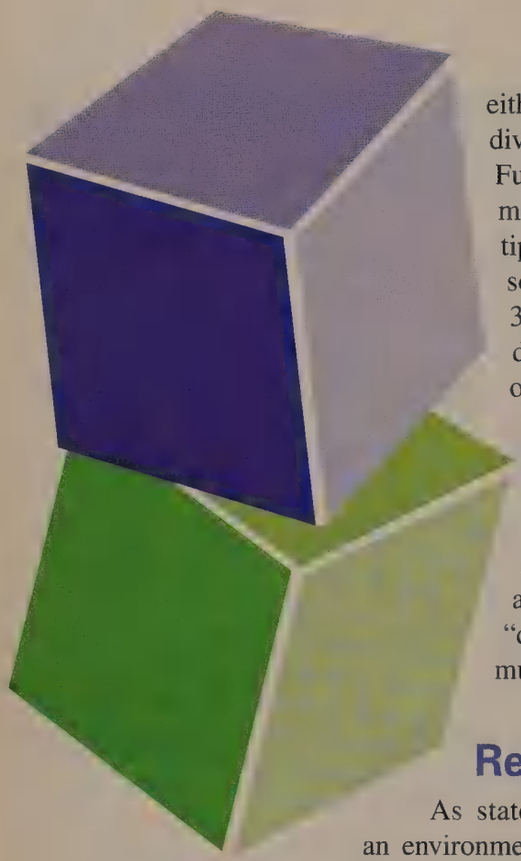
Beyond the Lesson

Included with the activity pages is "Divisibility and the Number 6" (see **activity sheet 7**) for investigating the divisibility test for 6. This test for 6 involves the known tests for 2 and 3 and is readily accessible to students by means of pattern recognition. After completing the table, our students quickly saw that whenever a number is divisible by 6, it is also divisible by both 2 and 3. This led them immediately, and accurately, to claim that "if a number is divisible by 2 and 3, it is divisible by 6."

During a class discussion, we talked more generally about using known divisibility tests for other numbers. We told our students, "We just discovered that when a number is divisible by both 2 and 3, then it is also divisible by 6. Do you think that if a number is divisible by both 2 and 4, it must be divisible by 8?" Several students thought it would be divisible by 8 until one student said, "It works for some but not all numbers." Students discovered that numbers such as 4 and 12 are divisible by 2 and 4 but not by 8. We discussed why 2 and 3 work for 6 but 2 and 4 do not work for 8. The important fact is that 2 and 3 do not have any common factors other than 1, but 2 and 4 have a common factor of 2. This dialogue then led us to a more general discussion about possible tests for other numbers. For example, the tests for 3 and 4 could be used for 12, but the tests for 2 and 6 cannot be used. During our conversation, students presented other divisibility tests: Use the tests for 3 and 5 for 15; use the tests for 2 and 9 for 18; and use the tests for 3 and 8 for 24 (but do not use the tests for 4 and 6 for 24).

You might think that we forgot about the divisibility test for 10, but once the students understand why the tests for 2 and 5 work, the test for 10 immediately follows. That is, when a natural number is decomposed into the sum of a multiple of 10 and the ones digit, students observe that the only time it makes sense for the number to be divisible by 10 is when the ones digit is 0.

Envisioning a sorting machine such as the one this article describes can provide a springboard into other mathematical concepts. For example, in the context of using this machine to sort numbers that are divisible by a given number from those that are not, you can introduce or reinforce the concept of a *function*. That is, a number enters the machine as the *input*, and it can have only one of two *outputs*,



either "divisible by the number" or "not divisible by the number" (see **fig. 8**). Furthermore, we can use the sorting machine to investigate common multiples. If we start with empty bins and sort the natural numbers for groups of 3 and then take the numbers that are divisible 3 and sort them for groups of 5, the numbers that are now in the "divisible by 5" bin are common multiples of 3 and 5. From this process, the smallest value of the numbers in the "divisible by 5" bin is the least common multiple of 3 and 5, or 15; and the numbers in the "divisible by 5" bin are all the positive multiples of 15.

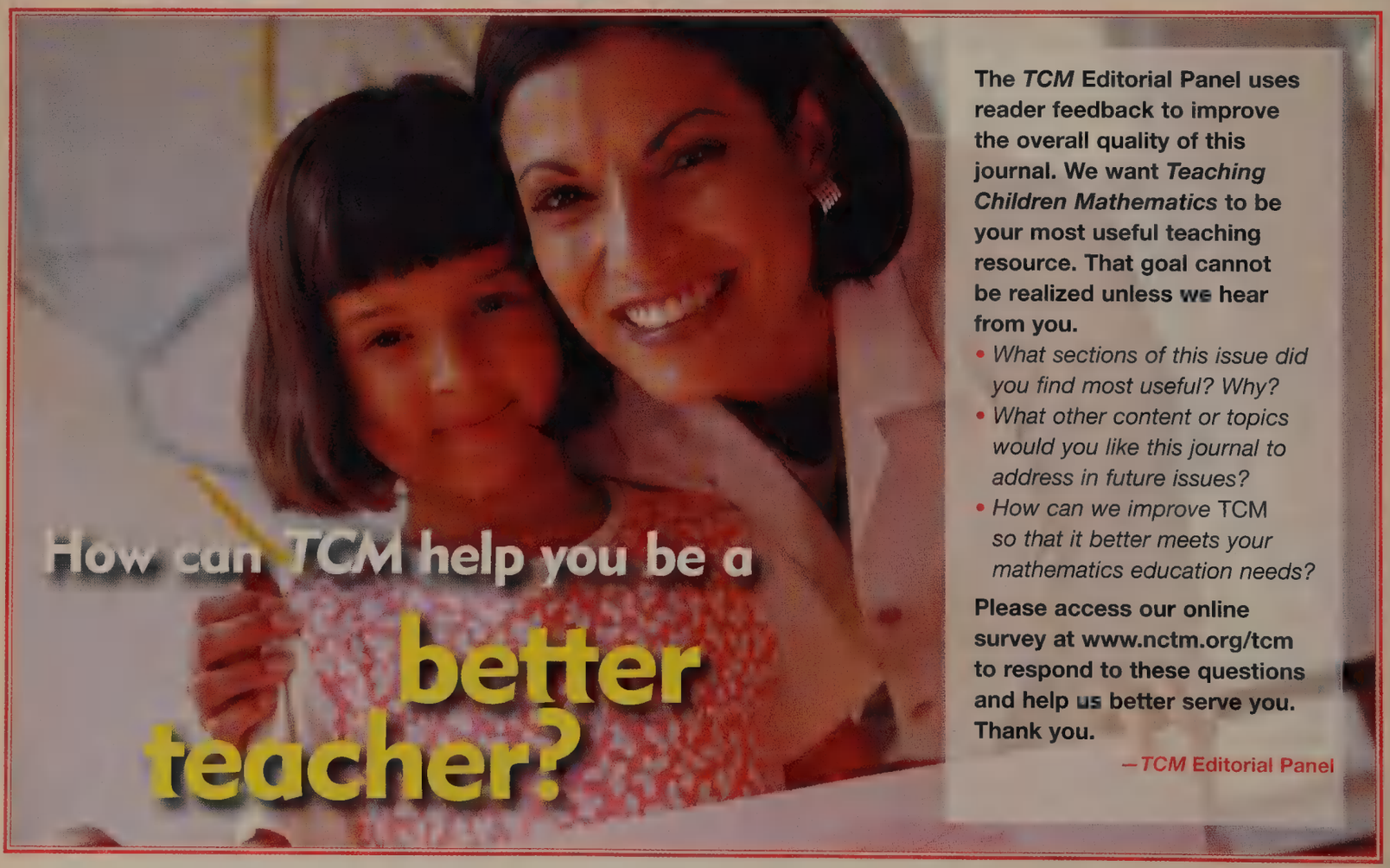
Reflections

As stated earlier, these activities provide an environment for students to begin to gain an appreciation of the desired outcomes stated in the Reasoning and Proof Process Standard. The activities in this investigation led our students to discover the divisibility tests by using both pictorial and tabular pattern recognition. Moreover, the patterns that students observed guided them to understand the validity of the divisibility tests. During these activities, students were engaged in a dialogue that required them to evaluate one another's mathematical understandings and claims. Through such activities and group discussions, students come to realize that reasoning and proof are fundamental aspects of mathematics.

Reference

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.

Note to readers: Full-size (8.5 x 11) activity sheets can be downloaded from **www.nctm.org/tcm**. Locate the February 2009 issue and then look for the link to the "Investigations" activity sheets.



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- *What sections of this issue did you find most useful? Why?*
- *What other content or topics would you like this journal to address in future issues?*
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—*TCM* Editorial Panel

Activity Sheet 1. Discovering Divisibility

DIVISIBILITY AND THE NUMBER 2 Name_____

To help you get started, the senior researchers suggest the following: For each number in the table below, your team will shade—on the base-ten blocks page for the number 2—as many groups of 2 as possible, using different-colored pencils. (Divide the numbers among your team members.) For each number, record the number of different groups of 2 that can be formed. Also record the number of squares that you cannot put into a group of 2.

No.	No. of Groups of 2	No. of Unshaded Squares
a) 16		
b) 13		
c) 21		
d) 24		

No.	No. of Groups of 2	No. of Unshaded Squares
e) 19		
f) 28		
g) 12		
h) 25		

When a natural number is divided by 2 and the remainder is 0, the number is divisible by 2. Using the information in the tables above, which of the numbers in the tables should be sorted into the bin that contains only the numbers divisible by 2? Why?

If your team were to tell someone ■ shortcut for determining if a number is divisible by 2 without dividing, what would it be? Once your team determines ■ shortcut, consult with ■ senior researcher before your team continues.

Using your team's shortcut from above, is 280 divisible by 2? Explain.

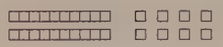
Write a five-digit number that is divisible by 2, and explain why it is.

Write a six-digit number that is not divisible by 2. Why is it not?

From the February 2009 issue of TEACHING CHILDREN Mathematics

Base-Ten Blocks for Activity sheet 1

DIVISIBILITY AND THE NUMBER 2 Name_____

a) 16	
b) 13	
c) 21	
d) 24	
e) 19	
f) 28	
g) 12	
h) 25	

From the February 2009 issue of TEACHING CHILDREN Mathematics

Activity Sheet 2. Discovering Divisibility

DIVISIBILITY AND THE NUMBER 5 Name_____

Now your team might want to look at groups of 5 using base-ten blocks to see if you can determine a divisibility shortcut for the number 5. Use the numbers in the table. Starting with the largest blocks, shade as many groups of 5 as possible, using a variety of colored pencils. Record the number of groups of 5 in the space provided in the tables. Also record the number of squares that could not be put into ■ group of 5.

No.	No. of Groups of 5	No. of Unshaded Squares
a) 35		
b) 14		
c) 123		
d) 210		
e) 87		
f) 175		

When a natural number is divided by 5 and the remainder is 0, we say that the number is *divisible by* 5. Using the information in the table, which numbers should be sorted into the bin at the right for the numbers that are divisible by 5? Why?

If your team were to tell someone a shortcut for determining if ■ number is divisible by 5 without dividing, what would it be? Once you determine a shortcut, consult with a senior researcher before your team continues.

Using your team's shortcut from above, is 280 divisible by 5? Explain.

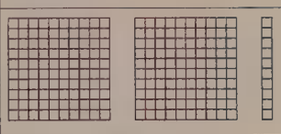
Write a four-digit number that is divisible by 5, and explain why it is.

Write a seven-digit number that is not divisible by 5. Why is it not?

From the February 2009 issue of TEACHING CHILDREN Mathematics

Base-Ten Blocks for Activity sheet 2

DIVISIBILITY AND THE NUMBER 5 Name_____

a) 35	
b) 14	
c) 123	
d) 210	
e) 87	
f) 175	

From the February 2009 issue of TEACHING CHILDREN Mathematics

Activity Sheet 3. Discovering Divisibility

DIVISIBILITY AND THE NUMBER 4 Name_____

Don't stop yet. Your team must determine a divisibility shortcut for 4. Again we suggest that you use the base-ten blocks page for the number 4 to help you complete the task. For each number in the table, shade ■ many groups of 4 as possible, using different-colored pencils. (You will only need two or three colors.) Record the number of shaded groups of 4 and the number of squares that could not be put into ■ group of 4.

No.	No. of Groups of 4	No. of Unshaded Squares
a) 32		
b) 132		
c) 26		
d) 126		
e) 360		
f) 234		
g) 136		
h) 242		

Knowing what it means to be divisible by 2 and by 5, what does your team think it means to be divisible by 4? Which of the numbers in the table are divisible by 4? Why? Before you continue, check your work with a senior researcher.

What does your team notice about the coloring of the flats? How does it differ from the longs and the units?

If your team were to tell someone a shortcut for determining if a number is divisible by 4 without dividing, what would it be? (Hint: Think about your answer to the previous question.)

Using your team's shortcut from above, is 732 divisible by 4? Explain.

Using your team's shortcut from above, is 526 divisible by 4? Explain.

When your team discovered its shortcut for 4, it investigated only numbers less than 1000. To help you determine whether or not the team's shortcut will work for larger numbers, think about the base-ten blocks diagram for the number 1526 and answer the following question. Shade groups of 4 within the modified-1000 block. Will there be any unshaded squares? Explain. (Hint: Does your team really need to do all of the shading? Instead, think about how many flats are in ■ modified-1000 block and how the shading of the modified-1000 block relates to the shading of a flat.)

Making use of your team's answer to the previous question, is 1500 divisible by 4? Explain.

Using what you learned about 1500, explain why your team's shortcut would or would not work for determining whether 1526 is divisible by 4.

Write the smallest four-digit number that is divisible by 4, and explain why it is.

Write the largest four-digit number that is divisible by 4, and explain why it is.

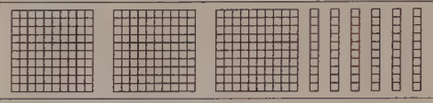
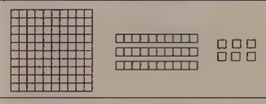
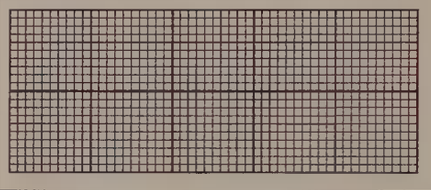
Will your team's shortcut work for ■ five-digit number? Explain.

Will your team's shortcut work for any whole number, regardless of the number of digits? Explain.

From the February 2009 issue of TEACHING Mathematics

Base-Ten Blocks for Activity sheet 3

DIVISIBILITY AND THE NUMBER 4 Name_____

a) 32	
b) 132	
c) 26	
d) 126	
e) 360	
f) 234	
g) 136	
h) 242	
modified-1000	

From the February 2009 issue of TEACHING CHILDREN Mathematics

Activity Sheet 4. Discovering Divisibility

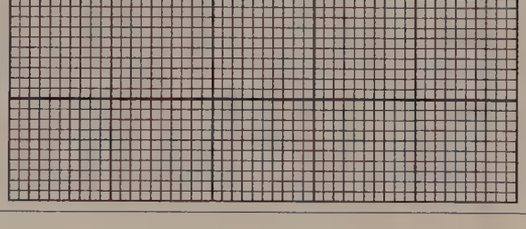
DIVISIBILITY AND THE NUMBER 8 Name_____

To sort the numbers that are divisible by 2 from those that are not, your team's shortcut had us consider the right-most digit and its remainder when divided by 2. To sort the numbers that are divisible by 4 from those that are not, your team's rule had us consider the two right-most digits as a number and its remainder when divided by 4. We want to investigate whether a similar rule can be made for 8. Using the base-ten blocks sheet, determine and record which base-ten blocks can be shaded into groups of 8, leaving no unshaded squares.

Base-Ten Block	No. of Groups of ■	No. of Unshaded Squares
■) unit		
b) long		
c) flat		
d) modified-1000 block		

Think about your team's answers in the table and how you determined sorting rules for the numbers 2 and 4. State a sorting shortcut for the number 8 and explain it. Before continuing, check with a senior advisor.

Now use your shortcut to sort these numbers into the correct bins above: 1456; 4343; 234; 7888; 12744; 10000; 8886.

a) unit	
b) long	
c) flat	
d) modified-1000	

From the February 2009 issue of TEACHING CHILDREN Mathematics

Activity Sheet 7. Discovering Divisibility

DIVISIBILITY AND THE NUMBER 6 Name_____

Your team has discovered several divisibility shortcuts, so determine if any of them can be used to develop a divisibility shortcut for the number 6. Since $2 \times 3 = 6$, your team may want to investigate how divisibility by 2 and divisibility by 3 possibly relate to divisibility by 6. Check to see if the numbers in the table are divisible by 2, 3, and 6. For the numbers 2 and 3, use the divisibility shortcuts that you already know. Remember that when ■ number is divided by 6 and its remainder is 0, the number is divisible by 6.

No.	Divisible by 2	Divisible by 3	Divisible by 6
a) 39			
b) 64			
c) 124			
d) 192			
e) 204			
f) 155			
g) 342			
h) 284			
i) 426			
j) 155			

Looking at the completed tables, what patterns does your team see?

If your team were to tell someone a shortcut for determining whether a number is divisible by 6 without dividing, what would it be? Consult with a senior researcher before your team continues.

Using your team's shortcut from above, is 36,726 divisible by 6? Explain.

Write a nine-digit number that is divisible by 6, and explain why it is.

Write a seven-digit number that is not divisible by 6, and explain why it is not.

Considering what your team has learned about the divisibility for the number 6, do you think this same idea works for all numbers? For example, if a number is divisible by 2 and 4, is it divisible by 8? Can your team find a number that is divisible by 2 and 4 but not by 8?

From the February 2009 issue of TEACHING CHILDREN Mathematics

Activity Sheet 5. Discovering Divisibility

DIVISIBILITY AND THE NUMBER 9 Name_____

As a team, use base-ten blocks to find a divisibility shortcut for 9. Using different colored pencils, shade as many groups of 9 as possible within each type of block. Start with the largest blocks. **Do not combine the unshaded squares.** Record the number of groups of 9 in the table below. Also, separately record the number of squares that remain unshaded in the flats, longs, and units.

No.	No. of Shaded Groups of 9	No. of Unshaded Squares				Additional Groups of ■	Unused Squares	Divisible by 9?
		Flats	Longs	Units	Total Unshaded			
a) 81								
b) 65								
c) 128								
d) 297								
e) 252								
f) 368								

Next, combine the unshaded squares from different types of blocks into additional groups of 9 and complete the table. Be sure to give reasons for whether or not the number is divisible by 9.

What does your team notice about the unshaded squares of the flats, longs, and units? How do these relate to the number? What is true about the digits for the numbers that are divisible by 9? (Hint: For which numbers can the unshaded squares be used to form complete groups of 9 leaving no squares unshaded? What does this tell your team?)

If your team were to tell someone a shortcut for determining if a number is divisible by 9 without dividing, what would it be? Consult with ■ senior researcher before your team continues.

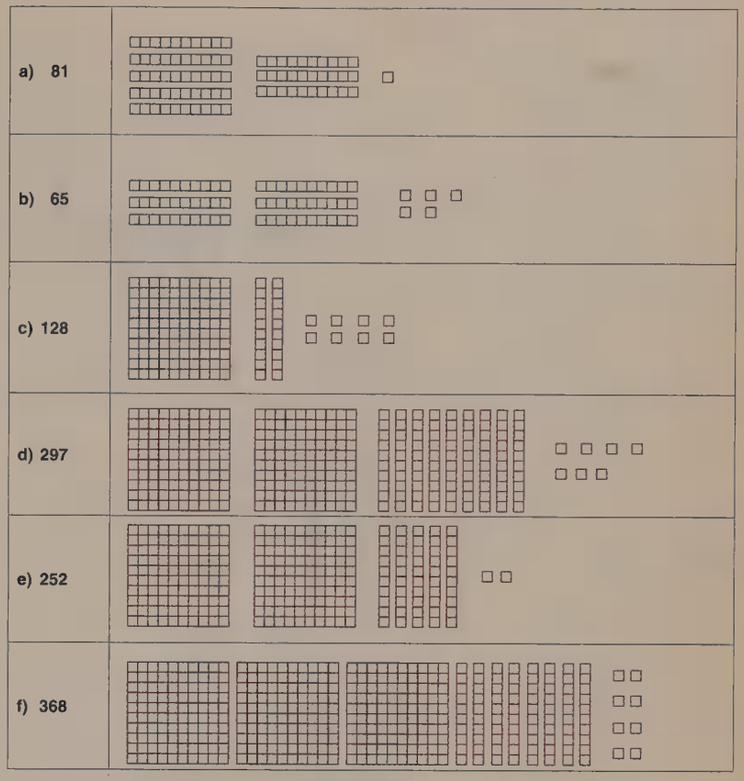
Using your team's shortcut from above, is 36,495 be divisible by 9? Explain.

Write an eight-digit number that is divisible by 9. Why is it?

From the February 2009 issue of TEACHING CHILDREN Mathematics

Base-Ten Blocks for Activity sheet 5

DIVISIBILITY AND THE NUMBER 9 Name_____



From the February 2009 issue of TEACHING CHILDREN Mathematics

Activity Sheet 6. Discovering Divisibility

DIVISIBILITY AND THE NUMBER 3 Name_____

Since your team has already discovered several divisibility shortcuts, see if any of them can be slightly changed to help your team with the number 3. Again use different colors of pencils to shade as many groups of 3 as possible on the page of base-ten blocks. In the table below, record the number of groups of 3 for each problem. Look for shading patterns as in the previous divisibility activity sheets. Also separately record the number of squares that remain unshaded in the flats, longs, and units. After doing so, determine the number of additional groups of 3 that can be formed from the unshaded squares. (This time you may combine the unshaded squares from different blocks.) In the table, record the number of additional groups of 3 and the number of unused squares.

No.	No. of Shaded Groups of 3	No. of Unshaded Squares				Additional Groups of 3	Unused Squares	Divisible by 3?
		Flats	Longs	Units	Total Unshaded			
a) 39								
b) 65								
c) 124								
d) 192								
e) 204								
f) 158								

Which of the numbers in the table can be placed in the bin of numbers divisible by 3? Why?

Did any of the shadings from the previous divisibility activity sheets have similar patterns to the groups of 3 shadings? If so, how can this help your team determine ■ new divisibility shortcut for the number 3?

If your team were to tell someone ■ shortcut for determining if a number is divisible by 3 without dividing, what would it be?

Using your team's shortcut from above, is 36,735 divisible by 3? Explain.

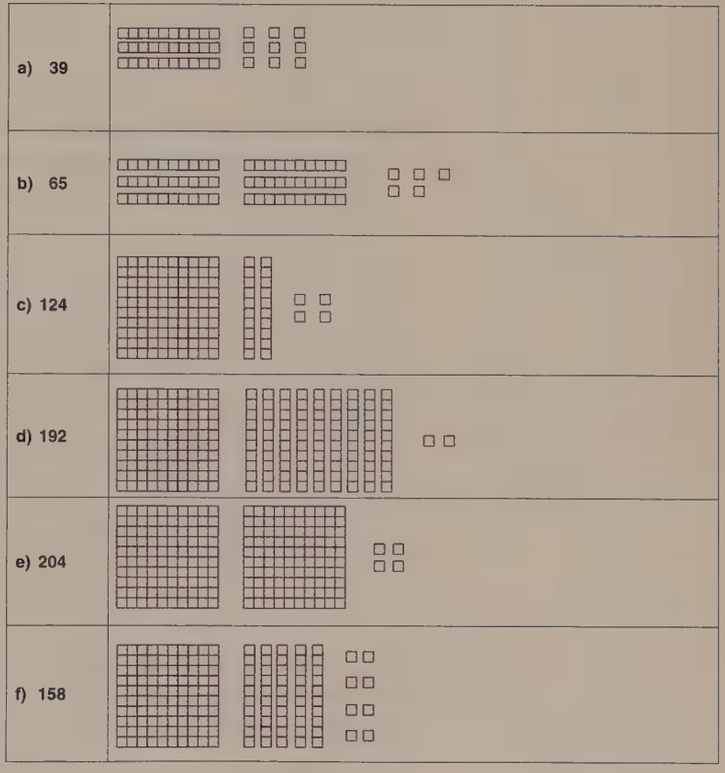
Write a seven-digit number that is divisible by 3, and explain why it is.

Write an eleven-digit number that is not divisible by 3, and explain why it is not.

From the February 2009 issue of TEACHING CHILDREN Mathematics

Base-Ten Blocks for Activity sheet 6

DIVISIBILITY AND THE NUMBER 3 Name_____



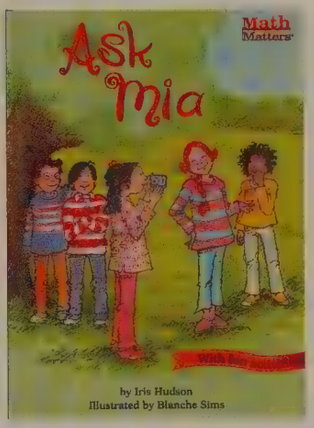
From the February 2009 issue of TEACHING CHILDREN Mathematics

For the Love of Books

Books

For Students

Math Matters: Ask Mia, *Iris Hudson, 2006. 32 pp., $5.95 paper. Grades 1–3. ISBN 1-57565-188-2. The Kane Press; (212) 268-1435; www.kanepress.com.*



Ask Mia, by Iris Hudson and illustrated by Blanche Sims, is a book that will encourage children to think and talk about mathematics and reading. The story revolves around a single mathematical concept: pictographs, a timely concept. The story line addresses a topic that is familiar to most 6–8-year-olds, "a bossy student." The fast action helps to keep students engaged while teaching them to understand how to create and interpret pictographs. This book can be read independently by students or read aloud to younger children.

The author suggests ways to extend children's reading skills: read together, connect with mathematics, and apply mathematical concepts. The book provides students with different pictographs and asks them to share what they see, encouraging students to develop critical thinking skills to teach them to create their own pictographs. This book encourages students to use real-life situations to help them develop deep understanding of mathematical concepts.

I used this book in my first-grade class and found it very helpful. Students were able to create other pictographs based on the information presented.—*Diana Duffey, Willow Glen Primary School #1, Francis, WI.*

For Teachers

From NCTM

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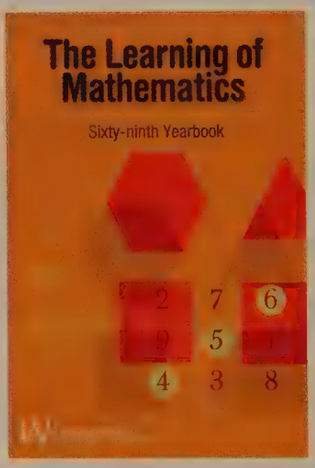
The Learning of Mathematics: Sixty-ninth Yearbook, *2007. 370 pp., $44.95 cloth. ISBN 978-0-87353-596-0. Stock #13199. National Council of Teachers of Mathematics; (800) 235-7566; www.nctm.org.*



Not since 1990—when the first NCTM yearbook on learning was written—has a yearbook focused on what is referred to in the preface as the "product" that schools produce: learning. As one of six principles in *Principles and Standards for School Mathematics* (NCTM 2000), learning is a central goal, and yet "discussions in our field more frequently address means of achieving that goal" (p. xi). When the 1990 yearbook was written, *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) was new, and notions about how to actively engage all students in learning were quite simple. This new yearbook recognizes that the learning process is complex and that truly meeting the learning needs of *all* students takes a more finessed pedagogy than organizing students in groups or posing problems.

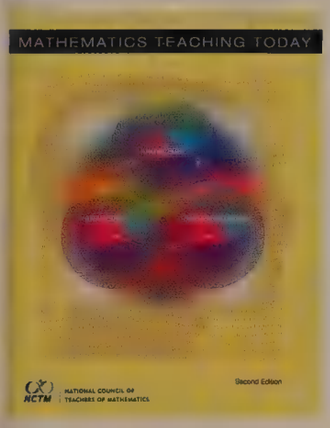
Prices on software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

This yearbook provides examples of how mathematics educators are connecting learning theory with the realities of classroom practice in diverse settings. The book presents findings about content-specific student learning trajectories that support teachers with thinking about how to "best" promote learning with understanding. Taken together, the chapters provide ideas and methods that inform instructional decisions to engage all students in high-level thinking, reasoning, and mathematical discourse. The book falls into the following five parts, each of which includes an overview and several illustrative chapters:

- Perspectives on How Students Learn
- Issues Related to Students' Learning in School Contexts
- Measuring and Interpreting Students' Learning
- Teachers' Learning of Mathematics
- Reflections on Mathematics Teaching and Learning

Based on the content foci, the material is most relevant for teachers of grades K–8 and mathematics teacher educators.—*Nicole Miller Rigelman, George Fox University, Newberg, OR 97132.*

Mathematics Teaching Today: Professional Standards for Teaching Mathematics, *2007. xiv + 208 pp., $35.95 paper. ISBN 978-0-87353-598-4. Stock #13201. National Council of Teachers of Mathematics; (800) 235-7566; www.nctm.org.*



This book presents a vision for effective mathematics teaching and the support components required to achieve and sustain that vision, adding an essential element to the guidance provided by curriculum standards documents. An exploration of four standards to improve the practice of teaching mathematics includes a thorough discussion of each component. Every chapter provides in-depth background to help stimulate informed discussion. Vignettes of environments in which mathematical learning is taking place help spotlight individual practices and teachers. The appendix provides correlation between the vignettes and *Principles and Standards for School Mathematics* (NCTM 2000). Generous margins highlight important textual points, with additional room for readers' notes. The focus throughout is on the total environment in which teaching and learning take place. Opening the opportunity to bring the focus on practice to a professional learning community (in a formal book study at the building or district level), a valuable chapter addresses reflective practice questions. The extensive bibliography provides yet another resource for continued discussions. I highly recommend the book for any educator or community member interested in shifting the paradigm of mathematics instruction from preparation for the past industrial age to preparation for the age of the future.—*Shirley Ferguson, Lebanon School District, Lebanon, NH 03077.*

Perspectives on Design and Development of School Mathematics Curricula, *2007. xiii + 199 pp., $29.99 paper. Grades K–12. ISBN 978-0-87353-599-1. Stock #13233. National Council of Teachers of Mathematics; (800) 235-7566; www.nctm.org.*

This NCTM resource provides detailed reviews of fifteen exemplary mathematics programs, fourteen of which are funded by the National Science Foundation (NSF). Authored by each program's director or associate, individual executive summaries detail the foundational design principles of the recommended program. The originating concept for this publication was the 2005 NSF K–12 Mathematics and Science Curriculum and Implementation Centers Conference. The programs at the conference—and now in this volume—review the following features of their curricula: Goals, Design Principles, Development Process, Lessons Learned, and Implications.

Readers will note the common goals and features of the innovative programs, such as the focus on "big ideas" across grade levels, incorporation of technology, evidence of active student engagement, embedded assessment, and evidence of equity and access. The volume is organized by grade bands.

The intended audience includes graduate students and faculty, mathematics coordinators, and teachers. In fact, the audience may well expand to principals, department chairs, and curriculum committees as a basis for evaluating existing curricula and researching programs worthy of further investigation in service of well-documented curriculum recommendations.

I recommend this publication with enthusiasm. As the volume states, traditional curricu-

lum change patterns have been characterized by *revolution* or *evolution*. Continued curriculum innovation and change are currently guided by the National Research Council report, *On Evaluating Curriculum Effectiveness* (2004). The closing chapter of *Perspectives on Design and Development of School Mathematics Curricula* realistically references the current challenges and limits of curriculum change in school programs and key issues, such as teacher professional development—where "adoption" decisions often rest.—*Marylou T. Clarke, Dartmouth Public Schools, Dartmouth, MA 02748.*

From Other Publishers

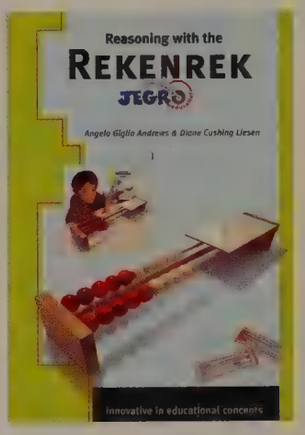
Panning for Gold: 15 Investigations to Enrich Middle School Mathematics, *Daniel J. Brahier, 2007. 240 pp., $24.50 paper. ISBN 978-0-325-01045-8. Heinemann; (800) 225-5800; www.heinemann.com.*

This activity book is an investigative approach for middle school mathematics concepts that provides fifteen discovery investigations in the concepts of numbers and operations, algebra, geometry, measurement, and data analysis and probability. Each lesson supplies connections to the NCTM Standards for grades 6–8, a synopsis of the lesson, a description of the activity, a hypothetical teacher dialogue with students, and extensions for the lesson. Teachers may use these as discovery lessons or as enrichment activities for scaffolding other lessons. Material lists are included, as well as excellent examples and instructions for using calculators.

In my opinion, chapter summaries on page v need to be reworded in specific areas. In chapter 7, for example, the summary does not detail the lesson or its connection to mathematics. Other chapter summaries are somewhat wordy. Each lesson needs to provide recommendations to grade levels and modifications for lower achieving students. It would be helpful if page vii included a key for NCTM Standards (for example: Shaded areas are connected to NCTM Standards).

However, overall this is an excellent activity book for mathematics in middle school. The content is sufficient, discovery-based, and teacher friendly, and the extensions are wonderful for mastery to distinguish achieving students. As a teacher, I am encouraged to use the resource because it is definitely easy to read and follow.—*Kelli Ballenger, Summers Middle School, Hinton, WV 25951.*

Reasoning with the Rekenrek, *Angela Giglio Andrews and Diane Cushing Liesen, 2006. 47 pp., $18.95 paper. ISBN 90-70190-24-9; Product #TB 23366T. Nasco Math; (800) 558-9595; www.eNasco.com.*



This teacher's guide on how to incorporate Rekenrek materials into mathematics lessons for students in grades K–3 is a type of arithmetic rack that has some visual similarities to an abacus. The lesson ideas in the book focus on using Rekenrek materials that contain two sets of ten beads. The purpose of these lesson ideas is to support student growth in developing efficient and flexible thinking strategies that go beyond using counting to solve addition and subtraction problems. The strategies highlighted in this book include the part-part-whole relationship, doubles, near doubles, and emphasizing the use of 5 and 10 as reference points.

The book has an easy-to-read manner with clear, concise lessons. However, the lesson ideas are not necessarily meant to be taught in the order in which they are presented. Instead, the reader will need to first identify strategies that students are currently using and then choose a lesson idea. The authors do, however, tell the reader when it is appropriate to introduce an activity. The lesson ideas also include step-by-step directions, a list of materials, teaching notes, and extended lesson dialogues. I appreciated the lesson dialogues that describe possible student responses and questions to help support students in the development of a particular strategy. I would recommend this book to educators who want to focus instruction on developing efficient, flexible strategies rather than just on memorizing basic facts.—*Heidi J. Higgins, Missouri State University, Springfield, MO 65897.*

Give Us Your Suggestions

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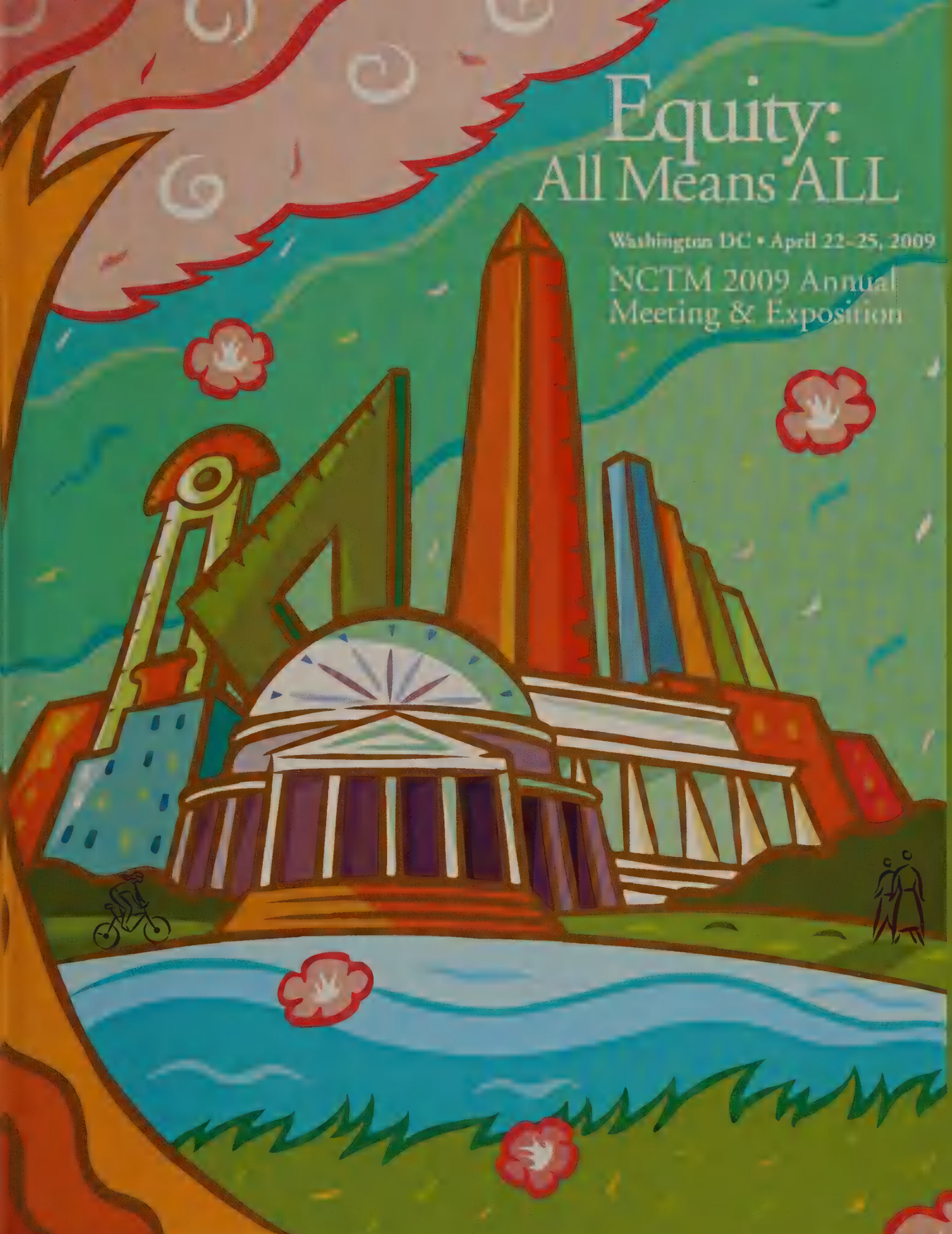

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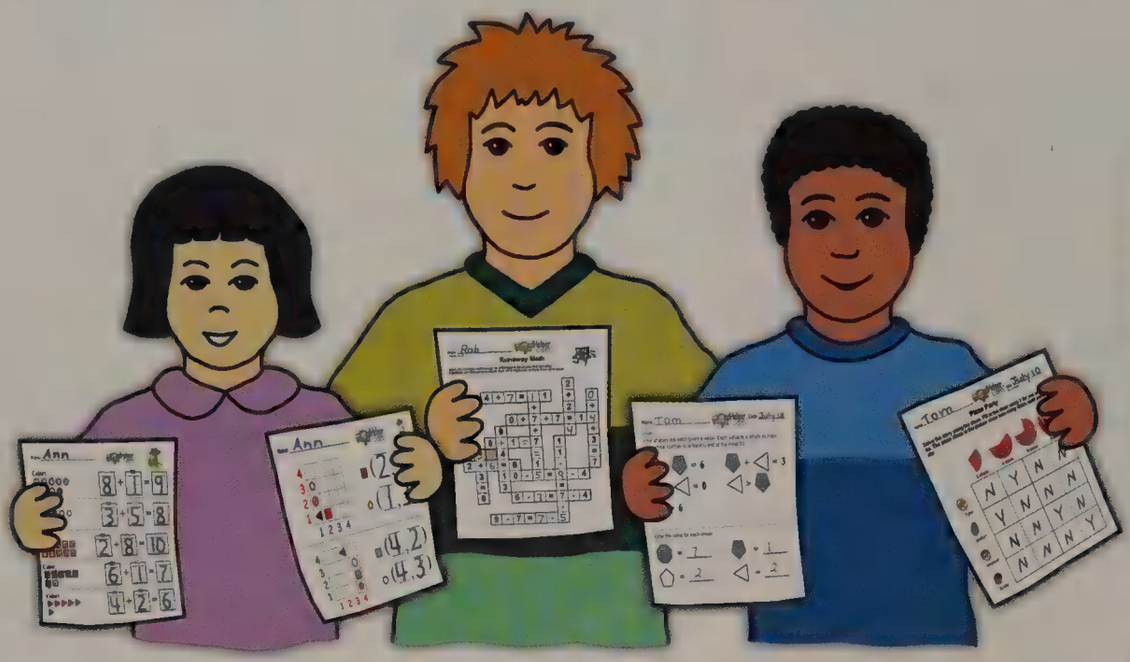

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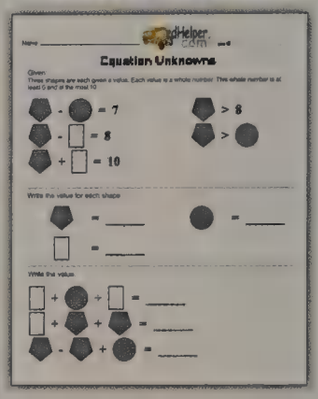

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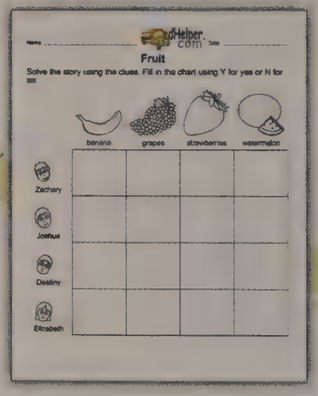
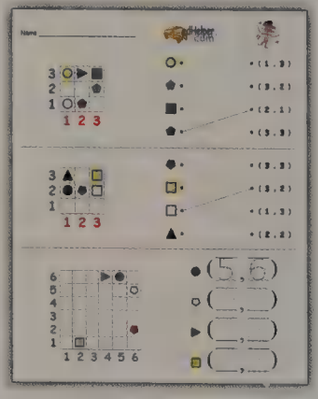
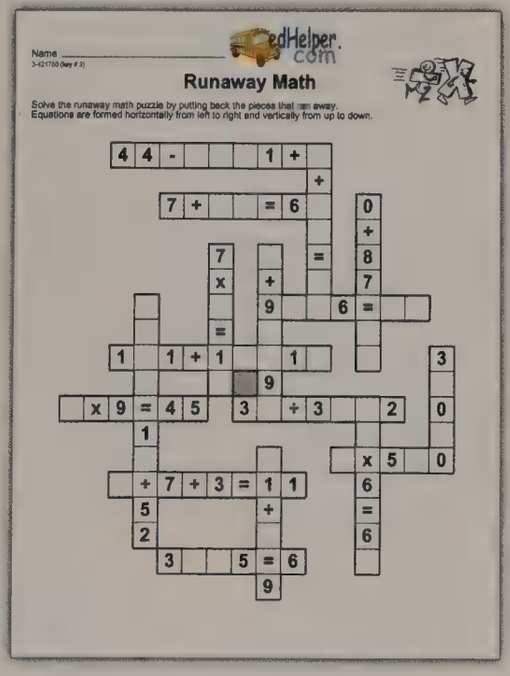
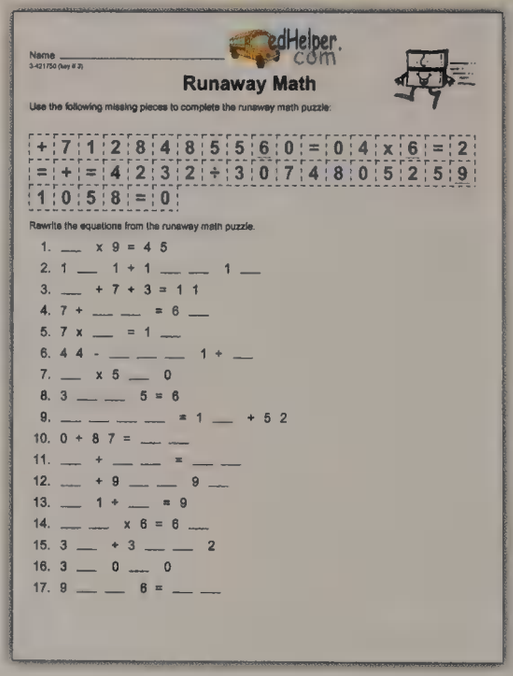


Daily Math



New Runaway Math


Runaway Math
Solve the runaway math puzzle by putting back the pieces that ran away.
Equations are formed horizontally from left to right and vertically from up to down.


Runaway Math
Use the following missing pieces to complete the runaway math puzzle:
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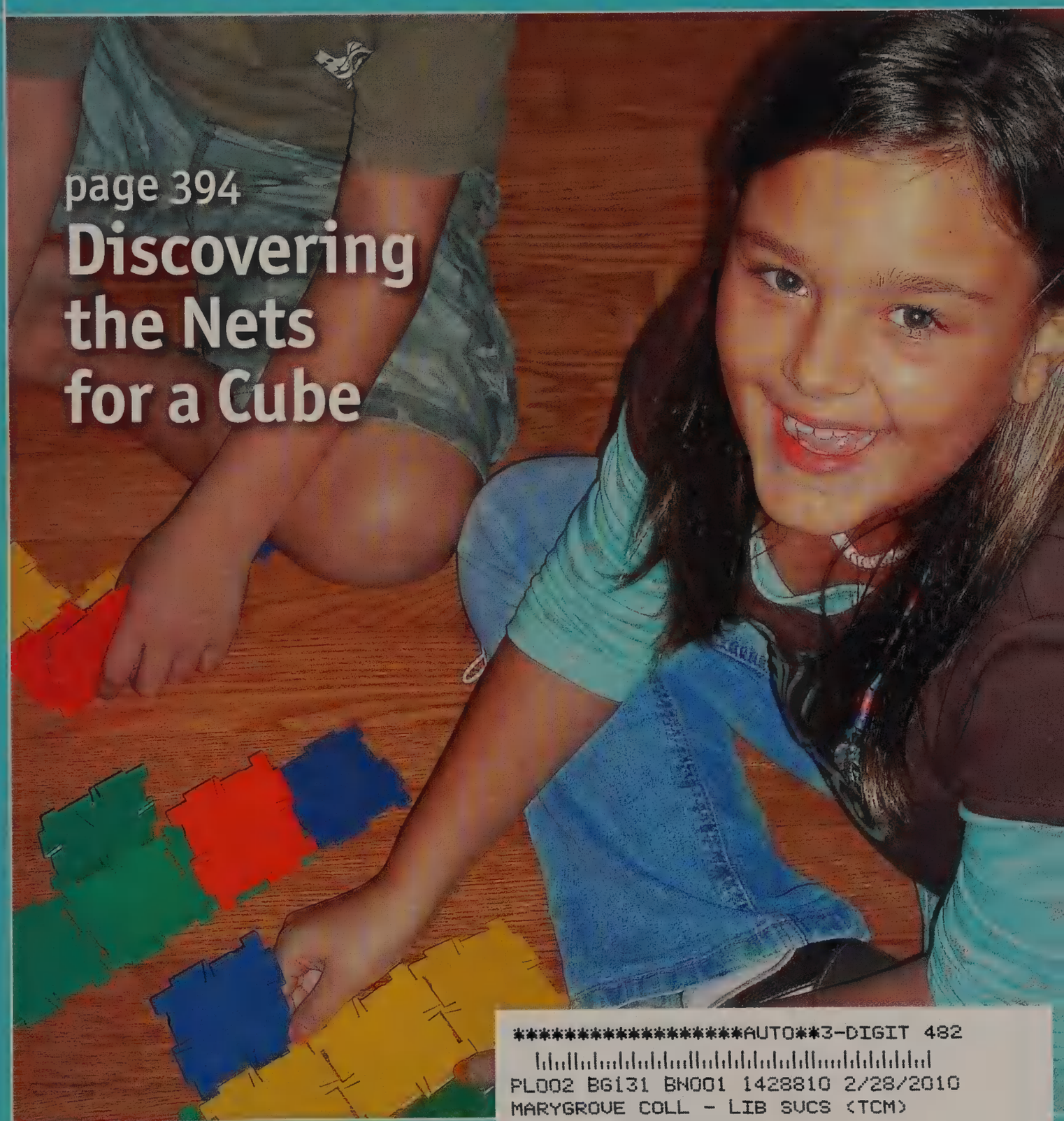

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Discovering the Nets for a Cube

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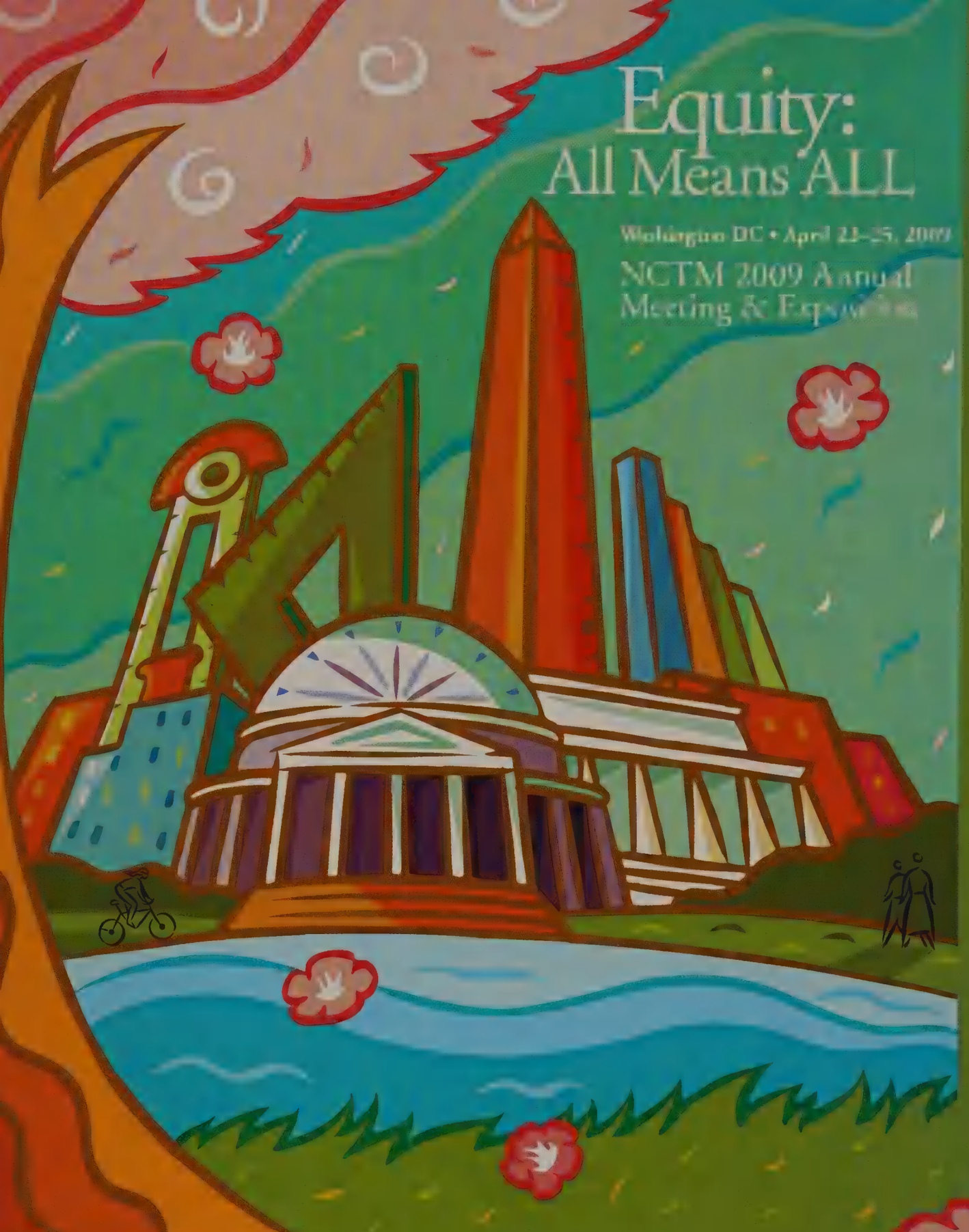

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TEACHING CHILDREN
Mathematics

Volume 15, Number 7,
March 2009

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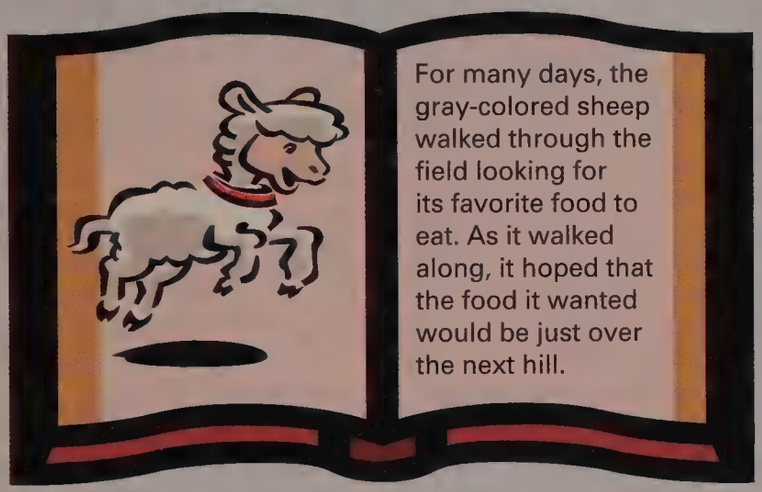
The Library Problem

Problem

This month our mathematics road trip takes us to the W. J. Niederkorn Library in Port Washington, Wisconsin. March is Reading Month, and we are celebrating at the library. We are searching for a familiar picture book that we know and enjoy or a new one that looks interesting. After locating just the right book, we take the time to read the story. One page has the following passage:


For many days, the gray-colored sheep walked through the field looking for its favorite food to eat. As it walked along, it hoped that the food it wanted would be just over the next hill.

Notice that some of the words on the page are longer and some are shorter. The minimum word length on the page of our book is two letters, and the maximum length is eight letters. You will help analyze the number of words and letters on the sample page.

Use a strategy to keep track of the word count and the letter count. What word length appears most frequently? What word length is used the least? What is the total word count for the page? What is the ratio of three-letter words to all the words on the page? What is the ratio of five-letter words to all the words on the page?

Now it is your turn to search your library for a picture book. Complete the following tasks to analyze the book you selected:

- Randomly pick two pages of the book.
- Construct a table, record the results, and tally the length of the words on each of the two pages.
- For each page, determine the length of the word that appears most often and thus has the highest ratio.
- For each page, determine the largest ratio for any word length. (For lower grades, determine the largest number of a particular word length.)
- Write a comparison of the two pages that you examined. Include word-length patterns that you see. Do the words on each of the two pages all have about the same number of letters, or does one page have many longer words than the other? Include the name and author of the book as well as a list of the words from the pages that you compare.

Extensions

- Are the results similar to another book by the same author? Be sure to check several different pages chosen at random.
- Is it possible to determine the difficulty of a picture book by looking at the tally marks on a table? To explore this question, find an easier book and a more difficult picture book to read. Compare and contrast the tally marks on the tables for each of the books.
- Find the word length mean, median, or mode for a particular page in this book. Use measures of central tendency to compare pages or books.

Edited by **Brian Schad**, *schad@aaps.k12.mi.us, a fifth-grade teacher at Lawton Elementary School in Ann Arbor, Michigan;* **Joseph Georgeson**, *jgeorgeson@usmk12.org, middle school mathematics department chair and teacher of eighth-grade students at the University School of Milwaukee in Wisconsin; and* **Sarah Bunten**, *sbunten@gmail.chelsea.k12.mi.us, a third-grade teacher at Pierce Lake Elementary School in Chelsea, Michigan.* *Each month this section of the "Problem Solvers" department features a new problem for students. Readers are encouraged to submit problems to the editors to be considered for future "Problem Solvers" columns. Receipt of problems will not be acknowledged; however, problems selected for publication will be credited to the author.*

Photograph by Joseph Georgeson; all rights reserved

The W. J. Middlekoop Library in Port Washington, Wisconsin, sets the context for our problem-solving journey through picture books

Many schools across North America celebrate the month of March as Reading Month. This month, "Problem Solvers" joins the reading celebration. The Library problem, which involves books, pages, and word counts, can open the door to understanding number, number relationships, and number representations. It also gives a context to studying statistics and data analysis in a meaningful way. The problem shows how a simple question or task can lead to many significant mathematical ideas.

Classroom Setup

Begin by sharing the book passage from the problem statement. Encourage your students to notice the length of the individual words on the printed page of our book. The minimum word length is two letters, and the maximum length is eight letters. Have students construct a chart or table and keep track of the letter count by using tally marks. For example, the first word is *For*, so we place a tally mark next to the number 3 on the table because there are three letters in this word. The second word is *many*, so we place a tally mark next to the number 4 because there are four letters in this word (see **table 1**).

Table 1

Tally Chart for Words

No. of Letters	Tally
1	
2	𝍸 /
3	𝍸 ///
4	𝍸 𝍸
5	𝍸
6	///
7	///
8	/

After tallying every word from the selected page, we can see which word length appears most frequently and which word length appears the least often. Adding all the tally marks together gives us the total number of words on the page.

Our book has a total of thirty-six words on this sample page. Ten of the thirty-six words have four letters. All of the other words have fewer than ten letters. The ratio of four-letter words is ten to thirty-six, which can be written symbolically as 10/36 or 10:36. This is the largest ratio of four-letter words to all of the words for this page.

Now let's compare our results with another randomly selected page. This time, five-letter words appear most often, nine of thirty-three times. Ask your students which ratio is larger, the 10/36 from the first page or the 9/33 from the second page. Determine together that ten of thirty-six is larger.

Your students' readiness will establish whether they are able to explore a book on their own or by working in pairs. Alternatively, you may want to do another whole-class activity by selecting a picture book that is a class favorite. After reading the selected book aloud to the class, turn to a page that you randomly selected before the lesson. Have a prepared transparency of the page (or something similar, such as the same page from a "Big Book" version of the story) so that each student can see and understand the problem you will present. On the board, construct a table. As a class, tally the lengths of words found on the page. Then analyze the results using the list of tasks presented in the problem.

Hints

- It is easier to start with the first word on the page and tally each word until you reach the last word rather than to tally the total number of two-letter words, three-letter words, and so on.
- This activity works better when students collaborate in pairs. One student counts the letters in the words while the other completes the table.
- For lower grades, have students select a book from the library that they are interested in reading. Collect the data as described in the problem. Have them determine which word length appears most often. How often does it appear? Repeat the activity with another page from the same book. The second selected page should have about the same number of words as the first. Compare the results of the two pages. Ask students which page has the largest number of a particular word length. If desired, repeat with a third page from the same book.
- For upper grades, have students select a book from the library that they are interested in reading. Complete the activity as described in the problem. You may want to have students randomly select three different pages from the book instead of two. This will provide them with more data to compare when they analyze their results at the end of the activity. To determine the ratios found on each of the pages, encourage students to use arithmetic, pictures, or manipulatives.

Data analysis (making observations from collected data) is a valuable skill for students and all citizens. In a democracy, numerical information is constantly presented; the ability to understand that information and give it meaning is essential. The Library problem gives students the opportunity to collect meaningful data, organize it, and then interpret the meaning it has for them. Other statistical concepts could be added by finding the mean, median, or mode word length for a particular page in this book. Those measures of central tendency could also be used to compare pages or even books. Other book data, such as the number of words in the sentences on a particular page or in a section of a book, can also be collected and analyzed.

Monitor students as they work on this problem. Note the methods that students use to determine their results. Listen closely as they discuss their findings. After the activity has been completed, have students use their comparison papers to discuss with the entire class how they worked the problem. Look for similarities or differences among the various books that were read. Conclude the discussion by stating the answers and the largest ratios that were found.

Where's the Math?

This problem considers the major areas of number and number relationships. Using fractions, ratios, decimals, and percentages in a meaningful way are basic to understanding much of mathematics. Understanding and moving among those representations are important skills. Additionally, this problem offers the opportunity to collect and analyze data in a real-world context. Similar methods are used to determine the reading level or readability of a book.

Students often find rational numbers—in the form of fractions, decimals, percents, or ratios—difficult to understand. Part of that difficulty often stems from the numbers being presented apart from a meaningful context. In the Library problem, students can understand the idea of comparing a part (the number of words with some particular number of letters) to a whole (the total number of words) in the form of a ratio. Ratios are comparisons, so it makes sense to compare the words in this manner. The connection between the concepts of ratio and division (and a common shared symbol for each) can also be made here in a meaningful way.

One way to compare ratios is to write them in the form of a decimal. Because a ratio is an implied division, it often makes sense to compare ratios by using division to convert them to decimal representations.

Another method for comparing ratios is to estimate their size in relation to benchmarks on a number line, such as 0, 1/2, and 1. For example, a set of fractions can be classified according to which of those three benchmarks each fraction is closest to. Using a number line gives students another context for looking at numbers.

Share Your Students' Work

Please try this problem in your classroom. We are interested in how your students responded to the problem, what problem-solving strategies they used, and how they explained or justified their reasoning. Include information about how you posed the problem, samples of student work, and photographs. E-mail your thoughts, reflections, scanned student work, and photographs to **Brian Schad** at schad@aaps.k12.mi.us by **May 1, 2009**; or send them to his attention at Lawton Elementary School, 2250 S. Seventh St., Ann Arbor, MI 48103. Include your name, grade level, and school name with your submission. Selected submissions will be published and acknowledged in a subsequent issue of *Teaching Children Mathematics*. ▲

(Solutions to a previous problem begin on the next page.)

Solutions to the Using Your (Number) Sense of Balance Problem

Exploring algebraic reasoning becomes fun and challenging when students can do so in context: Using Your (Number) Sense of Balance appeared in the March 2008 "Problem Solvers" section and was stated as follows:

Problem

In the balance puzzles in the figures, each shape is assigned a numerical value representing its weight. To solve these puzzles, make the total weight on each arm "balance," just as a sculptor might balance all the parts of a mobile.

Use the following rules for all the puzzles:

- Each shape has a unique weight and weighs more than 0 units.
- All weights are whole numbers.
- Different shapes have different weights, and identical shapes have the same weight. (Note: Each puzzle is a clean slate, so identical shapes may have different values from puzzle to puzzle.)
- The size of the shapes is not related to their weight.
- A shape hanging directly below the fulcrum does not affect the balance of the arms to the right or left of the fulcrum.
- The distance of the shapes from the fulcrum should not be taken into account when figuring out the values. (In making a real-life mobile, however, this would need to be considered.)

Problem 1. Figure out the weight of each shape in the mobile in **figure 1**. The total weight is 24 units. All shapes weigh less than 10 units.

Problem 2. Figure out the weight of each shape in the mobile in **figure 2**. The total weight is 60 units. All shapes weigh less than 10 units. Try to find more than one solution.

Problem 3. Figure out the weight of each shape in the mobile in **figure 3**. The total weight is 88 units. All shapes weigh 10 units or less. How many different solutions can you find?

Problem 4. Have each student create a mobile balance problem along with its solutions on a separate page. Ask pairs of students to exchange problems (without solutions) and try to solve them. Alternatively, collect students' problems and use 1–2 of them each week as fun puzzles for the entire class.

Variations

The first two problems are moderately difficult. Decide whether you will give students the clues for problem 2 (see **fig. 2**) ahead of time or reveal them only after students have wrestled with the problem for a while. The third problem may prove difficult for younger students and may be skipped. These problems are based on ideas from *In the Balance* by Lou Kroner (New York: Creative Publications, Wright Group/McGraw-Hill, 2000).

Figure 1

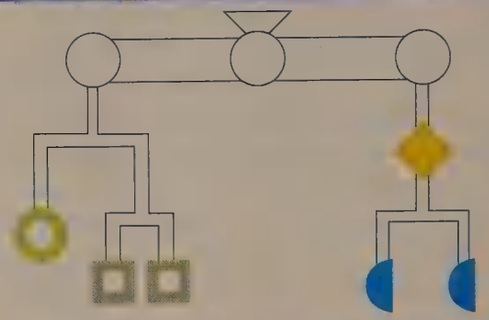


Figure 2

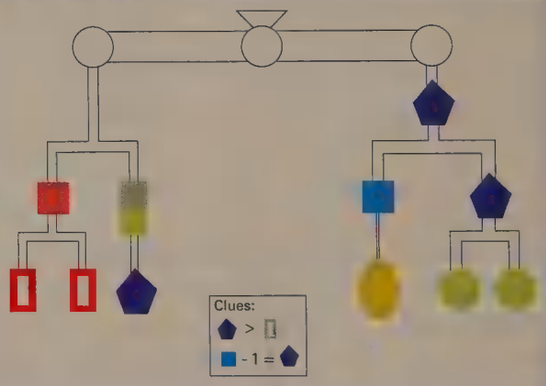
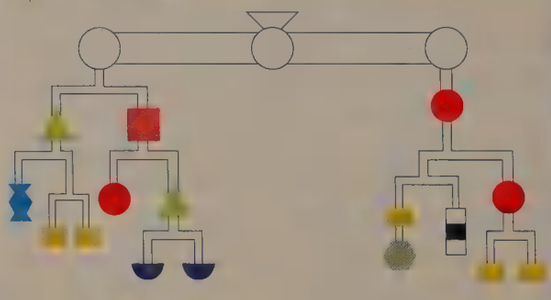


Figure 3



Several teachers in the Fullerton Elementary School District in California had their fifth and sixth graders try the Balance problem as part of a summer enrichment program. (Readers, avoid looking at the solutions in the last figure until you have tried these puzzles.) Teachers first showed students a picture of a two-pan balance and asked them what it means when the two sides are at the same height (level). Most students responded with statements such as, "The two sides are equal" or "Each pan weighs the same amount." The teachers discussed this idea of equivalence with students by asking them to consider the following problem:

$$5 + 3 = \_\_\_ + 6.$$

A few students quickly answered, "Eight." The teachers asked them to think back to the two-pan balance idea of equality: "Is five plus three equal to or the same as eight plus six?" This question helped students realize that two is the correct value to make the equation true.

Teachers then introduced the Using Your (Number) Sense of Balance problem. Using the images from the published article (see **figs. 1**, **2**, and **3**), the teachers printed the first and second problems on separate pages. In the first problem, where the total weight is twenty-four, students immediately understood that each arm of the mobile had to have a total of twelve. They found it challenging, however, that the different shapes had to have unique numerical values. Students worked methodically, trying different combinations that would sum to twelve but would use different digits. Many students successfully found valid combinations so that each main arm of the mobile had a total weight of twelve. For example, sixth-grader Kim was careful to show both visually and arithmetically that the two main arms of the mobile were equivalent (see **fig. 4**).

All the students correctly took into account that the two main arms of the mobile had to be equal (just as in the two-pan balance), but they did not always ensure that the two sub-arms on the left side were in balance (equivalent) as well. This error likely resulted because the condition was not clearly stated in the problem description. (Even experienced problem writers make errors. When using the Balance problem in the future, teachers should let students know that at any point where an arm splits into two or more sub-arms, the weight of each sub-arm must be equivalent to the others.)

In spite of this oversight in the directions, several students did in fact keep all the mobile arms in balance. Sixth-grader Daniel offered proof that his solution worked by writing a number sentence that included each of the weights (see **fig. 5**). Sharing his reasoning in words, fifth-grader Zachary explained that "the circle [on the left arm] has to be equal with the sum of the squares, so the squares are three and

Figure 4

Kim shows both visually and arithmetically that the two main arms of the mobile are equivalent.

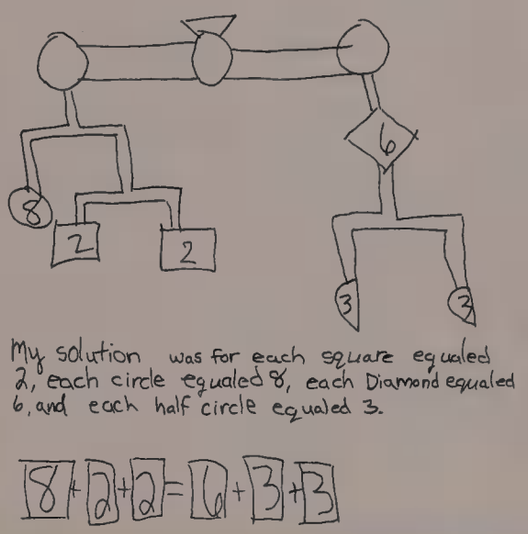
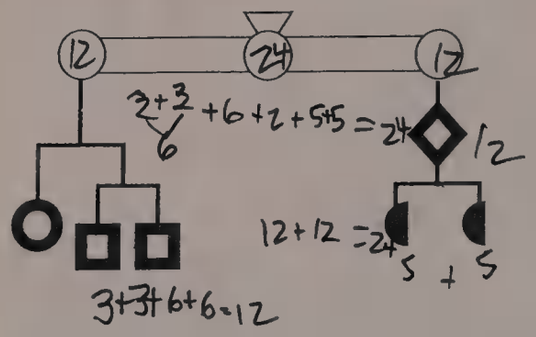


Figure 5

Daniel offers proof that his solution works by writing a number sentence that includes each of the weights.



Edited by **Mark Ellis**, *mellis@fullerton.edu, an assistant professor in the College of Education at California State University–Fullerton, where he prepares teachers of middle school mathematics and works with local classroom teachers in creating learning environments that support sense making in mathematics; and* **Cathery Yeh**, *catyeh@aol.com, who teaches fifth grade at Arnold Elementary School in Cypress, California. Each month this section of the "Problem Solvers" department discusses the classroom results of using problems presented in previous issues of* Teaching Children Mathematics.

Figure 6

Zachary found two unique solutions to the first problem.

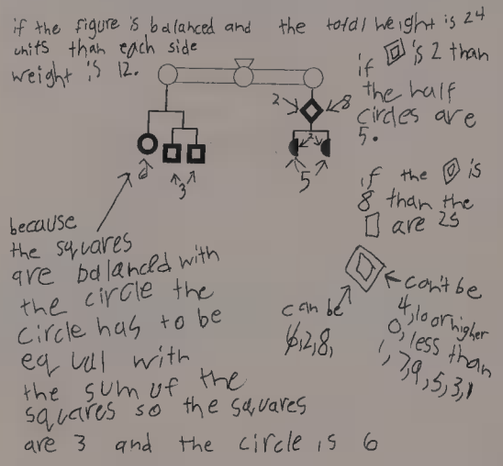

if the figure is balanced and the total weight is 24 units than each side weight is 12.
if ◇ is 2 than the half circles are 5.
if the ◇ is 8 than the □ are 25
because the squares are balanced with the circle the circle has to be equal with the sum of the squares so the squares are 3 and the circle is 6
can be 6,12,8,
can't be 4,10 or higher 0, less than 1, 7,9,5,3,1

Figure 7

Although Raymond solved one of the puzzles within the problem, his solution does not preserve the balance within the sub-arms.

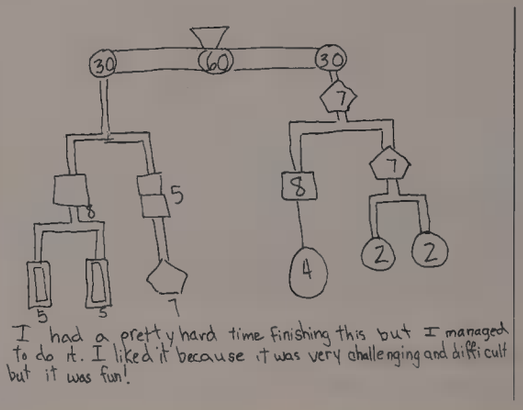

I had a pretty hard time finishing this but I managed to do it. I liked it because it was very challenging and difficult but it was fun!

Figure 8

Puzzle solutions

Figure 1 Solution

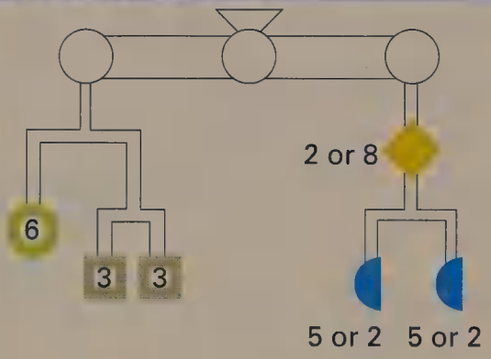


Figure 2 Solution

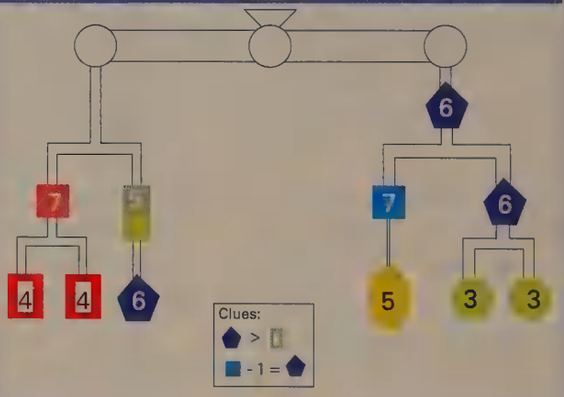
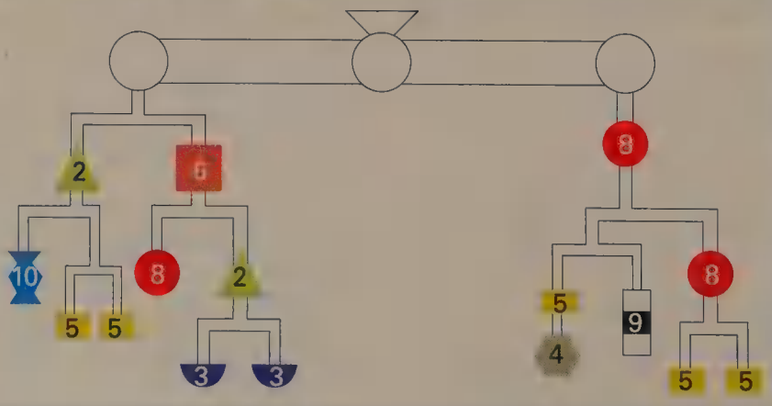


Figure 3 Solution



the circle is six" (see **fig. 6**). Zachary also discovered that "if the diamond [on the right arm] is two, then the half-circles are five each. If the diamond is eight, then the half-circles are two each." He found two unique solutions to the first problem.

The second mobile-balance puzzle proved to be more challenging for students. Everyone understood that each side of the mobile had to weigh thirty units because the total weight of sixty was cut in half. From that point, many used a guess-and-check strategy in trying to find a solution. The comments of sixth-grader Raymond sum up well what many students felt: "I liked it because it was very challenging and difficult, but it was fun!" His solution (see **fig. 7**) gives an equivalent weight of thirty for the left and right sides of the mobile, thus solving one of the puzzles within this problem. However, his solution does not preserve the balance within the sub-arms, an aspect that adds complexity to the problem. (See **fig. 8** for correct puzzle solutions.)

The students' understanding of equality was stretched through their work on the problem. They moved from thinking of the equal sign as an operator (e.g., 3 + 5 = ?) to recognizing it as representing a relationship in which two expressions have the same value (e.g., 3 + 5 = 2 + 6). This "relational" understanding of the equal sign will help students as they move from arithmetic to algebra.

Showing students a hanging mobile may help them understand that any time two or more sub-arms split, the weight below that split must be evenly shared among the sub-arms to stay balanced. Because of its challenging nature and the need for students to logically use a guess-and-check method, this activity is a great "brain stretcher" for them.

The authors wish to thank all the Fullerton Elementary School District fifth- and sixth-grade teachers who shared their students' work on this problem: Karen Alvarado, Alice Cha, Heidi Cockerill, Marcy Fry, Glenda Howell, Gordon Jones, Eileen Kroh, Devi Lam, Suzanne Lee, Hai Mee Park, Elena Perez, Dolph Petris, Kelly Pellizzari, Jesus Uribe, and Pamela Zinnel. Thanks also to Marty Bonsangue at California State University–Fullerton for helping with the data collection. ▲

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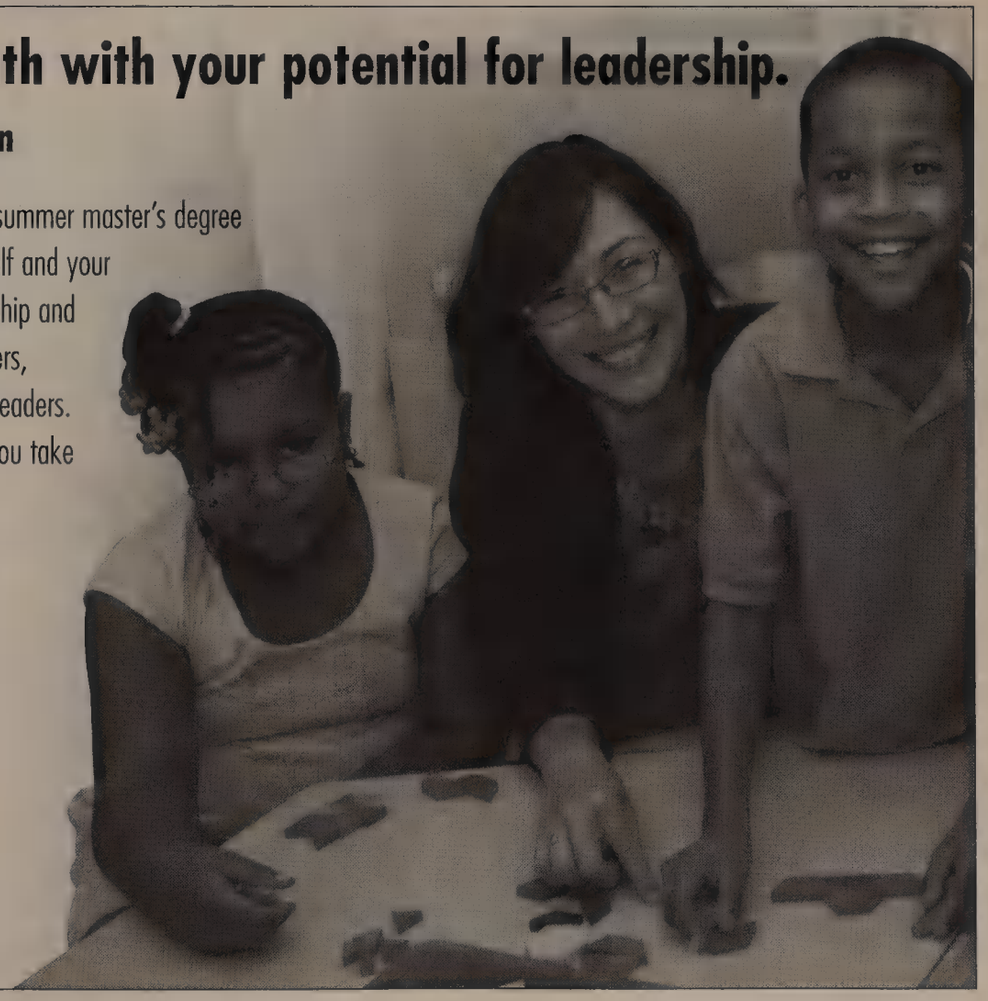

Mathematics Hiding in the Nets for a CUBE

Whether they are third graders or teacher candidates, students can learn about perimeter and area while having fun manipulating two-dimensional figures into three-dimensional objects.

A common mathematical activity for geometry students is to create nets for a cube. A net is a two-dimensional figure that can be folded into a three-dimensional object. Nets help students make the connection between the concept of two-dimensional figures and three-dimensional objects. Moreover, nets help children develop a sense of the three-dimensional world.

Third Graders' Thinking

In Mann's 2004 article, "Thinking beyond the Box: Responses to the Building a Box Problem," elementary school children explored the question, "How many different nets can you draw that can fold into a cube?" The children found many different nets for a cube. They also observed common characteristics of nets for a cube. Of

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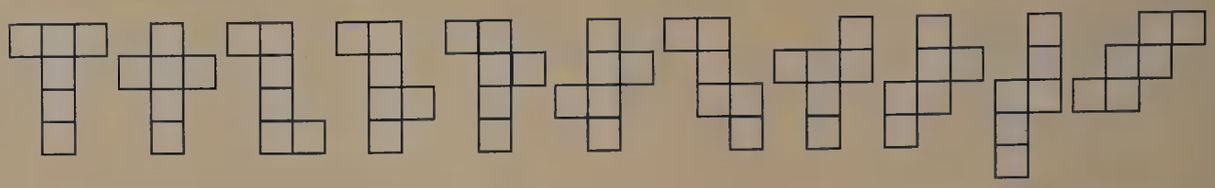
Figure 1

A cube has eleven nets.



all their discoveries, third-grader Morgan's conclusion about cube nets is particularly intriguing:

> Each net has 6 squares, and all of them have at least 4 all together and 2 on each side. If there were 3 in the middle and 3 on the side, it would not work. *All nets have a perimeter of 14 units.* They can be arranged by flips and turns. All nets start as 2-D shapes and become 3-D figures. *The area of each net is 6 square units.* The nets all have 6 square faces to make a cube [italics added]. (Mann 2004, p. 174)

The class of third graders continued investigating Morgan's claim that each net had the same area and the same perimeter. Students were amazed that although the nets were different shapes, they all had the same perimeter of fourteen units and the same area of six square units. **Figure 1** shows all possible nets for a cube, eleven noncongruent nets in total. By making connections between nets in two dimensions and cubes in three dimensions, the author of this article explains why the nets have the same area and, more interesting, the same perimeter.

Adults' Thinking

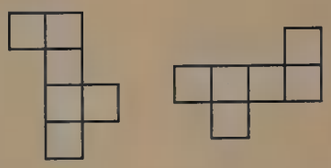
In a mathematics class for future elementary teachers, teacher candidates found the nets for a cube by using six square Polydrons™. Polydrons allow students to conjecture a net and verify if the net works by physically forming a cube from the net.

Net exploration

During the process of conjecturing a net and forming a cube, students recorded their results on grid paper for later group discussion. Whenever a student found a new net, the professor also recorded the results on grid paper posted in the front of the room. While finding common characteristics of the nets, students engaged in a rich discussion that went beyond analyzing area and perimeter. For example, a net suggested by one student was congruent through rotation and translation to one that had already been recorded in the class data (see **fig. 2**). The class debated whether or not to consider this new net as a different net. A justification such as "rotating the first net by 90 degrees and translating will make the first one congruent to the second" helped the class redefine the project from finding *different* nets for a cube to finding *noncongruent* nets for a cube. After the discussion, students' knowledge of transformational geometry became critical for evaluating whether or not a new net would be congruent to the ones they had already found.

Figure 2

Are these different nets or not?



The Cube Nets problem provided a rich set of examples that helped students address a misconception that some of them had regarding congruency: "If two figures have the same area and the same perimeter, wouldn't they be congruent?" All the nets have the same area and the same perimeter, but they are not all congruent to one another.

Net analysis

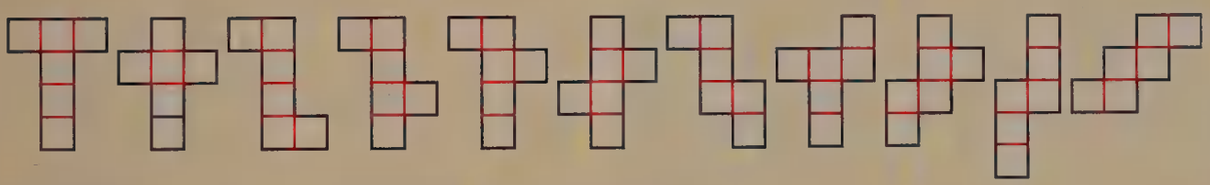
Why does each net have an area of six square units? An area of six square units in every net for a cube is not surprising. One student explained that a cube has six faces and that is why a net for it has an area of six square units. This reasoning is similar to that used by Morgan, the third grader: "The nets all have six square faces to make a cube."

Why do all the nets for a cube have a perimeter of fourteen units? Understanding the reason is challenging to justify. For these future elementary teachers, a careful analysis of all the nets was necessary (see **fig. 3**).

During the process of finding common characteristics of the nets, student teachers engaged in a rich discussion that went beyond analyzing area and perimeter to considering congruency.

Figure 3

The fold lines for the eleven nets for a cube are emphasized in red.



Students discovered that the fold lines are the five cube edges when a net transforms into a cube. After noticing that a cube has twelve edges in total and observing the five red line segments, a student asked where the seven remaining edges come from. This key question relates to the mathematical question, When each net transforms into a cube, what happens to each line segment that is the part of the net perimeter? Students found that each line segment would meet another line segment and form an edge of a cube (see **fig. 4**). A student explained:

> One pair of the fourteen units of the perimeter joins to form an edge of the cube. Another pair of the perimeter units forms another edge, and so on. The perimeter of fourteen units, therefore, will form the seven remaining edges so that the total number of the edges of a cube is twelve (five red fold lines plus seven, half the perimeter of fourteen units).

Figure 4

A net can be transformed into a cube.

The professor further challenged students with more questions: What happens if the perimeter is not fourteen units long? What if the perimeter is greater than or less than fourteen?

Students analyzed the two cases. If the net perimeter is greater than fourteen units, the number

Photograph by Tracy Hall-Ingram, all rights reserved

A net is a two-dimensional figure that can be folded into a three-dimensional object.

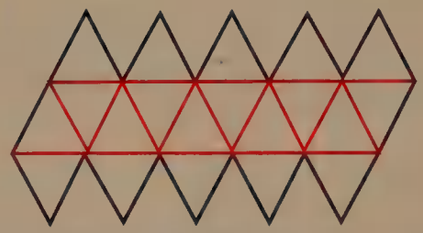
Photograph by Luanne M. Flom; all rights reserved

Figure 5

Students examine two nets for a tetrahedron and find that each net has three fold lines.

Figure 6

An icosahedron is a platonic solid with twenty faces.



of edges formed from such a perimeter is more than seven. This means that a cube would have more than twelve edges. If the net perimeter is less than fourteen units, the number of edges formed from such a perimeter is less than seven. This means that a cube would have fewer than twelve edges. Students concluded that neither of these two cases was possible for a cube. This explains why every net for a cube has neither more than nor less than a perimeter of exactly fourteen units in addition to the five fold lines inside each net.

From Cube Nets to Platonic Solids

Students extended the Cube Nets lesson to nets for other platonic solids. They noticed that the property of the same number of fold lines in each net and the same perimeter of every net can be extended to any net for platonic solids. When students examined the two nets for a tetrahedron, they found that each net has three fold lines (emphasized in red in **fig. 5**; the tetrahedron is known to have only two noncongruent nets). Each net comprises four equilateral triangles, which become faces of the tetrahedron as it folds. Both nets also have a perimeter of six units when the side length of the equilateral triangles is assumed to be the length of one unit.

When a tetrahedron net is folded, those three fold lines become three of the tetrahedron's edges. The perimeter of six units in each net forms the remaining three edges of the tetrahedron, producing a tetrahedron's six edges. On the basis of this extension, students conjectured that every net for each platonic solid has the same area in order to form the correct number of faces and also has the same perimeter in order to form the correct number of edges.

Before they folded the net and actually counted faces, edges, and vertices, the professor had students apply their conjecture to Euler's formula in order to find the number of faces, edges, and vertices for an icosahedron, a platonic solid with twenty faces (see **fig. 6**). Euler's formula is the relationship among the number of vertices (V), edges (E), and faces (F) of a polyhedron ($F + V = E + 2$). Together we found the following to be true:

1. This net is composed of twenty equilateral triangles. The number of faces of an icosahedron is, therefore, twenty ($F = 20$).
2. The net has nineteen line segments within it. They will be edges of the icosahedron when folded.
3. The perimeter of the net is twenty-two units, and each pair of the twenty-two will form eleven edges when the net is folded.
4. The total number of edges of an icosahedron is thirty, since $19 + 11 = 30$ ($E = 30$).
5. Euler's formula, $F + V = E + 2$, then, shows that the number of vertices of an icosahedron is twelve, since $V = E - F + 2 = 30 - 20 + 2 = 12$ ($V = 12$).

This article assumes that students understand the concept of the perimeter of a two-dimensional figure, an assumption that may need to be verified by teachers of young learners. Observe carefully how students find the perimeter of a particular cube net. Once they demonstrate a strong understanding about perimeter as the distance around an object, they can more meaningfully start reasoning about why the perimeters of the nets for a cube are the same.

Final Remarks

Classrooms where students like Morgan and her third-grade classmates actively discuss mathematical ideas and concepts are an exciting place to be. In such settings, discovery of mathematical relationships is an important goal for learning mathematics in the ways that NCTM's *Principles and Standards* (2000) advocate. Students conjecture, analyze, justify, and communicate why every net has six unit squares and why every net has a fourteen-unit perimeter. The mathematics discussed in this article helps students see why it is not just a coincidence but a significant connection that a two-dimensional net transforms into a three-dimensional cube. All cube nets must have the same area in order to form the correct number of cube faces; all cube nets must also have the same perimeter in order to form the correct number of cube edges.

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———. *Principles and Standards for School Mathematics.* Reston, VA: NCTM, 2000.

More Geometry for Young Learners

Principles and Standards calls for instructional programs that analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships (NCTM 2000, p. 41). The activities described in this article provide teachers with an excellent opportunity to design geometry lessons that nicely reflect the Standards. Start with an activity, for example, where your students try to find every possible net for a cube. If they are unable to find all eleven nets, have them visit NCTM's Illuminations Web site, where an applet provides twenty-four nets that may or may not form a cube. The discussion might start with the question, "How many nets for a cube are possible?"

As they use this Web site, help students justify why only eleven nets exist for a cube. A systematic analysis of the eleven nets is an authentic mathematical experience for young learners to reason and verify mathematical conjectures. Use questions to help students do an analysis:

- Which nets have four squares in the middle?
- Which ones have three squares in the middle?
- Which ones have two squares in the middle?
- And, more important, why are there only eleven possibilities?

Challenge students

As students explore the relationship between area and perimeter in the Cube Nets problem, address another important mathematical point: Can a net with an area of six square units and a perimeter of fourteen units always be folded into a cube? This question can challenge your advanced learners even as discussions on such examples benefit the whole class in understanding the necessary but insufficient relationship. If a figure is a net for a cube, then it has an area of six square units and a perimeter of fourteen units. This is what Morgan and her classmates discussed in their search for all nets for a cube. However, the reverse is not true. The fact that a figure has an area of six square units and a perimeter of fourteen units does *not* imply that the figure will be a net for a cube. **Figure 7** shows a net with an area of six square units and a perimeter of fourteen units that will not fold into a cube.

When students work with the eleven nets for a cube, observe carefully how they find the area of a net, another fundamental concept in geometry and measurement at the elementary level. Make sure that students, especially younger ones, can communicate why counting the number of unit squares in a net gives the area. Typical questions on area depend on a formula such as width times length for a rectangle, but the irregular shapes of the nets can help students revisit the idea of area from a nontraditional perspective.

Figure 7

This net with an area of six square units and a perimeter of fourteen units will *not* fold into a cube.

Mathematics

This update of the groundbreaking NCTM publication Professional Standards for Teaching Mathematics *provides more high-quality tools to improve instruction and learning*

Are there more multiples of 3 or multiples of 8? How can a diagram be used to represent $7 \times 1/2$ as well as $7 \div 2$? What do you know about quadrilaterals? Do numbers that are greater than 36 necessarily have more factors than numbers that are less than 36? If we have 49 sandwiches for our picnic, how many sandwiches can each child have? How many noses are in our class? How many ears are in our class? How many fingers are in our class? What kind of rule will fit the pattern we have observed?

These are just some of the types of questions used to help illustrate worthwhile mathematical tasks, rich discourse, and positive classroom environments for teaching and learning in *Mathematics Teaching Today: Improving Practice, Improving Student Learning* (NCTM 2007). If the task-discourse-environment framework sounds familiar, it is because the idea was previously shared in *Professional Standards for Teaching Mathematics* (NCTM 1991). Although much of the material in that landmark document remains sound and up to date, the NCTM Board of Directors determined that recent developments in the field—including the publication of *Principles and Standards for School Mathematics* (NCTM 2000)—warranted an update.

Mathematics Teaching Today is a resource for teachers, supervisors, teacher educators, families, business leaders, government officials, community members, and others with interest in promoting more and better mathematics for all children. In three chapters of Standards, the book outlines key aspects of high-quality mathematics teaching; a model for observing, supervising, and improving mathematics teaching; and guidelines for teacher education and continued professional growth. Those chapters are followed by an examination of the roles and responsibilities of people in the larger community who can support teachers, students, administrators, and teacher educators. Finally, reflection questions keyed to each of the sections in the three main chapters provide a tool to promote ongoing conversations and creative approaches for improving mathematics teaching and learning.

The Teaching Cycle

So, what is new in this collection of professional standards? *Mathematics Teaching Today* refers to a three-phase model of teaching practice: knowledge, implementation, and analysis (Reynolds 1992; Shulman 1987). This cyclic model (see the book cover depicted on p. 401) reminds readers that teachers draw on their knowledge of mathematics, students, and student learning as they plan for, enact, and make decisions about their teaching practice.

Teacher knowledge influences how lessons are implemented, and what occurs during class also deepens teachers' knowledge of their students, mathematics, and the effects of various teaching practices. Likewise, effective teaching demands the analysis both of student performance and of one's own teaching performance. When teachers reflect on their practice—by listening to students, ana-

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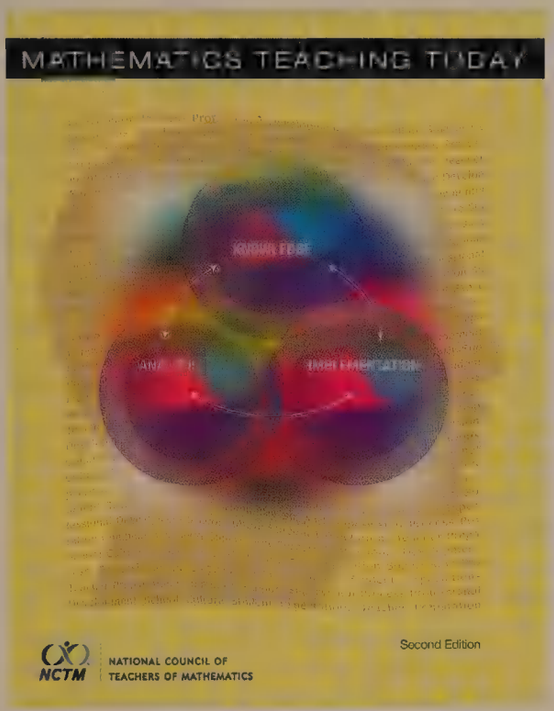
lyzing student work, and reviewing student responses to class activities—that reflective process enhances teachers' knowledge bases. And so the cycle continues.



Reflection

Consider the following example of the teaching cycle in action. Suppose that a fourth-grade teacher wants to help her students understand the relationship between factors and multiples. What knowledge will help this teacher prepare her lesson? What tasks will she pose to introduce these concepts? What sort of classroom environment will offer students the freedom to explore the relationship? How will she facilitate discourse to reflect student-initiated conjectures about observed patterns? How will she analyze students' reasoning? What responses will she formulate to encourage students to think more deeply and communicate more clearly? *Mathematics Teaching Today* provides vignettes that illustrate those aspects of the teaching cycle and that invite teachers to reflect on those issues as they relate to the teachers' own practice.

Teachers are not the only ones who analyze the teaching and learning of mathematics. Principals, department chairs, curriculum leaders, and mathematics coaches all observe teachers at work in their classrooms for various purposes. Whereas the *Professional Standards for Teaching Mathematics* suggested standards for evaluating teaching, *Mathematics Teaching Today* further emphasizes that observation occurs in the context of a cycle of continuous improvement; the book references models for such improvement. Regardless of the observer's particular objective, all teacher observation is ultimately aimed at improving teaching and learning. Thus, it is imperative that teachers be an integral part of the analysis of and reflection on their teaching. *Mathematics Teaching Today* describes how such observations may be conducted and what should be considered in defining the focus of those observations.

Observations should provide information on what the teacher and students are doing. When focusing on the teacher, observers can determine the teacher's command of knowledge and strategies for teaching mathematics as well as whether the teacher is providing adequate engagement for student learning. When focusing on the students, observers can determine whether the teacher has provided a context and opportunity for students to be engaged in significant and appropriate activities and whether the students use a variety of representations to demonstrate their mathematical thinking. Below are examples of what observers may look for as evidence that teachers and students are making positive contributions to the processes of teaching and learning mathematics.

Is the teacher—

- choosing "good" problems that invite exploration of an important mathematical concept and that allow students the chance to solidify and extend their knowledge;
- assessing students' understanding by listening to discussions and by asking students to justify their responses;
- assessing questioning techniques to facilitate students' learning and reasoning;
- encouraging students to explore multiple solutions;
- challenging students to think more deeply about the problems they are solving and to make connections with other ideas within mathematics;
- using multiple representations to foster a variety of mathematical perspectives;

- creating a variety of opportunities, such as group work and class discussions, for students to communicate mathematically; and
- modeling appropriate mathematical language and strategies with a disposition for solving challenging mathematical problems?

Are students—

- engaging actively in the learning process;
- using existing mathematical knowledge to make sense of assigned tasks;
- making connections among mathematical concepts;
- reasoning and making conjectures about problems;
- communicating their mathematical thinking verbally and in writing;
- listening and reacting to others' thinking and solutions to problems;
- using a variety of representations, such as pictures, tables, graphs, and words, for their mathematical thinking;
- using mathematical and technological tools, such as physical materials, calculators, and computers, along with textbooks and other instructional materials; and
- building new mathematical knowledge through problem solving and understanding?

Observation of mathematical teaching practices occurs in the context of a cycle of continuous improvement

Teacher Education and Professional Growth

The chapter in *Mathematics Teaching Today* that addresses teachers' education and continued professional growth reflects developments in the field that have occurred since the first edition. Of particular note is that recommendations for the content preparation of teachers at all grade levels are consistent with the current requirements set forth by the National Council for Accreditation of Teacher Education (NCATE 2006), the largest teacher education accreditation body. *Mathematics Teaching Today* also reflects recommendations from the Conference Board of the Mathematical Sciences (CBMS 2001), organized using the same grade-band structure that appears in *Principles and Standards* (NCTM 2000). As a result, this chapter now includes recommendations for the preparation of prekindergarten teachers. Expectations for them are the same as those for teachers of kindergarten through grade 5. Recommendations include demonstrated proficiency in number and operations, understanding of mathematical relationships and functions, ability to use spatial visualization and geometric reasoning, understanding of data analysis and probability concepts, and knowledge of measurement concepts and tools. Although these content expectations may seem high for teachers who work with three- and four-year-olds, such a level of mathematical knowledge is consistent with the high expectations of the National Board for Professional Teaching Standards (NBPTS 2001) for accomplished early childhood teachers.

Reflective Questions

Another new feature of the book is the collection "Questions for the Reflective Practitioner," designed to inspire thoughtful reflection and discussion by individuals or within groups of preservice teachers, practicing teachers, school teacher leaders, department chairs, principals, district-level administrators, teacher educators, professional developers, parents, community members, and others with an interest in and commitment to improving the teaching and learning of school mathematics. Although questions may be aimed at particular readers, everyone is invited to consider the issues they address from his or her own professional or personal perspective. Here are some of the reflection questions:

1. Reflect on your daily classroom routine. To what activities do you devote the most time? To what activities do you devote the least time? What messages might your classroom routines send to students about what is valued in your classroom? What messages might those routines send about the nature of mathematics?
2. What role does technology play in your classroom? How do students decide when technology would be an appropriate tool to use? What technology tools would you like your students to use more regularly? Which technology tools would you like to learn more about?
3. What are the benefits and limitations of using student work samples to assess the effectiveness

of specific lessons or units? What other information would be useful for documenting the quality of student learning?

4. How can you encourage and support collaborative planning and decision making to enhance teachers' professional growth and, ultimately, children's learning?
5. What role does problem solving play in basic content courses for teacher candidates or teachers? How are teacher candidates or teachers prepared to teach heuristics and explore multiple solution methods with their own students?

The dialogue resulting from discussions about the teaching and learning of mathematics, the supervision of teachers, and the education or continued growth of teachers ought to stimulate further investigation of issues or pursuit of strategies for improvement.

Positive Constants

What has stayed the same in this collection of professional standards? Many of the issues identified in *Professional Standards for Teaching Mathematics* are as relevant today as they were in the early 1990s. For example, inherent in the original title was a message that mathematics teachers are professionals. Teaching is a complex endeavor, and expert teachers should work collaboratively with other teachers and administrators to address issues of professional and school improvement.

The goal of more and better mathematics for all students cannot be reached by relying on teacher demonstrations, practice worksheets, inadequate class time, or six classes a day per teacher. Mathematics teachers need appropriate resources to challenge, motivate, and engage all students. Calculators, computers, and manipulatives are as integral to learning and doing mathematics as chemicals are to a chemistry laboratory. If mathematics teachers are to meet the needs of the increasingly diverse student populations in their classrooms, they need time to plan, study, reflect, develop curriculum, and confer with colleagues.

Likewise, the book delineates the roles and responsibilities of government policymakers, business and industry leaders, schools and school systems, colleges and universities, professional organizations, families, and communities. Each group has an opportunity to help improve mathematics education by supporting teachers and schools and by providing needed resources, including money, equipment, personnel, and time.

Finally, as in the original document, the beginning of each chapter spells out assumptions about teaching and learning, which are as valid today as when they first appeared:

- All students can learn to think mathematically.
- *What* students learn is fundamentally connected with *how* it is learned.
- Teaching is a complex practice that cannot be reduced to a formula or to isolated components.
- Because teaching is complex, the continuous improvement of teaching is complex.
- The education of mathematics teachers is an ongoing process. Being a teacher implies a dynamic and continuous process of growth that spans a career.

Just as its predecessor did, *Mathematics Teaching Today* raises issues, asks questions, provides a vision, and challenges readers to do their part to build on existing strengths, identify areas for growth, and make and enact plans for improvement. We are confident that those questions and issues will spark a renewed, collaborative conversation leading to better mathematical experiences and understanding for all children.

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ALIGNING THEORY WITH PRACTICE

Integrate elementary school children into university mathematics methodology courses? Yes! Doing so gives preservice teachers invaluable field experience in aligning theory with practice.

University mathematics methodology courses sometimes include preservice teachers who have little, if any, experience interacting with children. However, integrating methodology courses into an elementary context has the potential to offer preservice teachers a chance to see student learning at a deeper level (Lowery 2002). Allowing preservice teachers to learn theory-aligned pedagogy while interacting with elementary children can help them better understand how children learn and develop (Linek et al. 2003). Community-based field experiences enable preservice teachers to learn from experiences and reflections (Gallego 2001). For example, Gee (2006) described a method of integrating children and preservice teachers in a way that encouraged learning in both participant groups. In a longitudinal study by Cady, Meier, and Lubinski (2006), preservice teachers who had been in a program with children were better prepared to teach effectively.

Higher education has an impact on how teachers instruct in mathematics (NCTM 2000). Conversations and observations with preservice teachers in the university setting indicated that many of them were not learning to teach mathematics in an ideal manner. Few, if any, preservice teacher–child interactions and problem-based experiences took place outside the university classroom. Consequently, many of the preservice teachers were not experiencing how to teach mathematics in a pedagogically appropriate way. The program of study for the majority of the preservice teachers placed their mathematics methodology course before any student teaching experiences. Field assignments rarely placed them in a teaching situation that did not emphasize drill-based, rote memorization mathematical teaching. Although problem-based learning was modeled in their courses and they experienced numerous investigations, focusing on how children learn was less effective with no children actually present (Linek et al. 2003).

Such conditions led to a major concern: How could a methodology course be designed to include elementary children and to model theory supported by practice? This article describes the way that elementary students were integrated into a mathematics methodology course. We provide a model for others to implement such a program with the idea that experiences (Gallego 2001) and reflections (Bischoff, Hatch, and Watford 1999) are the best teachers.

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Coordinating the Course

To support aligning theory with practice, a university course was offered off campus at a local elementary school with children present. Through community contacts, this school was selected as the coordinat-

ing school because of its principal's reputation as a person who strongly supports students' academic health. He was given the flexibility to include any students he wished. He chose students whose teachers indicated that the children needed additional support in mathematics. Because of scheduling issues, he selected students from primary grades.

Two groups of twenty-five students in the second and third grades and two groups of six to eight preservice teachers per week were organized and placed into two separate primary classrooms that were connected by a pod. The preservice teachers circulated between the grades throughout the weeks, which allowed for exposure to two grade levels rather than just one. The pod floor plan made navigating from one classroom to the next easy and allowed the preservice teachers access to the professor with little effort.

The class took place immediately following the children's regular school day. The elementary students were asked to attend this mathematics-based class for nine weeks (the quarter system), one day per week for one hour. Because the class met after school and not with a master teacher, the curriculum was not based on the needs of the teacher or school; instead, it was based on the perceived needs of the preservice teachers and the children. This format offered great freedom in relation to course content.

Structuring the Course

The course was designed with an emphasis on the way that children learn mathematics and how to structure lessons to bring about optimum learning. The preservice teachers were given several assignments to help them focus on the issue of student learning and problem-based lesson planning, including readings (Van de Walle 2007), verbal and written weekly reflections, and a formal lesson plan with an in-depth reflection.

Adult classes were two-and-a-half hours. Preservice teachers spent the first hour with the children. After walking the children to buses, day care, or waiting parents, the adults convened for the remaining portion of the class time to debrief, review the reading, and review the model lesson. The course had the same content and course number as the traditional on-campus course; interested preservice teachers simply signed up for the off-campus version after e-mails went out to advertise the alternative course option.

Beginning the Course

The university course ran for ten weeks; children did not attend the first week. During the first class, the instructor explained the course structure, and the preservice teachers had time to prepare for their first interactions with the children. **Figure 1** provides an

Figure 1

The outline of the course components emphasizes the content and reading topics.

Week	Course Content	Topic (Van de Walle 2007)	
1	Introduce course, create emergency lesson plans and centers; no children attend	Exploring What It Means to Do Mathematics	*Foundations and Perspectives*
2	Children begin attending; teach at centers	Developing Understanding in Mathematics	
3	Teach Navigating and Mission lessons	Teaching through Problem Solving and Planning in the Problem-Based Classroom	
4	Preservice teachers' individual lessons begin—open topic	Building Assessment into Instruction	
5	Individual lessons—Algebra	Algebraic Thinking: Generalizations, Patterns, and Functions	*Development of Mathematical Concepts and Procedures*
6	Individual lessons—Fractions	Developing Fraction Concepts	
7	Individual lessons—Fractions	Computation with Fractions	
8	Individual lessons—Measurement	Developing Measurement Concepts	
9	Individual lessons—Geometry	Geometry Thinking and Geometric Concepts	
10	Individual lessons—Probability	Exploring Concepts of Probability	

overview of the course with a focus on content and reading topics. Preservice teachers separated into two groups that were randomly placed into a grade to teach their individual lessons. On the basis of the mathematical content of their readings, they then selected a week to teach. After reviewing the syllabus, explaining the course requirements, and establishing expectations, the instructor began preparing the preservice teachers to teach the elementary students. They worked in groups to create three or four centers (Van de Walle 2007) for their first meeting with students and to formulate centers for an emergency lesson plan in the event that one of their colleagues failed to teach a lesson. (This contingency planning proved to be important, as several student teachers did not show up when scheduled to teach their lessons.) Center sessions ran approximately fifteen to twenty minutes. Children rotated through the centers, which allowed the preservice teachers to get to know each child before they began to teach the whole class.

Teachers were provided texts from the Navigating through Algebra series (Cuevas and Yeatts 2003; Greenes et al. 2001) and the Mission Mathematics II series (Hynes and Blair 2005; Hynes and Hicks 2005). They were asked to select a lesson to teach for the second meeting. The books provided models with a problem-based format similar to the structure referenced in the university text (described below). Teachers organized the lesson on their own. Two of them usually acted as teachers as the rest circulated among the children to support the lesson.

Reading and planning

Course readings were divided into two sections (see **fig. 1**). Readings in the first four weeks focused on foundations and perspectives. Readings in the second section focused on the development of mathematical concepts and procedures aligning with the Van de Walle (2007) text. The first set of readings emphasizes what it means to learn mathematics and how to teach mathematics by highlighting problem-based learning. The second set concentrates on supportive methods to teach major mathematics content areas.

Individualized lesson plans began the fourth week of the quarter and gave preservice teachers the opportunity to teach a problem-based, whole-class lesson connected to the reading topic. Each teacher selected a week to instruct the children using the before-during-after lesson plan structure described by Van de Walle (2007, pp. 41–48) and Friel and Carboni (2000) with some additional elements (such as state standards, goals, special education modifications, and specific strategies for English language learners).

Alternative assessments potentially allow preservice teachers to approach evaluation using conceptual analysis (McDuffie, Akerson, and Morrison 2003) and emphasizing students' mathematical knowledge. To incorporate alternative assessments, teachers implemented one of the observational assessments described in their text (Van de Walle 2007, pp. 84–87). The observational assessments provided the teachers with means above and beyond

activity sheets, quizzes, and tests to appraise students' mathematical knowledge.

The professor limited her participation when the teachers presented lessons, acting as more of a course facilitator when teachers requested help. Keeping lessons focused on the preservice teachers and their students was important, as in traditional classroom teaching. When opportunities arose to help scaffold and guide the teachers, the professor modeled techniques in the *during* stage of the lesson while the teachers who were not teaching that lesson assisted the children. Such a high teacher-to-student ratio minimized discipline problems and allowed the preservice teachers to focus solely on the mathematics.

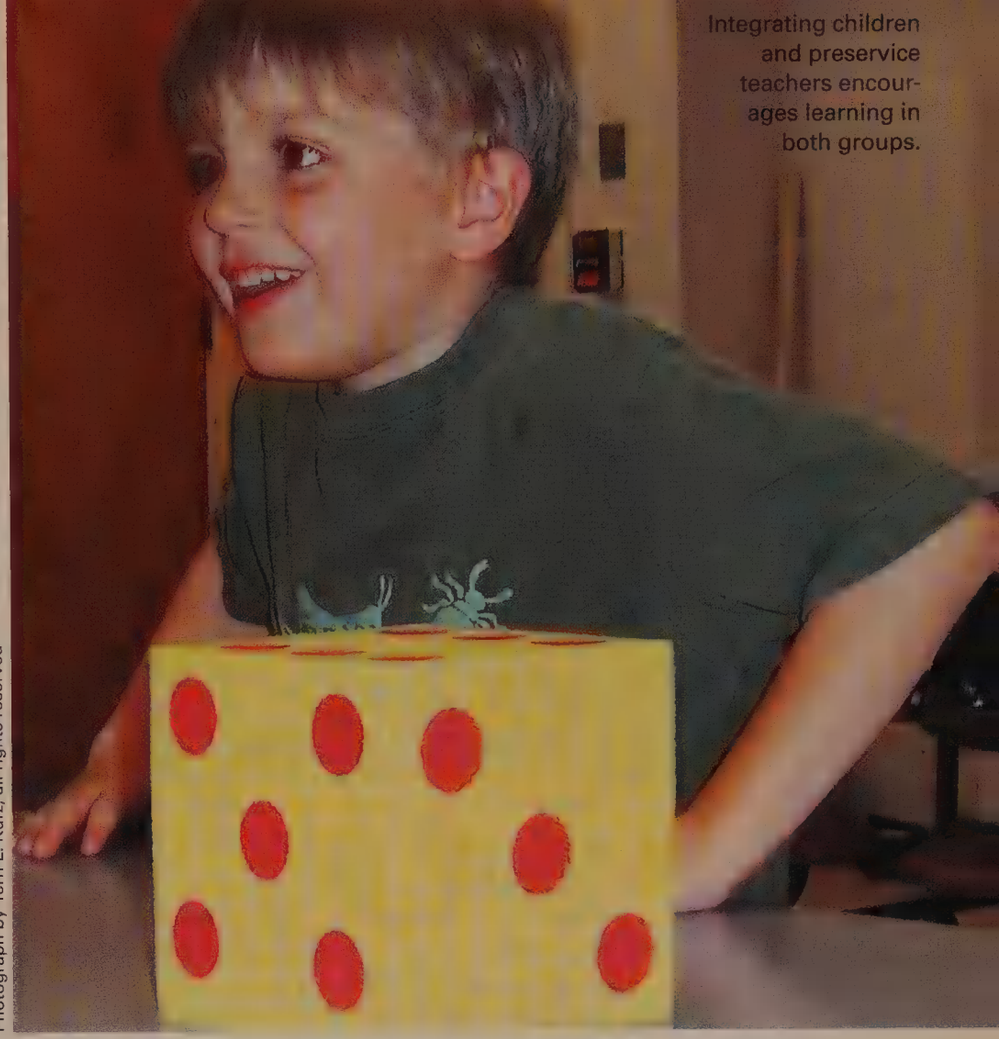
Reflecting

Reflections have been shown to be a critical element of preservice teachers' learning and growth (Lowery 2002), and in this course the preservice teachers reflected in groups, as a whole class, and through individual writing. After reconvening to an empty intermediate-age classroom (where the desks were larger), the teachers grouped together on the basis of grade level and discussed their observations of the lesson and student learning. They were prompted to answer the following questions:

- What went well? What should be done again, and why?
- What should be improved, adapted, or changed? Why?
- Were the students thinking and learning? Elaborate.
- "What did you learn about the students you worked with today?" (Gallego 2001, p. 318).

These questions were designed to provide minimal structure and to allow the teachers to analyze their experiences and observations. After each group had time to discuss the questions, the teachers reconvened to a whole-class structure and had an open-ended discussion on their observations. Both analyses took about forty-five minutes total and were constructive, valuable course elements. Many preservice teachers were unaware of pedagogical issues, and the discussions were their opportunity to evaluate theory aligned with practice. For example, some teachers had never seen children use manipulatives. Through dialogue, they were able to analyze the affordances and constraints of teaching mathematics with manipulatives.

The remaining class time was configured as a typical mathematics methodology course and was used to discuss readings and explore other mathematical ideas.



Integrating children and preservice teachers encourages learning in both groups.

Photograph by Terri L. Kurz; all rights reserved

Writing activities that give ample time take the pressure off preservice teachers and allow them to reflect on what they have learned (Flores and Brittain 2003). In written reflections, the teachers were able to integrate ideas about learners with ideas about instruction (Davis 2006). They were asked to write a somewhat open-ended reflection every week; it was due electronically within four days of when the lesson was taught. They were asked to focus the reflection on what they observed and discovered in relation to how children learn mathematics. The reflections also became the basis of dialogue between the preservice teachers and the professor. The teachers often asked questions, seeking clarification on the mathematics in the lesson or expressing other concerns, such as classroom management or whether something was done correctly in class. Such issues are common with students in field-based learning. In a study by Moore (2003), preservice teachers were shown to focus on procedural concerns, classroom management, and lesson planning even when the structure of the field-based experience emphasized examining theory and practice.

After each teacher presented an individual lesson to the students, a personal reflection took place (instead of the weekly reflection). Self-reflection is an important and valuable element to effectively

Photograph by Terri L. Kurz; all rights reserved

Until they took this course, some of the preservice teachers had never seen children use manipulatives.

guide preservice teachers in becoming quality teachers (Bischoff, Hatch, and Watford 1999). The individualized self-reflection inquiries arose from the ideas of monitoring a constructivist classroom as outlined by Taylor, Fraser, and Fisher (1997); the questions emphasized understanding and interpreting how children learn mathematics.

Adapting the Course

Although this course was initially used with a relatively small number of preservice teachers (fourteen) and on an academic quarter system, it can easily be adapted. The course has been taught several times. When higher numbers of preservice teachers enrolled, another group of (first-grade) children was included. If few preservice teachers enroll, some can pair up for individual lesson plans and can teach a lesson before the individuals do.

A semester system might provide even better opportunities for preservice teachers. The text could be used to guide more centers just like the centers implemented during the first meeting with the children. Furthermore, lessons from the Navigations or Mission series or the *Field Experience Guide* (Bay-Williams 2007) based on Van de Walle's text could be presented to the children before the individual lesson plans begin. Such modifications would give preservice teachers opportunities to interact with students and to teach problem-based lessons before jumping in to design and teach their own lesson plans (as is necessary with the quarter system). The semester system may provide increased opportunity, analysis, and depth because of the greater time commitment. Additionally, the readings on foundations and perspectives could be expanded; doing so would probably enhance the knowledge of what encompasses quality problem-based learning.

In an ideal setting, a relationship would be established with the school to offer the course off campus for at least one academic year with the same elementary school children. The professor would be a consistent fixture and bring in new preservice teachers as the quarters (or semesters) progress. This setting would add stability for the elementary students, their school, and the professor (with regard to planning and scheduling for the academic year). A yearlong program would also provide an interesting topic of research in terms of the potential mathematical sense gained by participating elementary school children.

Conclusion

This course structure went beyond placing elementary children with preservice teachers. First and foremost, it allowed preservice teachers to learn about teaching mathematics in a nonthreatening environment without the formal observations and evaluations common in student teaching situations. Although written lesson plans were graded, teaching the lesson was not. During lesson delivery, preservice teachers could make mistakes and learn without grade penalties, thus opening the doors for discussion and self-critique. Group discussions, individual reflections, and whole-class analyses allowed preservice teachers to observe each lesson's strengths and weaknesses and potentially learn from them (Lowery 2002; Flores and Brittain 2003).

Moreover, preservice teachers experienced how to structure and teach a problem-based lesson using manipulatives. The course allowed teachers to view lessons and to teach using appropriate framework to produce optimum problem-based learning. Teachers also experienced the proper use of manipulatives and discovered how manipulatives support student learning. In traditional university courses, preservice teachers use manipulatives without children present. In this course, teachers could observe children's thinking processes as they used manipulatives and conversed with the teachers. Also, because of the preservice teachers' planning, they could use alternative assessments to evaluate a lesson's effectiveness on student understanding.

In a university setting, professors seldom observe student teachers in their placements, and little information about mathematics lesson specifics are observed or taught. University supervisors and master teachers are often unaware of appropriate

techniques of teaching problem-based mathematics. This methodology course placed preservice teachers directly with their mathematics methodology professor to gain insight and guidance from her experience and research. This course allowed for a more controlled learning environment that easily aligned the theory of learning mathematics with the practice of creating and teaching problem-based lessons. Additionally, because lessons were not tied to a school calendar, the preservice teachers were able to deal with more content area topics (albeit with less depth) than would be possible in a ten-week observation or teaching block in a regular classroom.

This course structure addresses the problem of aligning theory with practice in a university mathematics methodology course. The alternative course structure addresses the need to provide elementary school children (Lowery 2002) in a university setting with support learning (Cady, Meier, Lubinski 2006; Gee 2006). Written and verbal reflections provide preservice teachers with opportunities to analyze teaching and practice (Lowery 2002; Flores and Brittain 2003; Davis 2006). Implementing the course structure provides a way to align theory and practice (Linek et al. 2003). With little effort beyond preparing for a traditional mathematics methodology course, preservice teachers gained more exposure to problem-based teaching and learning from this structure.

This course may also support preservice teachers' development regarding how to teach pedagogically aligned curriculum. Further research needs to be done and should focus on how aligning theory and practice with the support of elementary student interactions might change preservice teachers' knowledge of student mathematical learning. Investigating the impact of this course structure on the participating preservice teachers' teaching habits once they have their own students would also be an interesting study.

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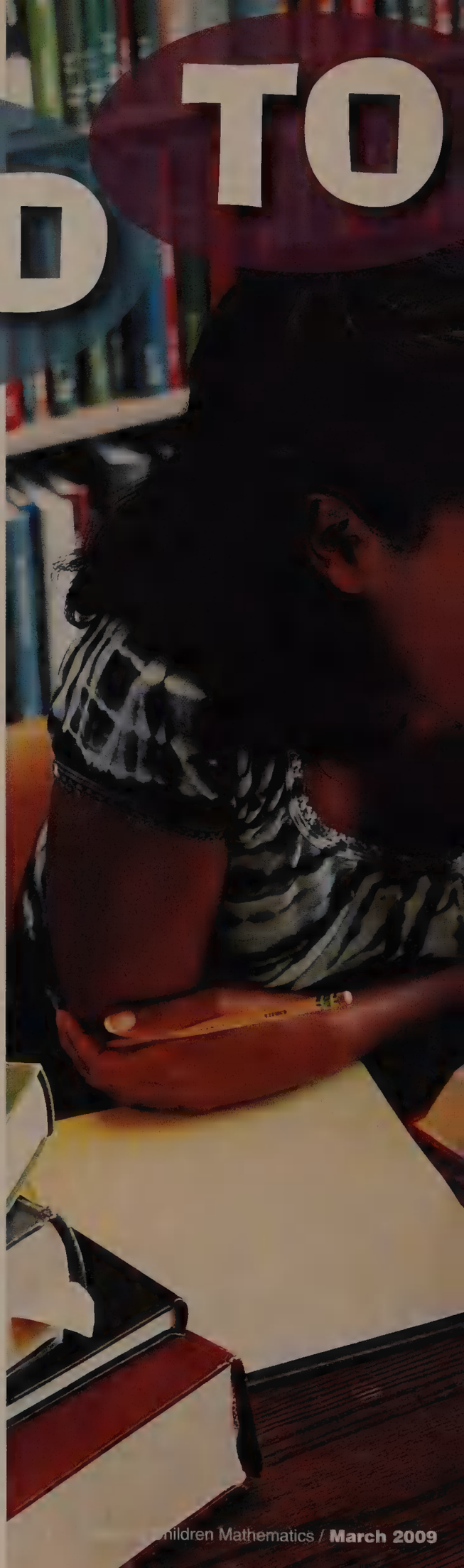

Engaging in dialogue about the implications of mathematics demonstration classrooms becomes a form of professional development for teachers in communities of practice.

Teachers' professional identities are associated with an ongoing commitment to professional development and the explicit aim of improving practice in order to increase student learning (NCTM 2000). During the last century, professional development initiatives took many forms, including laboratory schools (Shadick 1966), lesson study (Bruce 2008), video clubs (Van Es and Gamoran Sherin 2008), and so forth. Demonstration classrooms can be traced back to the 1920s (Spencer 1926). For the purpose of this article, a *demonstration classroom* is a professional development initiative—that is, for in-service teachers—that is situated in a community-of-practice framework.

Contemporary views of professional development have been linked to notions of communities of practice, recognizing that teachers, just as students, learn best when they learn from one another, in familiar settings, and where opportunities for sharing and continued support exist (Darling-Hammond and Bransford 2005).

A Current Example

In 2004–2005, members of Canada's Waterloo Region District School Board (WRDSB) introduced the demonstration classroom to their teachers. This form of the professional development model involved opening regular elementary classrooms to groups of teachers for observation. The WRDSB selected two classrooms in the district one first grade and one fourth grade—on the basis of the teachers' exemplary pedagogical practices in mathematics education (e.g., teaching through problem solving, use of manipulatives, learning through discussion, etc.).

According to Donegan and Shantz (2007, p. 24), demonstration classrooms adhere to the following beliefs:

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BE THERE!
LINEAR ALGEBRA
with concurrent examples

- Demonstration of instructional practice is an effective tool to support teacher learning and instructional change.
- Deconstruction of the demonstration lesson through facilitated dialogue and discussion is a critical part of the process.
- Multiple visits over time inspire change and growth most effectively.
- The effectiveness of a demonstration model requires ongoing monitoring and assessment.

Donna Kotsopoulos and a group of elementary teachers had the opportunity to visit the fourth-grade mathematics demonstration classroom at W. T. Townshend Public School. Following the visit, Kotsopoulos sat down with the classroom teacher, Duane Heide, to learn more about how demonstration classrooms have been implemented.

Kotsopoulos: How did you come to be a demonstration classroom teacher, and what special experiences did you bring to this role?
Heide: As a student in elementary school, I struggled with learning mathematical concepts. When I became a teacher, I found that I was able to relate to some of the difficulties of my students, and I attempted to present the mathematics in ways that all students could understand.

A few years into my teaching career, I had the opportunity to co-teach [in] a fifth-grade classroom with another teacher. He was a "master" teacher of mathematics, and I learned—through observation—effective teaching methods, use of math manipulatives, and teaching mathematics through problem solving.

As my enjoyment of teaching mathematics increased, I became a mathematics teaching coach for other elementary teachers in my school district. When the opportunity came up to be a demonstration classroom teacher, I was immediately interested. This is my second year as a demonstration classroom teacher; the first year, I taught sixth grade. I have grown immensely as a mathematics teacher through my role; the experience has had reciprocal benefits. Sharing classroom experiences through conversations with the visiting teachers, ongoing professional development by the WRDSB, and, of course, the problem-solving approach with my students have all contributed to my own development as a teacher.

Teachers learn best when they learn from one another

Kotsopoulos: How many visitors have you had in your classroom this year?
Heide: Elementary teachers from the district visit my classroom twice during the school year. I have a maximum of six visitors for each session, with three visits occurring each week. One hundred seventy-five teachers have participated in visits to my classroom this year.
Kotsopoulos: According to Donegan and Shantz (2007), the purpose of a demonstration classroom is to provide an effective tool to support teacher learning and instructional change. Can you describe how this is achieved?
Heide: Prior to the beginning of class, I meet with the visiting teachers to explain the lesson, my rationale for choosing or developing the problem that [I] will … present to the students, and to discuss multiple strategies that could be used to find the solution to the problem. When the students arrive in the morning, I have the visiting teachers introduce themselves to the students. [For] the duration of the morning, visiting teachers only observe and do not directly interact with students.

Unfamiliar Pedagogy

Heide went on to tell Kotsopoulos that the type of mathematics teaching and learning demonstrated in his classroom is often very different from what visiting teachers have experienced. Heide first introduces a problem to his elementary students, giving them time to ask questions and think about strategies and manipulatives that they can use. He groups the students in homogenous pairs, preselecting them to ensure that students receive the support that they require. While students are working on the problem, the visitors are free to move around the classroom, recording observations in an observation journal.

At the end of the lesson, students convene as a whole group for a math congress to discuss various strategies that they used to find the problem solution. Heide facilitates the presentation of the students' ideas and strategies, and students defend their solutions, ask questions for clarification, and make important mathematical connections.

After Heide dismisses his students, he holds a forty-minute lesson debriefing session for the visiting teachers to share their reflections about student learning, teaching strategies, and the problem-solving lesson structure. Many comment that the teaching and learning that they just observed are unlike their own fairly traditional approaches. One of the most important aspects of this session is

"After this session, I felt that I have been given the tools to get where I need to go"

that all participants set goals, such as a teaching strategy that they plan to implement in their own classrooms. The entire process is geared toward maximizing support for instructional change within a community of practice: sharing, listening, and thinking. The process is intended to make teachers feel safe about discussing their own mathematical teaching challenges, while at the same time empowering them to make important—and sometimes radical—changes to their practice.

Kotsopoulos inquired about the sorts of questions that the teachers raise as a result of Heide's sessions. She also asked him about the resources he uses to develop a problem-solving approach to teaching mathematics.

Heide: Many teachers are unfamiliar with this teaching pedagogy, and after the first visit they tend to ask one of two questions: "Where do we find the problems to present to the students?" or "How can the entire mathematics curriculum be presented to the students when teaching through and about problem solving?" The WRDSB has created a database of mathematics problems that teachers can both contribute to and access. This is one of my starting points when looking for problems. Other resources I consult include grade-level mathematics textbooks, provincial curriculum documents, professional teaching resources, and various mathematics Web sites.

Empowered Teachers, Empowered Students

After the first visit, Heide sends a follow-up e-mail in which he includes a list of the resources that he uses. Many teachers express concern about their ability to cover all aspects of the mathematics curriculum. He encourages them to try to mentally step back and choose problems that will naturally lead students to understand mathematics' big ideas and allow them to see connections between mathematical concepts. Kotsopoulos asked about Heide's opportunities to interact with teachers after they visit his demonstration classroom twice.

Heide: Following each visit, I inform visiting teachers that I am available to come to their schools in the afternoons in a mathematics coaching role. Many visiting teachers have invited me to support different aspects of their mathematics teaching practices. I have worked alongside teachers to create mathematics unit plans, look more deeply at assessment and evaluation, create a bank of problems for a particular mathematics strand, engage in lesson study, and to become familiar with manipulatives and resources that support all students.

Kotsopoulos: Were children specially selected for this classroom?

Heide: When the class list was created, no student was specifically selected to be in the mathematics demonstration class. My class has a heterogeneous mixture of socioeconomic status, mathematical abilities, and behaviors. Participating teachers have told me that they really appreciate that the demonstration classroom represents a typical classroom.

Kotsopoulos: How do you perceive the experiences of the children in the classroom?

Heide: When students were asked about their experiences in the classroom toward the end of the year, the results were positive and very reflective. Students commented that they enjoy using different strategies and ... like that they could develop their own methods of problem solving. Students expressed that they appreciate working with peers rather than working thorough textbooks. The mathematics textbook was used infrequently in our classroom during the year. Many students expressed confidence in choosing appropriate strategies to assist them in solving problems. Their perspectives were consistent with what I perceived as the classroom teacher. I sensed that they looked forward to sharing their ideas and strategies with their classmates during the math congress. The aim in my classroom is to empower students to believe in themselves as young mathematicians.

Kotsopoulos: If you could build on this program, in what ways would you do so?

Heide: I would like to see all teachers open up their classrooms to their colleagues. Teachers have so much to learn from watching each other teach. It is what communities of practice in education should be all about. Providing opportunities to observe each other in teaching is very powerful. The demonstration classroom is particularly powerful because visiting teachers experience it in real time. It is a real teacher. These are real students.

Kotsopoulos: What impact is the demonstration classroom having on mathematics instruction?

Heide: Through the informal discussions I have had with visiting teachers, I have found that many have begun to embrace teaching mathematics through problem solving. I am pleased when I hear

REFLECT AND DISCUSS

You Had to Be There!

Reflective teaching is a process of self-observation and self-evaluation. It means looking at your classroom practice, thinking about what you do and why you do it, and then evaluating whether it works. By collecting information about what goes on in our classrooms and then analyzing and evaluating this information, we identify and explore our own practices and underlying beliefs.

The following questions related to "You Had to Be There!" by Donna Kotsopoulos and Duane Heide are suggested prompts to aid you in reflecting on the article and on how the authors' ideas might benefit your own classroom practice. We encourage you to reflect on the article independently as well as to discuss it with your colleagues.

- Which aspects of your teaching would you want to share with another teacher, and why? Which aspects of teaching would you want to observe?
- Think about the problems you use in your mathematics program. How do these problems enable students to see connections between mathematical concepts? How might you adjust the problems in order to maximize the opportunities for students to make connections?
- Heide's mathematics program was structured around problem solving. Reflect on your own mathematics teaching: To what extent do your students engage in mathematical tasks related to problem solving?
- Students in Heide's class convened for a math congress at the end of every problem-solving session. The purpose of math congress was to give students an opportunity to share their strategies so they could learn from one another. What opportunities do you give students to share their problem-solving strategies with you and with other students? How might you increase these opportunities or make them more effective?
- Reflecting on practice during the debriefing component was an important aspect of visiting the demonstration classroom. Think about ways that you reflect on your practice and how such reflection shapes your decisions.
- Compare the effectiveness of using a demonstration classroom experience for professional development to the effectiveness of using videotaped classroom discussions.
- Reflect on the forms of professional development that you have engaged in as a teacher. Which ones have helped your teaching, and why? What role can you take in professional development in your district?

The Editorial Panel invites you to tell us how you used "Reflect and Discuss" as part of your professional development. We appreciate the interest and value the views of those who take the time to send us their comments. Submit letters to Teaching Children Mathematics *at* **tcm@nctm.org**. *Please include "Reader's Exchange" in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit are subject to abridgment. Letters are also edited for style and content.*

of teachers moving away from a "watch me and learn" model of teaching mathematics to facilitating mathematics discussions in their classrooms. Some visiting teachers have shared that they have a renewed interest in teaching mathematics. It has been enormously rewarding.

Concluding Thoughts

During the debriefing session following the lesson that Kotsopoulos observed in the demonstration classroom, she talked with some visiting teachers who were on their second visit. Consistent with Heide's perspective, the visiting teachers' perspective was that demonstration classrooms are a form of professional development that significantly impacts their practice. One teacher observed, "I often leave some professional development sessions feeling that I am not a good teacher, and I have so far to go; but after this session, I felt that I have been given the tools to get where I need to go."

Another visiting teacher discussed the implications for problem solving in her class: "[The visit] helped to reinforce the problem-solving practices I am already using and encouraged me to make those practices a more regular part of my math program."

A third teacher described the importance of the debriefing session: "I found the reflection part after the lesson was the most beneficial. It helped me reaffirm some good practices that I am already using." As Donegan and Shantz (2007) explain, facilitated dialogue and discussion are critical components of the demonstration classroom experience.

Present-day approaches to sharing classroom practices tend to focus on digital representations (e.g., video clips) of these settings (Bruce 2008; Van Es and Gamoran Sherin 2008). Although other important media for sharing classroom practices exist, experiencing a setting in person may be more authentic in terms of bringing about communities of practice (Darling-Hammond and Bransford 2005). As one visiting teacher explained, "I think it was very helpful to actually witness the concept or idea of problem solving."

Another visitor echoed these comments: "It was the first time I had seen anyone teach through problem solving. Going through step by step how the teacher would approach the problem with the class and seeing it done was very beneficial."

Indeed, even from Kotsopoulos's perspective, experiencing Heide's teaching and student learning in real time was "powerful." The demonstration classroom visit also afforded other benefits: As an

observer alongside the visiting teachers, Kotsopoulos made note of classroom configurations, student work on the walls, teacher-selected print sources available to students, desktop "mathematics kits" available at all times for students, and so forth. Video may not have fully captured the experience of being physically present in this setting.

The visiting teachers taught various elementary grades, not fourth grade only; and administrators also visited, suggesting that demonstration classrooms are not grade specific or limited to those teaching the same grade as the demonstration classroom.

Demonstration classrooms have a history dating back more than eighty years, but they remain a real-time professional development experience that cannot be replaced by video representations of teaching. The saying "You just had to be there" captures the complexities associated with mathematics teaching and learning, changing or improving practices among teachers, and facilitating and sustaining communities of practice.

The authors wish to thank the Waterloo Region District School Board; Inge Ford, principal of W. T. Townshend Public School; and WRDSB Mathematics Consultants John Lee and Christine MacTavish for their support and willingness to share information about demonstration classrooms.

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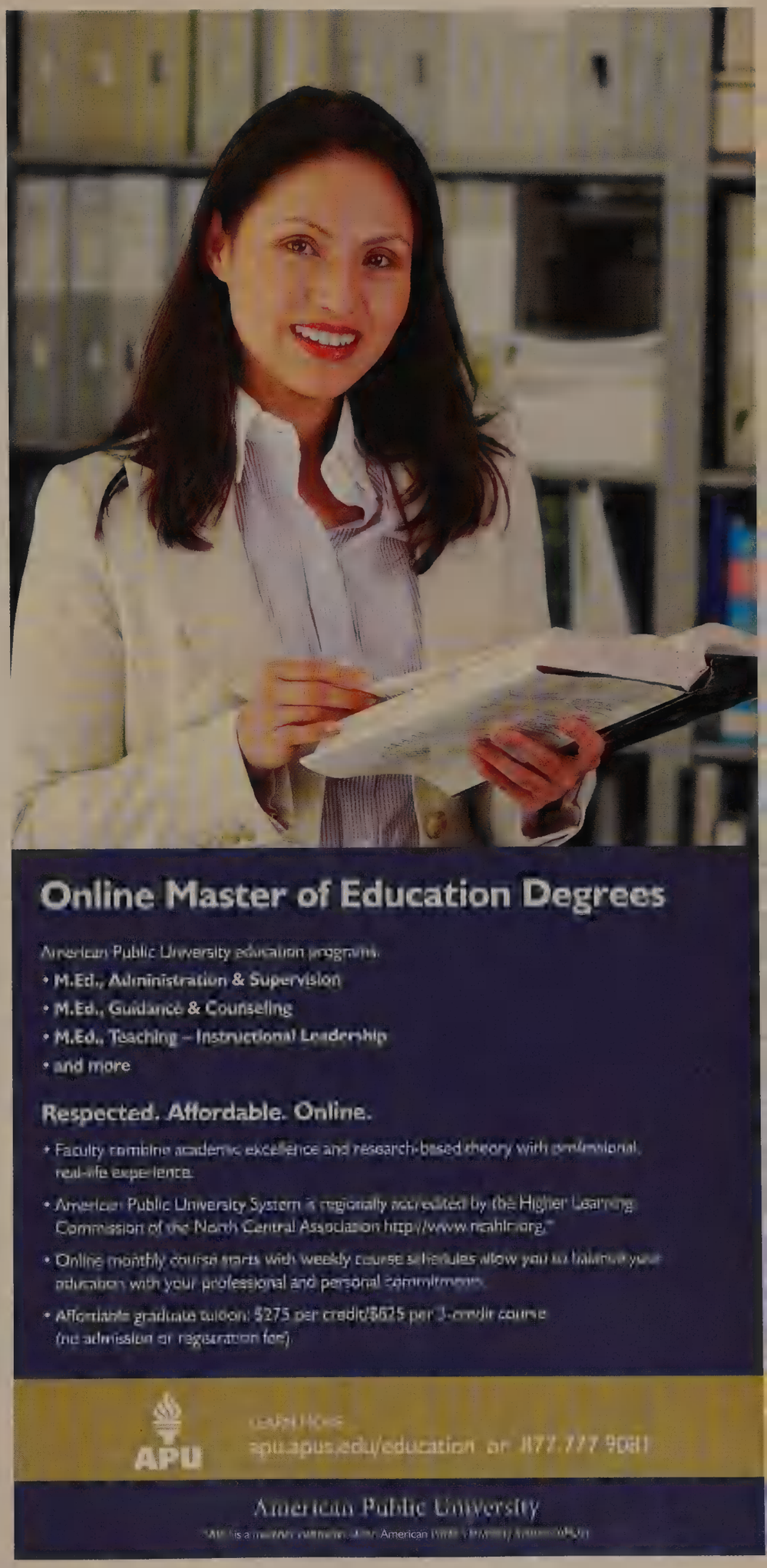
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Probability

"Math by the Month" activities are designed to engage students to think as mathematicians do. Students may work on the activities individually, in small groups, or as problems of the week for the entire class. Because no solutions are suggested, students look to themselves for mathematical justification, thereby developing confidence to validate their work.

The following activities are designed to help you and your students investigate probability through mathematical investigations involving number sense and operations, statistics, data collection, communication, and representation. ▲

WEEKLY ACTIVITIES

PROBABILITY: K–2 — MARCH 2009

2 **Will it happen?** Fold a sheet of paper in half and open it again. Label the left side of the paper *Likely* and the right side *Unlikely*. With a partner, think of ideas for events that will probably happen. Write your ideas under the *Likely* heading. Do the same for events that you think will probably not happen. Write these ideas under the *Unlikely* heading on your paper. What makes some events more likely than others? Share your ideas with your classmates.

9 **Pick a coin.** Place 10 coins in a bag using 2 different types of coins (for example, pennies and nickels). Shake the bag and ask a friend to take a coin out without peeking. Record the coin's name and put it back into the bag. After you repeat this 10 times, ask your friend to predict which 2 types of coins are in the bag. Repeat 10 more times. Does your friend still agree with his prediction? If not, ask how he would change it. Now it is your turn. Let your friend put 10 coins in a bag using a combination of 2 different coins to test your prediction skills. Challenge: Also predict how many of each type of coin are in the bag.

16 **How odd!** You will need a deck of cards numbered 1 through 10. Discuss with your partner which cards in the deck show even numbers and which cards show odd numbers. Shuffle the cards and place them face down. Take turns drawing cards until they are all gone. How many cards should each player have? Each of you sorts your cards into a stack of even cards and a stack of odd cards (four stacks in all). If odd cards are worth 1 point and even cards are worth 2 points, how many points are all your cards worth? The player with the higher number of points wins. Play this game several times. What is the probability of selecting an even number? What is the probability of selecting an odd number? Discuss with your partner whether this is a fair game.

23/30 **Gently toss a penny.** Did it land on heads or on tails? Which way will it land more often: heads or tails? Toss your penny 9 more times. Record the results of each toss. Post your results on a class chart to compare your results with the results of your classmates. Ask your teacher to help you figure the class total of heads and tails. Did the penny land more often on heads or on tails? How do your data compare to the class data? Why do you think the class data look different from some individual data?

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Edited by **Dana Islas**, *who teaches kindergarten at Pueblo Gardens Elementary School in Tucson, Arizona. Readers are encouraged to submit problems to be considered for future "Math by the Month" columns to the department editor at dana.islas@tusd1.org. Receipt of the problems will not be acknowledged; however, those selected for publication will be credited to the author. Readers are also invited to submit creative solutions and adapted problems to* **tcm@nctm.org** *for potential publication. Please include "Reader's Exchange" in the subject line. Because of space limitations, responses beyond the 250-word limit are subject to abridgment. Submissions are also edited for style and content.*

PROBABILITY: 3–4

MARCH 2009

Here fishy, fishy. Of 17 fish in a bowl, 2 are black, 5 are gold, 3 are spotted, and 7 are silver. What is the probability of catching a fish of each color if you put the fish back in the bowl each time? If you caught a black fish, put it in a separate container, and then caught 2 more fish, which colors would these 2 fish most likely be? Explain your thinking. Which color of fish is most likely to be the last one in the bowl? Explain.

Build a city. Divide a spinner into 6 equal sections labeled *cube*, *sphere*, *cylinder*, *pyramid*, *cone*, and *rectangular prism*. Gather these same six 3-D shapes or blocks. Spin 10 times. After each spin, take the shape that matches your spin and build a city. Now draw your city. What is the probability of the spinner landing on each shape? What is the probability of its landing on a shape with curves, on a shape with 2 or more faces, or on a shape with 6 faces? How can you change the spinner labels to increase or decrease the probability of its landing on a specific shape?

Make a pizza. You and 2 friends make a pizza and cut it into 10 slices. Each of you puts toppings on 3 slices. You put extra cheese and pepperoni on yours. Max uses olives and mushrooms, and Olivia chooses pineapple and sausage. The group decides to place all the ingredients on the 1 remaining slice. You each take turns spinning the pizza and then—without looking—take the first piece you touch. You take turns until only 1 slice remains. The last slice will go to your little sister. Olivia spins first. What are her possible outcomes? What is the probability of her getting the slice with everything? Which slice is your little sister most likely to get? Explain your reasoning. What are the chances that you will get at least 1 pizza slice with your favorite toppings on it?

Oddly, we evenly roll along. Obtain a number cube and make a recording sheet with 2 columns and 20 rows. Roll the cube and use a tally mark to record the number for each roll. Did you roll more odd numbers or more even numbers? Compare your results with a friend's and then combine your results. Did you get more odd or even numbers? Why is this so? Gather data from other classmates. How do their data compare to yours? Compare the class data to your data.

WEEKLY ACTIVITIES

PROBABILITY: 5–6

MARCH 2009

I predict a kiss. If you drop 5 Hershey's Kisses® from the same height, predict whether more will land on their sides or on their bottoms. Now drop them and record the results. Repeat 5 times. What do you discover? What percentage of the time do the kisses land on their bottoms? Are your results similar to or different from your classmates' results? Now try dropping Hershey's Kisses with Almonds. Again, predict first and then drop the kisses and record the results. Figure the percentage of times your kisses landed on their bottoms and on their sides. Compare your results of plain candies with almond candies. Explain why you think these results happen.

Rock-Paper-Scissors. With 2 friends, decide who will be player 1, 2, and 3 to play Rock-Paper-Scissors. Play 20 times and record the results of each game. Player 1 gets a point if all 3 players show the same sign. Player 2 gets a point if 2 players show the same sign. Player 3 gets a point if all players show different signs. Is this a fair game? Explain your thinking. If the game is fair, what makes it fair? If it is not fair, how could you make it fair? Play again to test your ideas.

It's not fair! How you can tell if a game is fair? Share your thinking with your teacher for feedback. Next, create a fair game with a friend. Record your game rules and play the game many times to be sure it is fair. If it is not fair, make adjustments to make it fair. When your game is fair, explain what makes it fair. Invite others to play your game. Record their outcomes. Gather the data to check if your game is still fair. Challenge: Repeat this activity but create a game that is unfair and explain how to make it fair.

Which sums are more likely? If you roll a 6-sided number cube, what are the chances of getting an even number? Predict and then roll 20 times. Record the results. Now roll 2 number cubes and add the results. Are you more likely to get an even or an odd sum? Predict and then roll 20 times. Record the results. Now use 3 number cubes. Are you more likely to roll an even or an odd sum? Predict and then roll 20 times. Record the results. What did you discover? Why do you think this happens? How does increasing the number of cubes influence the likelihood of rolling an odd or an even sum? Explain.

What Makes a Video

Wisely chosen classroom video excerpts can strengthen teachers' abilities to understand and analyze students' mathematical reasoning.

By Katherine A. Linsenmeier and Miriam Gamoran Sherin

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For many years, we have been using video to help preservice and practicing teachers explore the mathematical ideas that students raise in their classrooms. In our classes and workshops, teachers watch short video clips of real classrooms—their own and others—and then discuss the student thinking portrayed. The ability to analyze and understand students' mathematical reasoning is an essential aspect of teaching expertise and is one that can be strengthened through viewing and discussing video (Sherin and Han 2004). Recent research shows that teachers who learn to analyze student thinking via video also become more effective at responding to student ideas during instruction (Cohen 2004).

Despite the potential of video to support teacher learning, we have found that not all video is equally valuable for prompting teachers to examine students' mathematical thinking. This concern led us to consider the question, What makes a video clip interesting?

Productive Video Clips

Here we describe three types of video clips that we have found to be productive for teachers to examine. We also describe three dimensions that can be used to characterize the student thinking portrayed in a video clip. Familiarity with these dimensions will help teacher educators who want to identify effective video clips to use with teachers. Moreover, the information will be useful for teachers who want to be able to select productive video excerpts to view with colleagues.

Others have written about different formats to use when showing teachers video excerpts and about different approaches to facilitate video-based discussions, so our purpose here is to focus on the issue of *selecting* video clips—in particular, selecting clips that are likely to lead to productive discussions of student mathematical thinking.

The *"What?"* clip

Our first instinct is often to share video clips that present us with a dilemma, that is, clips that leave us confused—but intrigued—about the student ideas they contain. We call such clips *"What?"* clips

because they prompt us to ask, "What just happened there?" At the same time, in order for the confusion to be engaging, rather than off-putting, a video clip must contain enough evidence of student ideas that we are able to make some reasonable hypotheses about what is going on. Confusion due to a lack of entrée into student ideas is not "good" confusion, whereas confusion due to the complexity and innovation of student ideas is. In our experience, *"What?"* clips often involve students obtaining the incorrect answer to a problem; mathematical mistakes, particularly in the context of reasoning and problem solving, seem to be ready fodder for exploration.

For example, in one clip, Mr. Novak's class is working on the following problem: If one inch represents fifty miles, then one-half inch represents how many miles?

When Matthew responds that the answer is seventy-five, Novak asks Matthew about his answer (see **fig. 1**). Matthew seems to understand that half of fifty is twenty-five, yet he also says that half an inch represents seventy-five miles. How can he hold these two opposing ideas?

In discussions of this video clip, the confusing nature of Matthew's thinking leads teachers to become engaged in exploring his mathematical understanding. One group of teachers focused on Matthew's mixing of the concepts of "one half" and "one half more" and mentioned that his confusion is one that they have seen before. In particular, the teachers wondered if Matthew's mistake is due to an incomplete conceptual understanding of fractions as part of a whole or due to merely misinterpreting the mathematical language of fractions.

The *"Wow!"* clip

While dilemmas often prompt us to want to explore a video clip, they are not the only events that do so. In some powerful video clips, the student thinking is easy to follow, but the students' innovation

Figure 1

An example of a *"What?"* clip: "What is Matthew thinking?"

Problem on the board: If 1 inch = 50 miles,
then 1/2 inch = ____ miles.

Matthew:	[It's] 75 ...
Mr. Novak:	Talk us through your thinking here.... What are we looking for, Matthew?
Matthew:	Like, what's one-half more. Like, what a half equals ...
Mr. Novak:	We know one is fifty....
Matthew:	I just put fifty and then, till a hundred, and that's fifty, sixty, seventy, eighty, ninety, and one hundred. I just did half of fifty, and it would be like seventy and eighty, and I just added five so it would be equal in the middle.

still makes us say, *"Wow!"* In other words, with a *"Wow!"* clip, we are able to follow and understand the mathematical reasoning portrayed, but we still regard that reasoning as creative and unique. Rather than simply using a strategy that has been taught, students discover their own method of working through a problem. In fact, one teacher opened the discussion of a *"Wow!"* video clip by exclaiming, "I would never have thought of doing it that way!" *"Wow!"* clips are interesting because they provide us with new insights into how students think about the mathematics presented in the classroom.

Figure 2 is an excerpt from one of our favorite *"Wow!"* clips. In this clip, Mr. Epson has students share their methods for solving a problem about ratios. Tim's explanation involves converting fractions to their equivalent percents, a strategy Epson has never used in class. By the end of the clip, Tim's work is clear. In one discussion of this clip, a teacher explained, "[Tim] knew that a fourth was half of a half, and then he went over to the numbers [in the problem]." The teachers had an in-depth discussion of Tim's understanding of the relationships among fractions, ratios, and percents, and they also explored whether Tim's ideas would be useful for other students to consider.

Figure 2

An example of a *"Wow!"* clip: "Wow! Tim has a really creative approach!"

Problem on the board: If 1/2 inch represents 30 miles, then 1/4 represents _____ miles.

Tim:	I knew that one-half is fifty. I also knew that one-fourth would be twenty-five. So then I knew that twenty-five was half of fifty. I did what's half of thirty, and I got fifteen.
Mr. Epson:	Huh? You want to come up to the board and write that out for us?
Tim:	I knew that this [pointing to the one-half] equaled fifty, and I knew that that [pointing to the one-fourth] equaled twenty-five.
Mr. Epson:	But, doesn't it say that half an inch on the map represents thirty miles?
Tim:	One-fourth equals twenty-five. One-half equals fifty. And then three-quarters equals seventy-five, and then the whole equals one hundred. I know that twenty-five is half of fifty. So, I went on and did thirty divided by two, and that got me fifteen.
Mr. Epson:	Okay, I see. You made one hundred your whole.
Tim:	That's the easy way!

The *"Hmm ..."* clip

Whereas *"What?"* and *"Wow!"* clips both present student thinking that is out of the ordinary, clips that contain much more routine thinking can be intriguing as well. *"Hmm ..."* video clips are, in a sense, less dramatic versions of *"What?"* clips; rather than an initial reaction of intrigue and shock, *"Hmm ..."* clips lead us to pause and ponder. Although they portray students engaged in routine, even algorithmic, mathematical reasoning, something about the students' ideas is confusing. Perhaps, as you will see in the next example, students have made a mistake that prompts teachers to explore the mathematics underlying the algorithm used. *"Hmm ..."* clips are important to consider when selecting video to prompt discussion because they remind us that student thinking does not always have to be innovative to be interesting.

In one *"Hmm ..."* clip, a pair of students is playing a card game designed to help them practice their multiplication facts. The students play many rounds of the game, every round consisting merely of each student turning over a pair of cards, multiplying the two values shown, and stating the product. The person with the highest product wins that round and gets to keep all four cards. Although the mathematical ideas in this clip are not as sophisticated as in a *"What?"* or a *"Wow!"* clip, the students' mistakes and decisions, which are often confusing, prompt us to want to understand what they are thinking. For example, why do students get different answers to the same multiplication problem in different rounds of the game? Why do students sometimes challenge one another's answers and other times let incorrect answers go? Even if these students have not mastered basic multiplication facts, what do they understand?

Figure 3

Teachers discuss a *"Hmm ..."* clip: "Hmm .... I have to think about this."

Wanda:	[I noticed] the way he was counting on the card in the later [round].... He was counting the groups.... I think it was three times six.
Daniel:	So, what was he doing?
Yvette:	He was counting it that many times.
Wanda:	Which I thought was interesting.
Daniel:	When he counted six and then another six, I mean ...
Frances:	Yes, and he was keeping track. You noticed he was counting with one—one, two, three, four, five, six. And then he was keeping track, so he knew he counted six one time, six the second time, six the third time.
Daniel:	Well, then he knows what the meaning of multiplication is.
Wanda:	Which is ... exactly. I mean, that's the base.
Yvette:	He just hasn't retained the facts.

Figure 3 provides an excerpt from a discussion about the multiplication game video clip. In this discussion, the teachers recognize that one of the students in the video is counting groups of six and using his fingers to keep track of how many groups he has counted so far. The teachers' conclusion—that the student

Table 1

Three Dimensions of Student Thinking

Dimension	Description	Questions to Consider
Windows	Evidence of students' mathematics thinking	▪ Is student written work visible? ▪ Do students explain their ideas verbally? • Do we see students' gestures or facial expressions?
Clarity	Ease of understanding students' ideas	▪ Am I confused about what students are doing or saying? *or* • Do I understand the students' ideas or methods?
Depth	Nature of students' mathematics thinking	• Are students involved in routine tasks based on memorization and rote recall? *or* ▪ Are students engaged in mathematical reasoning and problem solving?

Table 2

Types of Video Clips and Dimensions of Student Thinking

Clip Type	Description	Dimensions of Student Thinking				
		Window		Clarity		Depth
What?	*"What just happened?"*	Evidence of student ideas	*and*	Student ideas are confusing	*and*	Students engage in mathematical reasoning or problem solving
Wow!	*"I never thought of that!"*	Evidence of student ideas	*and*	Student ideas are clear	*and*	Students engage in mathematical reasoning or problem solving
Hmm...	*"There's something interesting here."*	Evidence of student ideas	*and*	Student ideas are confusing	*and*	Students engage in routine, algorithmic thinking

understood what it means to multiply but had not yet retained the multiplication facts—may have powerful implications for future classroom instruction.

Key Dimensions

In order to recognize and distinguish among different types of video clips, we find it helpful to look at three aspects of how student thinking is portrayed (see **table 1**). First of all, each clip type described here contains a *window* into (i.e., evidence of) what students are thinking. For example, Tim provided a verbal explanation and wrote some of his work on the chalkboard. In the card game example, students did not do a lot of talking, but we are able to see them counting on the cards.

Second, a video clip can be characterized by its *clarity*, that is, by how easy it is to understand students' ideas and methods. *"What?"* and *"Hmm ..."* clips, in which student thinking is confusing, may prompt us to want to understand what a student meant by a statement or how a student arrived at a particular answer. On the other hand, we are often prompted to discuss a video clip, even when the student thinking is clear, if the students' ideas are particularly insightful; such is the case with *"Wow!"* clips.

Third, video clips can be characterized by the *depth* of the student thinking portrayed. Both *"What?"* and *"Wow!"* clips involve students who are engaged in mathematical reasoning and problem solving; in these clips, students demonstrate their own unique and personal ways to solve a problem. *"What?"* and *"Wow!"* video clips often lead to exploration of the substantive and innovative mathematical ideas that they contain. In contrast, *"Hmm ..."* clips portray students relying on rote recall of facts and on routine algorithms that have likely been presented to them by a teacher. It is important to remember that, in some cases, we can be prompted to have rich discussions about video that contains routine mathematics; in the case of *"Hmm ..."* video clips, students' ideas become a jumping-off point for delving more deeply into the mathematics. To summarize, **table 2** characterizes the three video clip types in terms of these dimensions of student thinking.

Video Clips for Discussion

We have introduced three types of video excerpts that can promote productive discussion of students' thinking about mathematics. We have also described three aspects of video to attend to when

Table 3

Steps in Selecting a Video Clip of Student Thinking

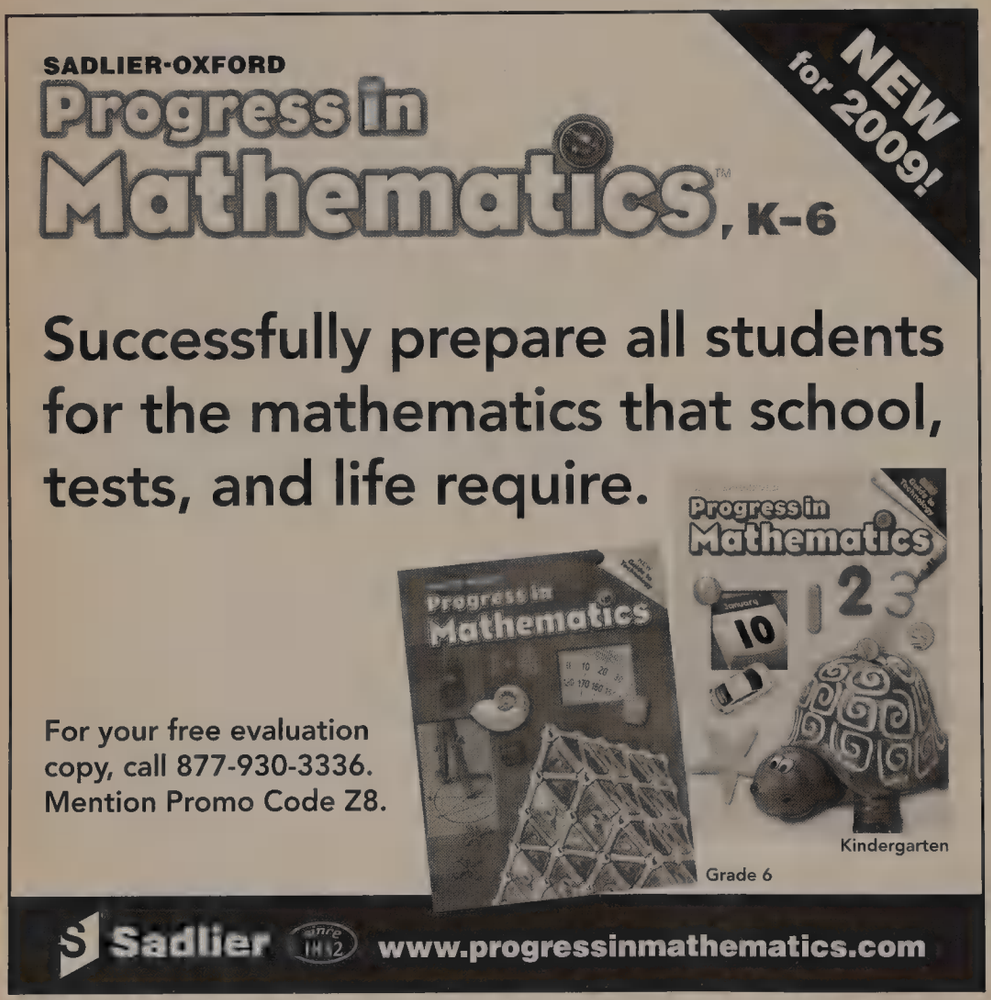
Step 1: Window	Locate portions of classroom video that contain evidence of student thinking.
Step 2: Clarity and Depth	Find an excerpt with a useful combination of clarity and depth—one that makes it a *"What?"* *"Wow!"* or *"Hmm ..."* clip.
Step 3: Prepare for the Meeting	▪ Make copies of the problem or worksheet used in the video. ▪ Make a list of possible discussion questions.

selecting excerpts to share with teachers: the *windows*, *depth*, and *clarity* of student thinking.

To conclude, we offer a few practical suggestions to help you get started in selecting and using video clips of students' mathematical thinking (see **table 3**). Once you have videotaped a classroom lesson or two, you will need to find an excerpt to share. Evidence of student ideas is always key; start by finding a passage with ample *windows* into students' thinking. Next, decide if the excerpt you are considering has the appropriate combination of *depth* and *clarity* to be a *"What?"* a *"Wow!"* or a *"Hmm ..."* clip. Sometimes a little editing can make a clip much more effective. For example, you might need to cut two minutes that students spend silently working on a problem before they begin sharing their ideas.

Of course, a productive discussion is not based on an interesting video clip alone but also on how you engage your fellow teachers with the video. Providing teachers with some background information about the lesson will allow them to spend their time focused on what the students are thinking about the problem rather than focused on understanding the context for the students' work. We almost always bring copies of the problem or activity sheet that students are working on in the video, and we often have teachers solve—and discuss—the problem before watching the video clip. This "prediscussion" can get teachers to start thinking about complexities of the mathematics that they might otherwise miss (Seago 2004).

Furthermore, prepare a list of some questions you plan to ask. What is it about this video clip that you find interesting? You need not stick to your list of questions, but they provide a starting point for your discussion and can remind you to bring up issues that first-time viewers might miss. Finally, remember that you yourself do not need to have an answer to each of your questions; the best questions often do not have a single answer. As we see it, the goal of discussion is to explore student ideas, not to reach the "correct" conclusion.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CALL FOR MANUSCRIPTS FOR THE OCTOBER 2010 FOCUS ISSUE

Teaching Mathematics in a Flat World

Author Thomas Friedman's "flat world" phrase points to our new century's globally connected nature, exponential technical advances, and instantaneous communication across the planet. Former U.S. Secretary of Education Richard Riley keenly observed, "We are preparing students for jobs that do not exist, using technologies that have not been invented, in order to solve problems we cannot even imagine at this time."

Teaching Tomorrow's Citizens Today

As we hurtle through this rapidly advancing flat world, what does it mean to "know," "do," and "teach" math? Attempting to prepare students for a dynamically changing environment, what mathematical knowledge and essential understanding do elementary teachers need today to help them better educate the workforce and leaders of tomorrow?

The *Teaching Children Mathematics* Editorial Panel is seeking articles to address themes related to the mathematical knowledge of these teachers. We strongly encourage submissions that reflect the value of teacher knowledge directly within the classroom and its influence on student learning and achievement. Moreover, we welcome articles related to preparing preservice teachers as well as practicing teachers. The following topics are to guide—not limit—authors in addressing one or more such issues.

Teachers' Essential Mathematical Knowledge

- Identify the skills, concepts, and dispositions elementary teachers must know to promote both the procedural fluency and the conceptual understanding of their students.
- Provide evidence regarding content areas or specific topics that enable teachers to become more confident and competent in teaching mathematics.
- Describe the *essential* mathematical understandings necessary for elementary teachers and how these may differ from typical mathematics requirements.
- Explain how these essential concepts allow teachers to empower *all* their students with the mathematical skills and innovative problem solving needed in today's flat world.
- Demonstrate the connection between teachers' essential knowledge and the elementary curriculum in state or national standards or within NCTM's *Curriculum Focal Points.*

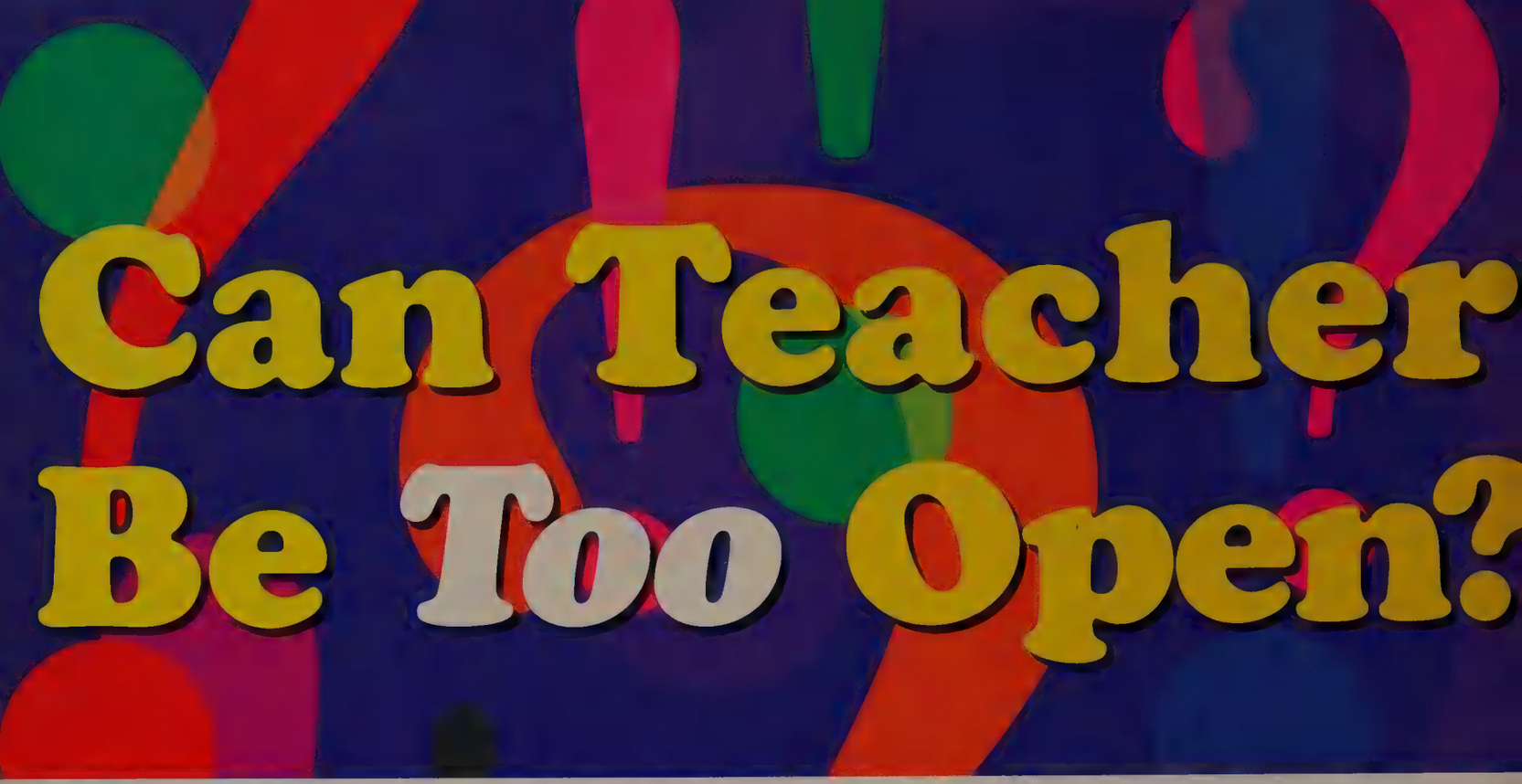
How to Best Learn

- Provide examples of innovative courses or professional development programs that have been successful in increasing teacher content knowledge and understanding.
- Describe effective, collaborative efforts—among school districts, institutes of higher education, and other agencies—for preparing elementary teachers to teach mathematics.
- Explain obstacles to increasing teachers' mathematical knowledge and how they have been effectively addressed.
- Clarify the importance of "mathematical habits of mind" and pedagogy in relation to learning and teaching mathematics in a flat world.
- Describe technology tools and other resources that support teachers in their work and in their acquisition of essential knowledge.

Impacting the Classroom

- Explicate the effects of an increase in a teacher's mathematical knowledge on the teacher's curriculum selection, teaching practices, and assessment choices.
- Discuss the connections between teachers' mathematical knowledge and student learning and achievement in mathematics.
- Describe specific classroom examples or experiences that demonstrate the importance of teacher knowledge in learning a particular mathematical concept or idea.
- Explore beliefs that teachers hold with regard to the value and type of mathematical knowledge needed in their classrooms.
- Explore the roles of math coaches and elementary math specialists in advancing the understanding of mathematics for both teachers and students.
- Examine the role of teacher collaboration in aiding teachers to reflect on and improve their mathematical knowledge and classroom practice.
- Discuss the level of mathematical understanding required for teachers to successfully address the needs of diverse learners and differentiated instruction.

Manuscripts should not exceed 2500 words; include figures and photographs at the end. Submit completed manuscripts to *Teaching Children Mathematics* by accessing **tcm.msubmit.net** by July 31, 2009. On the cover page, please state clearly that the manuscript is being submitted for the October 2010 *TCM* Focus issue. Author identification should appear only on the cover page. For manuscript preparation guidelines, visit my.nctm.org/eresources/submission_tcm.asp.



Can Teacher Be Too Open?

Over the last two decades, many elementary teachers have been asking questions such as the following in their mathematics classes:

- What do you notice?
- Why?
- What would make sense?

As have educators in other contexts, many elementary teachers have moved toward a shared belief that open-ended, probing, and inexplicit teacher questions are more likely to promote student reasoning and engagement with substantial mathematics than narrow, directed, and explicit questions (Lampert 2001; Mewborn and Huberty 1999; NCTM 1991, 2000; Vacc 1993). Originally, educators proposed using such inexplicit questions in response to the many fact-seeking questions being asked in mathematics classrooms, such as "What do you get when you add seven plus three?" or "Should you multiply or divide to solve this problem?"

As part of an effort to change these fact-oriented questioning practices, *Professional Standards for Teaching Mathematics* presented a list of inexplicit open questions as good examples of teacher dialogue (NCTM 1991, pp. 3–4). These questions, which require reasoning and justification rather than a recitation of facts, can be considered inexplicit because they allow for a wide range of not only *possible* answers but also *kinds* of answers. Students could respond to "What do you notice?" by talking about a pattern, the correct answer, the way the problem was written, a similarity to a previous problem, or many other topics. Whereas explicit questions direct students to focus on specific ideas and answer in particular ways, inexplicit questions often provide no clues about the kind of answer the teacher desires.

For example, Lampert wrote that she began lessons in her fourth-grade classroom with questions such as, "OK, who has something to say about A?" as a way of teaching students "that mathematical talk can have a broad range, and not just be about right and wrong answers to teachers' questions" (2001, p. 145). The question—"Who has something to say about A?"—is inexplicit because, as Lampert pointed out, students could comment on the difficulty of the problem, announce the correct answer, or discuss a solution strategy. In contrast, the question "Who can show how they solved problem A?" is explicit because it narrows the response possibilities.

Although inexplicit open questions allow greater flexibility in terms of how students can answer, a lack of guidance on what constitutes an appropriate response can also discourage some students from participating in discussions. Those who lack confidence may stay silent because they do not know what kind of answer the teacher expects.

This tension between asking questions that invite thinking and asking questions that encourage

By Amy Noelle Parks

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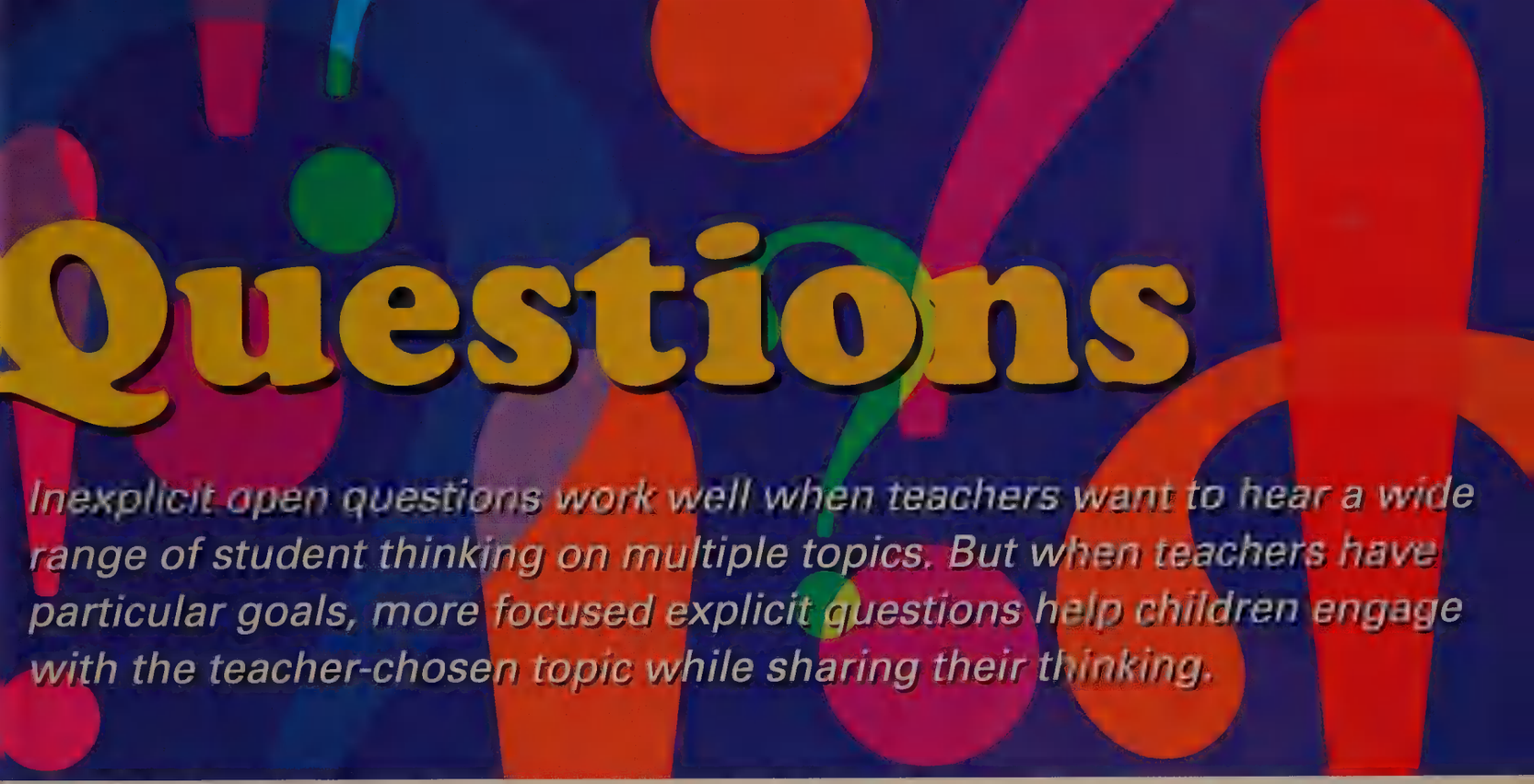

Questions

Inexplicit open questions work well when teachers want to hear a wide range of student thinking on multiple topics. But when teachers have particular goals, more focused explicit questions help children engage with the teacher-chosen topic while sharing their thinking.

all children to participate was one I noted when I was teaching third grade and making the transition toward Standards-based mathematics instruction. Often when I asked a question such as "Why?" or "What do you notice?" I found myself calling on the same three students again and again, while others stared blankly at me.

When I left the elementary classroom to become a researcher, my interest in teacher questioning continued. I decided to explore the use of inexplicit and explicit questions as part of a larger study in which I spent a year observing an ethnically diverse, urban third-grade mathematics class taught by a reform-oriented teacher. In this class, I found that inexplicit open-ended questions often made it harder for some children to engage with the mathematics under discussion. However, efforts to make open-ended questions more explicit and context specific often allowed these same children to participate competently in the practices of reasoning and justification.

Marcus's Story

Marcus routinely solved all problems asked of him, participated frequently in small groups, and quickly mastered his multiplication facts. However, in whole-class discussions, he often appeared to be struggling when responding to inexplicit questions because he could not immediately identify the kind of response the teacher desired.

At the beginning of a lesson on rounding, the teacher wrote the following on the board and asked students to write observations in their journals:

730	700
73	100
703	700

After a few minutes, the teacher asked, "Who can say what they observed?" [*Following a short pause, three other students raised their hands, but she called on Marcus.*]

Marcus: It's even.
Teacher: Why?
Marcus: Odd?
Teacher: Why?
Marcus: Even?
Teacher: Now they're back to even. When I ask, "Why?" do I mean you're wrong?
Marcus: No?
Teacher: You need to tell me why. If you don't know, say, "I just guessed. I don't know."
Marcus: I don't know.

On the board, the teacher drew a number line from 0 to 800 with slashes at each hundred. She asked students to look at the numbers and think about rounding. Marcus correctly placed 73, 730, and 703 on the number line. He correctly drew arrows from these numbers to the hundreds numbers they rounded to. Although privately complimented on his written work, Marcus again did not raise his hand when the teacher asked students what they had done.

In the previous conversation, Marcus had interpreted the teacher's question "Why?" as a cue to change his answer, although that was not what the

Inexplicit questions allow students more flexibility with answers, but the lack of guidance regarding an appropriate response also discourages some students from answering

teacher intended to communicate with her response. It is unclear if the teacher was asking Marcus why he said the numbers were even as opposed to odd; why he chose the category of even or odd as important; or why he chose to focus on the numbers in the second column (which were all even) as opposed to the numbers in the first column. Because the question did not narrow possible responses for Marcus, he had a wide range of interpretations to choose from, including the one he chose: that "Why?" meant he should change his answer. Given his prompt response to the number line, Marcus likely could have made a statement about the relationship between the numbers on the board and the nearest hundred; however, the inexplicit question left him struggling over the teacher's intent rather than the mathematics.

The question "Why?" can be a powerful follow-up to a student response. Asking for an explanation rather than telling students that they are right or wrong can provide opportunities for students to share their thinking. In describing her practice, the teacher in this episode had talked about her desire to move away from correcting students and toward asking them to explain their answers. However, in this particular case, the why question did *not* prompt Marcus to explain his thinking.

Although the question that launched this episode, "Who can say what they observed?" was very similar to Lampert's question, "Who has something to say about A?" the teacher in this episode—unlike Lampert—had particular content in mind that she wanted to make part of the discussion (i.e., rounding to the nearest hundred). Thus, rather than pursuing Marcus's thoughts about even numbers when he finished speaking, she redirected the class toward the subject of rounding. If the teacher had been willing to follow the student's thinking—even in an unexpected direction—then inexplicit open questions would have served her purpose. However, because her goal was to direct students toward particular content, more explicit open questions may have helped students like Marcus to recognize her intentions.

For instance, in this case, the teacher could have opened with the question, "Who can use rounding to explain the relationship between these two columns of numbers?" which would have directed the children's attention toward the particular mathematics that interested the teacher at the moment. Similarly, after Marcus raised the issue of even or odd, more explicit questions could have linked his observation to the intended content, perhaps by asking students why they thought all the numbers in the second column were even and all the numbers in the first column were not. In cases where teachers want to hear a wide range of student thinking on multiple topics, inexplicit open questions work well because they cast a wide net. However, in cases where teachers have particular goals, more focused explicit questions can help children engage with the topic the teacher has chosen while still sharing their thinking.

Caitlin's Story

In contrast to inexplicit questions, those in which the teacher was more specific about her intentions seemed to encourage more children to participate competently. These were not questions that called for lower-level thinking or recitation of basic facts. Rather, they were questions that provided language support for the kind of response the teacher wanted.

When the teacher asked, "Who can say whether Sienna's answer is correct?" [*pointing to the board, where 700 + 30 + 1 was written under 730*], several students raised their hands. "Caitlin?" A student who rarely participated in whole-class discussions, Caitlin spoke with confidence:

Caitlin: No. No, it's wrong.
Teacher: Caitlin, can you say why you disagree with Sienna?
Caitlin: There's an extra one. 700 plus 30 is 730. There should be a zero, not a one.

Caitlin publicly evaluated another student's expanded answer and provided a reason for her correction, thus demonstrating knowledge of place value as well as an ability to put her thoughts into words. The teacher's question focused Caitlin's attention on the answer to a particular problem and gave her specific directions about the kind of answer expected. Instead of simply asking, "Why?" the teacher asked Caitlin to say why she disagreed with Sienna's answer. This question communicated to Caitlin the kind of response that was expected. Although the question directed toward Caitlin was explicit, it could still be considered open because Caitlin could have chosen many ways of explaining her disagreement with Sienna. She could have

commented that there is no "1" in 730, or she could have presented a written addition problem, demonstrating that Sienna's numbers did not add up to 730. In this case, the teacher's explicit open question allowed Caitlin to competently share her mathematical thinking rather than to puzzle over the teacher's intentions.

What Can Teachers Learn?

A number of specific strategies seemed to be at play in asking explicit open questions. Often, the teacher attached a student's name to the question, which worked to invite particular children to speak. Even children who did not choose to volunteer in discussions typically responded when called by name.

Additionally, explicit open questions tended to identify what the teacher saw as important in the problem under discussion (i.e., rounding and Sienna's answer). These questions also seemed to follow other questions. That is, rather than opening a discussion, they were almost always used in the midst of a conversation or else in response to a bit of mathematical work. For example, rather than putting a problem on the board and asking students to talk about what they noticed, the teacher might first ask a student to show his or her work on the problem and then ask specific questions regarding that work, such as, "Can you tell me why you're adding 23 and 23?" This kind of question still requires students to justify their thinking but offers a starting point in the problem to begin talking. Inexplicit open questions ("Why?" or "What do you notice?") can help teachers' questioning techniques focus more on student thinking rather than on correct answers. However, building on these questions to make them specific to particular contexts by naming the mathematics under consideration or by identifying the kind of response desired can help more children engage in these discussions.

Another strategy this teacher used to make questions more explicit and to provide more support was to ask introductory yes or no questions to which the class responded as a group. Often students responded by giving a thumbs up or a thumbs down. Answering an open-ended question such as "Why?" in this way is impossible; however, when this teacher asked yes and no questions, she found ways to encourage students to think. For example, she asked, "Is there a way to solve this problem without regrouping?" and "Did he find all the arrays?" All the children responded to such questions (admittedly, because the teacher expected them to do so). Some children looked carefully around the room before making a decision; however, the teacher frequently called on these visibly uncertain students to explain why they had answered yes or no. Buoyed by the knowledge that many of their classmates had interpreted and answered the question in similar ways, these students were often able to articulate their thoughts more clearly than when asked a more inexplicit question or when asked an unexpected question.

One afternoon the class worked on this problem:

> When we made soup, we had
> 50 carrots broken into 2 pieces,
> 15 celery stalks broken into 4 pieces,
> 9 potatoes cut into 10 pieces, and
> 2 onions cut into 50 pieces.
> How many whole vegetables did we have?
> How many vegetable pieces did we have?

Early in a discussion about this problem, Jerome, another student who rarely participated in whole-class discussions, shrugged and refused to answer when the teacher asked him to "tell the class something" he had done to solve the problem. However, a few minutes later, the teacher asked students to put their thumbs up or down to show whether they should add fifty carrot pieces to the total number of vegetable pieces in the soup. After looking around the room and seeing that most students had their thumbs pointed down, Jerome pointed his down as well. The teacher called on him and asked whether they could just write "50" in their list.

Jerome: You have to double it! Double it!
Teacher: Why do you have to double the carrots?
Jerome: Because it said to cut them in half.

Jerome had solved this problem in his journal before the discussion began; however, when asked to say "something" about his solution, he had nothing to say. When asked a specific question about what he had done and given the additional support of seeing that his classmates agreed, he participated with both competence and confidence.

Conclusion

Learning to ask questions that require students to reason and justify their thinking can be challenging for teachers. Drawing on published lists of recommended questions is a good way for teachers

to build a repertoire of questions. However, such questions are recommended frequently in part because they can be used at any grade level and in any lesson. In some ways, this advantage of being adaptable to many contexts can also be a disadvantage because these questions offer little support to students who may be trying to interpret their teachers' words and intentions.

The teacher questions in the episode with Marcus ("Who can say what they observed?" and "Why?") could be used in many lessons at many grade levels; however, they did not encourage Marcus to articulate his thinking in a meaningful way. Using strategies to revise the questions (e.g., "Who can talk about the way the numbers in the first column are being rounded?" or "Why did you think it was important to focus on whether the numbers were even or odd?") might have helped Marcus engage more productively in the discussion.

Teachers cannot cut out a list of context-based questions and post them on their overhead. Nor can they recycle the same questions from lesson to lesson. Learning to adapt questions to direct students' attention to particular features of the mathematics while still allowing for students' own thinking and creativity is a greater challenge than building a repertoire from a preestablished list. However, by finding ways to direct children's attention toward the mathematics and clarify the kind of responses they are seeking, teachers invite more students to participate competently in their mathematical discussions.

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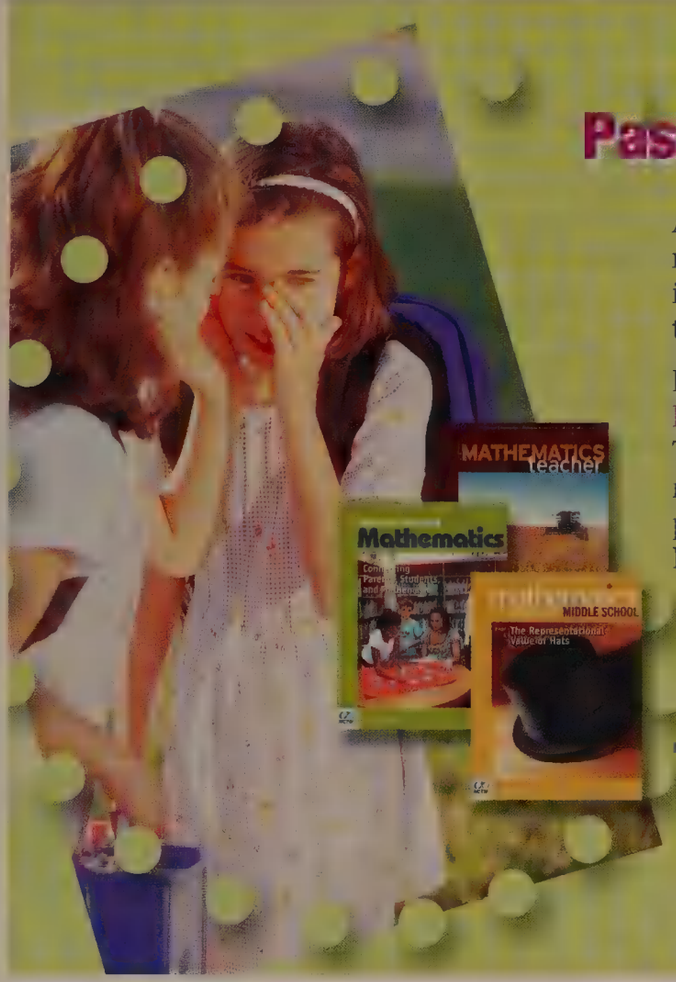

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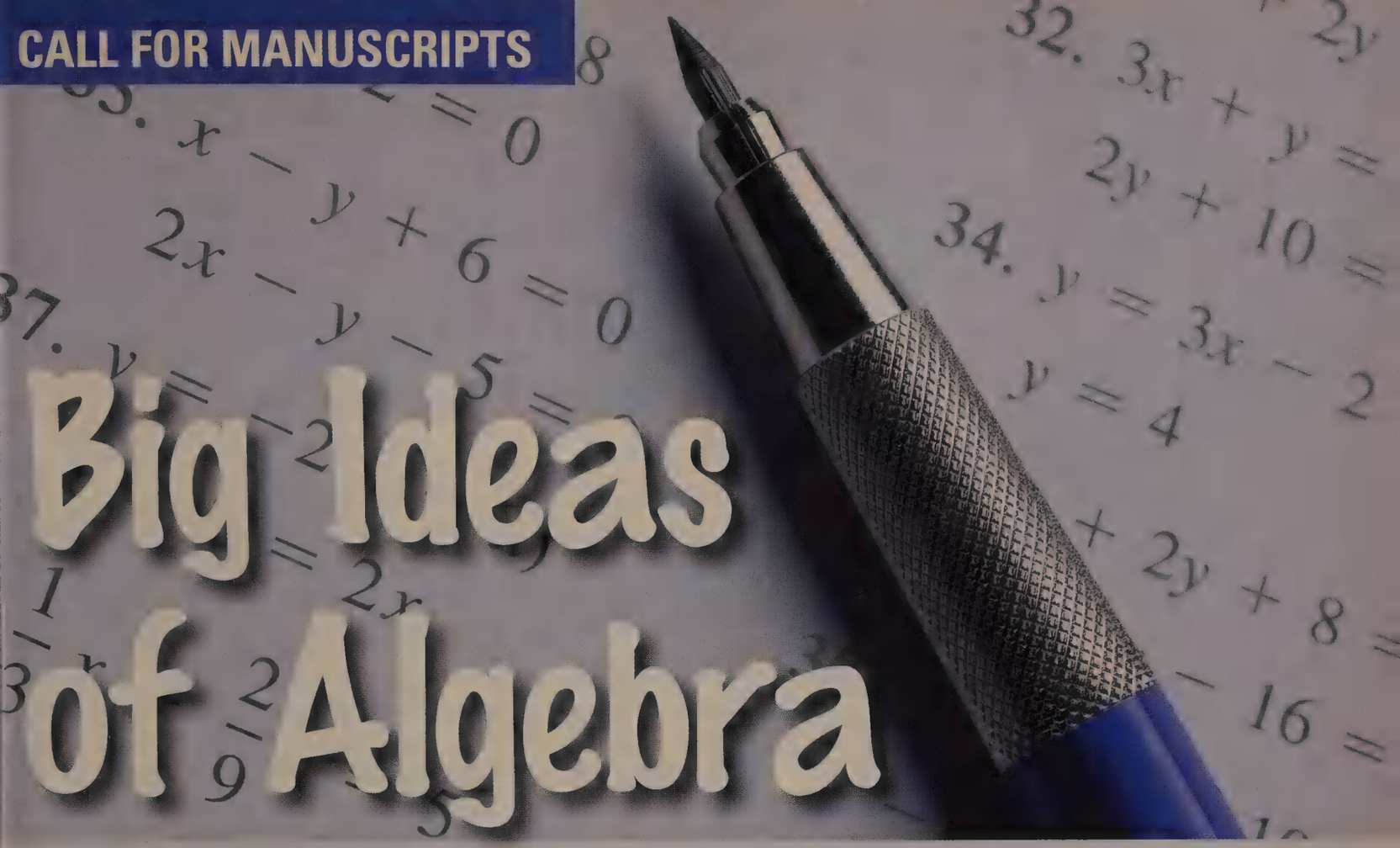

Big Ideas of Algebra

"By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more-sophisticated work in algebra in the middle grades and high school" (NCTM *Principles and Standards for School Mathematics*, p. 37).

The Editorial Panel is searching for manuscripts that share ideas and lessons to make the big ideas of algebra (Equivalence, Patterns and Functions, Variables, and Graphing) accessible to all students. What do you do in you classroom to make algebraic thinking come alive for your students? To teach algebraic thinking, how do you—

- use concrete materials;
- relate concrete materials to symbols, tables, graphs, and other representations;
- incorporate children's books, familiar contexts, science, and problem solving; or
- link arithmetic and algebraic thinking?

How do you establish a classroom or learning atmosphere that stimulates learning and algebraic thinking and supports students? Which activities and questions do you use to help students do the following?

- Investigate the meaning of the equal sign.
- Explore the concept of equality, or balance, as central to understanding equations and inequalities.
- Transform expressions and equations into equivalent forms.
- Create, describe, extend, and interpret patterns.
- Perceive the regularity and predictability of patterns.
- Discover repeating patterns as a foundation for considering growing patterns and number sequences.
- Use patterns as a vehicle to stimulate thinking about variables and functions.
- Explore and develop understanding about functions.
- Develop ways to determine the output when the input is given for a particular function.
- Develop ways to determine the input when given the output for a particular function; expand understanding of how to construct inverse operations rules to "undo" what other operations perform.
- Understand assorted uses of variables.
- Determine what is changing when using variables.
- Conclude whether a change follows a particular pattern.
- Determine and communicate what is happening in graphs.
- Use coordinate graphs for the purpose of representing relationships between two quantities.
- Understand the process of plotting points in coordinate graphing.
- Play games that reinforce and deepen understanding of algebraic thinking and ideas.
- Make connections among arithmetic, geometry, and algebra.

Typed, double-spaced manuscripts should be no longer than 2500 words; attach figures and photographs at the end. Submit completed manuscripts to *Teaching Children Mathematics* by accessing **tcm.msubmit.net**. Author identification should appear only on the cover page. For manuscript preparation guidelines, visit www.nctm.org/publications/content.aspx?id=7696.

Kathryn B. Chval and Sarah J. Hicks

Calculators in K–5 Textbooks

> The vision for mathematics education described in *Principles and Standards for School Mathematics* is highly ambitious. Achieving it requires solid mathematics curricula, competent and knowledgeable teachers who can integrate instruction and assessment, education policies that enhance and support learning, classrooms with ready access to technology, and a commitment to both equity and excellence. (NCTM 2000, p. 3)

As the quote above indicates, electronic tools to support the *teaching* of mathematics can be an important part of teachers' resources for promoting student *learning* of mathematics. The three articles in this month's "Contemporary Curriculum Issues" series collectively focus on issues relevant to improving the effectiveness of electronic tools in classrooms; the support needed for teachers to use calculators in interesting and useful ways; examples of ways that existing and emerging technologies can encourage, support, and enhance students' engagement with and learning of mathematics; and the impact of electronic computer algebra systems (CAS) on the learning of algebra. The series editors encourage you to read the "Contemporary Curriculum Issues" department articles in each of the three school journals, *Teaching Children Mathematics* (*TCM*), *Mathematics Teaching in the Middle School* (*MTMS*), and the *Mathematics Teacher* (*MT*). Each article explores effective, grade-appropriate uses of electronic tools in mathematics classrooms. Knowing what experiences with technology students may bring to your grade level and what the expectations are for grades following yours can help you build smoother transitions for the students. The *MTMS* article by Hollenbeck and Fey explores these questions: If you and your students had full use of existing mathematical and communication tools, how would it change the way you teach mathematics in the middle grades? How would it change the way that you assess student learning? How would it change the content of your curriculum? The *MT* article by Zbiek and Heid explores features of CAS, particularly when used as part of a classroom tool set, and addresses several ways in which CAS can cause the taught and learned curriculum to focus on big mathematical ideas that might have previously escaped students' attention.

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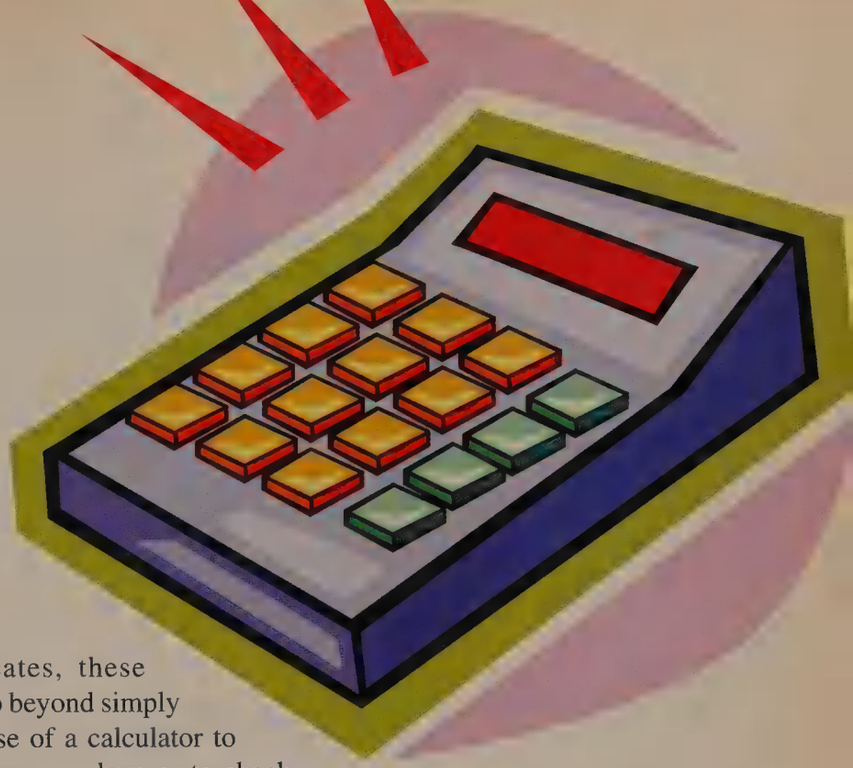
"Contemporary Curriculum Issues" provides a forum to stimulate discussion on contemporary mathematics curricular issues across a K–12 audience. NCTM concurrently publishes sets of three articles focused on a single curriculum issue. Each article addresses the issue from the perspective of the audience of the journal in which it appears. Collectively, the articles are intended to increase communication and dialogue on issues of common interest related to curriculum. Manuscripts on any contemporary curriculum issues are welcome. Submissions can be for one article for one particular journal, or they can be for a series of three articles, one for each journal. Readers are encouraged to send manuscripts suitable for this section by accessing the appropriate Web site: tcm.msubmit.net, mtms.msubmit.net, or mt.msubmit.net, or by contacting editor Barbara J. Reys (reysb@missouri.edu) for TCM, *Glenda Lappan (glappan@math.msu.edu) for* MTMS, *or Chris Hirsch (christian.hirsch@wmich.edu) for* MT. *Doubled-spaced typed manuscripts should not exceed 2500 words.*

The Classroom

For many years national organizations and prestigious committees have recommended using calculators in classrooms (NCTM 1989, 2000; National Council of Supervisors of Mathematics [NCSM] 1988; National Research Council [NRC] 1990). NCTM's recently released position statement again emphasizes the importance of calculators; notes that "curricula and courses of study should incorporate instructional technology in learning outcomes, lesson plans, and assessments of students' progress"; and explains why using calculators and other technological tools is important ("The Role of Technology in the Teaching and Learning of Mathematics" 2008).



With guidance from effective mathematics teachers, students at different levels can use these tools to support and extend mathematical reasoning and sense making, gain access to mathematical content and problem-solving contexts, and enhance computational fluency. In a well-articulated mathematics program, students can use these tools for computation, construction, and representation as they explore problems. The use of technology also contributes to mathematical reflection, problem identification, and decision making.

In fact, teachers and students use a number of technological tools, such as SMART Boards™, laptop computers, the Internet, personal digital assistants (PDAs), and calculators. At the same time, elementary teachers face the challenge of "keeping up" as new technologies are introduced and "old technologies" change. Additionally, teachers must work to coordinate their efforts to use technology with both their adopted mathematics textbook and their state and district curriculum standards.

Calculators have the potential to positively influence both the teaching and learning of mathematics. Yet this influence requires the availability of calculators, teachers who know how to use calculators as well as how to teach effectively with them, curriculum materials that facilitate their use, and supportive school and community environments. Unfortunately, too often these conditions do not exist in relation to the use of calculators in U.S. elementary schools. In this article, we report the expectations for calculator use based on an analysis of current mathematics curriculum materials and state standards documents at the elementary level. We also discuss concerns and recommendations related to the use of calculators in today's elementary mathematics curriculum context.

State-Level Curriculum Standards

An analysis of K–5 state mathematics standards (Chval, Reys, and Teuscher 2006) notes that thirty-six of forty-two state documents include a discussion of calculators in either introductory material (twenty state documents) or within statements of specific learning expectations (twenty-three state documents). Across these twenty-three state documents, 186 grade-level expectations (GLEs) refer to the use of calculators at the K–5 levels. **Table 1** describes the roles of calculators that the GLEs identify. State standards describe different uses of calculators for students. As the table indicates, these standards go beyond simply expecting use of a calculator to compute larger numbers or to check answers of computations solved using paper and pencil.

Although NCTM and others advocate for the use of calculators, their inclusion in curriculum materials is more sporadic. Do current curriculum materials incorporate the use of calculators? What are the implicit and explicit messages related to calculators in current elementary mathematics curriculum materials? Do current curriculum materials provide

Table 1

Roles of Calculators in GLEs within State Curriculum Documents

Role of Calculator	Description
Represent	Students use calculators to represent mathematical quantities and ideas, including different notations and graphs. They also connect physical models to mathematical language.
Solve problems or equations	Students use calculators to solve applied problems or equations.
Develop or demonstrate conceptual understanding	Students use calculators to build conceptual knowledge of mathematical ideas or to demonstrate understanding of these concepts.
Analyze	Students use calculators to compare, interpret, identify relationships, make predictions or make sense of data.
Compute or estimate	Students use calculators to compute or estimate.
Describe, explain, justify, or reason	Students use calculators to help them describe strategies, explain reasoning, or justify mathematical thinking.
Choose appropriate calculation method	Students determine if they should compute mentally, use calculators, or use paper and pencil.
Determine a calculated answer's reasonableness	Students determine if the answer acquired by using a calculator is reasonable.

problems and activities that require intentional calculator use for a variety of purposes? Do current curriculum materials provide adequate support for teachers to effectively use calculators?

To investigate these questions, fourth-grade student and teacher materials were examined from six curriculum series: Math Trailblazers; Everyday Mathematics; Math; Investigations in Number, Data, and Space; Math Expressions; and Think Math. For each series, both the extent and nature of calculator use were analyzed. In the next section, we describe the findings from that analysis.

Curriculum Materials

All six curriculum series mention the use of calculators. To determine the extent of use, each page that refers to calculators in both the teacher and student materials was identified. Because calculators are included in the list of necessary materials for specific lessons in some cases but not in others, such pages were not counted. In two series, the authors explicitly state that calculators cannot be used on some specific student pages. Student pages that prohibit the use of calculators were not included because the investigation focused on when and how calculators are to be used in the materials. **Table 2** summarizes the number of pages that refer to calculators for each curriculum series examined.

As the table shows, the extent of references to calculators varies considerably across the six curriculum programs. **Table 3** lists the percentage of days that mention calculators (either in the student or teacher materials) and indicates that some calculator use in the teacher materials is optional.

Note that students do not typically use calculators for an entire class period, so the percentages listed in the table should not be interpreted as a percentage of *time* that students use calculators but rather as a percentage of days that involve calculator use at some level. For example, in the course of an hour-long lesson, students may use the calculator for only one computation.

In addition to variation in the extent of calculator use, dramatic differences exist related to the nature of calculator use across the materials. For example, note the following:

- Everyday Mathematics and Macmillan/McGraw-Hill Math identify specific calculator models, whereas the other curricula do not. Everyday Mathematics includes directions for using both the TI-15 calculator and the Casio fx-55 calculator. Macmillan/McGraw-Hill Math uses the TI-15 calculator.
- Macmillan/McGraw-Hill Math, Math Trailblazers, and Everyday Mathematics include lessons on how to use specific calculator features.
- Both Math Trailblazers and Everyday Mathematics indicate when calculators should be used on assessments; the others do not.
- Teacher materials for the Math Trailblazers, Everyday Mathematics, and Investigations series indicate specific times when calculators should *not* be used.
- Everyday Mathematics and Math Trailblazers have extensive notes in the teacher materials related to how to use calculators and potential mathematical challenges that may occur in class discussions (e.g., the difference between the *subtract* key and the *negative* key). Everyday Mathematics and Math Trailblazers also discuss the use of calculators in family letters.
- Math Trailblazers and Everyday Mathematics

Table 2

Number of Pages Referencing Calculators in Student and Teacher Materials

Curriculum Series	No. of Pages in Student Materials	No. of Pages in Teacher Materials
Math Trailblazers	56	153
Everyday Mathematics	72	116
Macmillan/McGraw-Hill Math	46	9
Investigations	3	43
Math Expressions	1	8
Think Math	4	5

Table 3

Percentage of Days Involving Calculators

Curriculum Series	Percentage of Days Calculator Use Is Expected	Percentage of Days Indicating Optional Calculator Use
Math Trailblazers	62/160 = 39%	0%
Everyday Mathematics	27/126 = 21%	3/126 = 2%
Macmillan/McGraw-Hill Math	9/184 = 5%	14/184 = 8%
Investigations	2/160 = 1%	36/160 = 23%
Math Expressions	5/160 = 3%	3/160 = 2%
Think Math	3/155 = 2%	0%

include mathematical problems that address the eight calculator roles identified in the GLEs (see **table 1**).
- Across the six curriculum series, calculator use is described in different features of the materials such as enrichment activities, technology links, readiness activities, homework assignments, assessments, and classroom routines.

To illustrate the use of calculators within specific curriculum features, **Figures 1a** and **b** compare Everyday Mathematics and Math Expressions.

The Role of Calculators

What roles do calculators play in the different curriculum series? We illustrate some of the differences by examining specific problems.

Macmillan/McGraw-Hill Math has three primary features related to the use of calculators:

- Choose a Computational Method—where students decide if they should use paper and pencil, a calculator, or mental math (twenty-nine cases);
- You Can Also Use a Calculator—which provides an example of how to use calculators (thirteen cases); and
- Technology Links that help students learn how to use the TI-15 calculator (three cases).

Investigations includes a general statement to teachers about the use of calculators in classrooms:

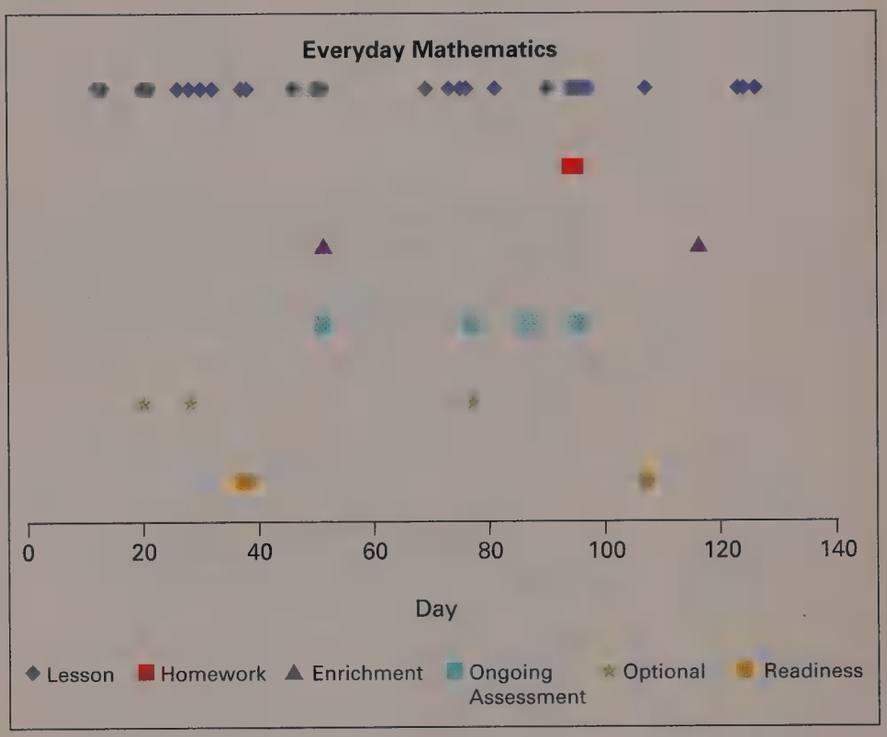
> Students need to learn how to use the calculator effectively and appropriately as a tool, just as they need to learn to read a clock, interpret a map, measure with a ruler, or use coins. They should use calculators for sensible purposes—just as you would do—not as a replacement for mental calculations or for pencil and paper calculations they are learning to do. Encourage students to use calculators to double-check calculations, as an aid if they have many calculations to carry out outside of math class, or to solve problems for which they can think about a solution but don't yet have the experience to carry out the computation. (*Implementing Investigations in Grade 4* [2008], p. 35)

Within specific Investigations units, a majority of the statements about calculators refer to Broken Calculator problems (thirty-two of the forty-three

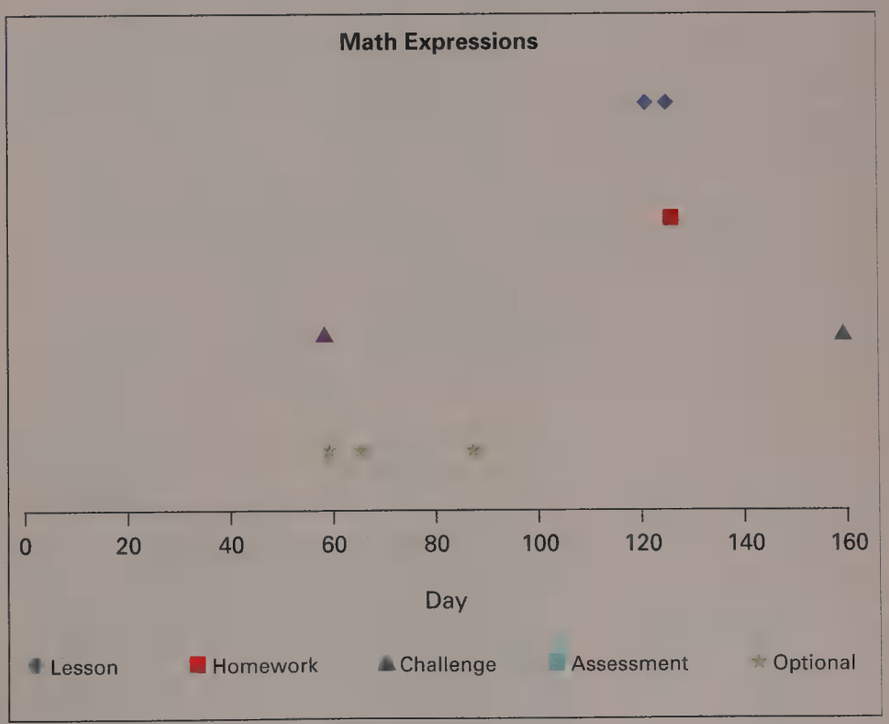
Figure 1

Using the two tables below, we can compare the number of days that involve calculator use in the Everyday Mathematics and the Math Expressions curricula.

(a)



(b)



references to calculators in the teacher materials and two of the three references in the student materials). The Investigations curriculum includes a "Ten-Minute Math" component that is separate from the daily lesson. In the second unit, an activity called "Broken Calculator" is introduced. The teacher materials explain:

> Students work to make a number appear on their calculator display without using particular keys, which are said to be "broken." The broken keys can be operations, numbers, or both. ... Note that Broken Calculator can be done with or without the use of calculators. If students have calculators, ask them to write the appropriate expression first and then use the calculator to check that their solution results in the given number. If students do not have calculators, they can be told to imagine that they are using a calculator to create the number. (*Implementing Investigations in Grade 4* [2008], p. 34)

Figure 2

Sample problem involving calculators from Think Math *Lesson Activity Book Grade 4* (2008, p. 154)

❸ Use a calculator to multiply these numbers between **6.5** and **6.6**.

Numbers between 6.5 and 6.6 →	Numbers multiplied by themselves
6.5	
6.51	
6.52	
6.6	

Paul multiplied a number by itself and got 43. His number must be between two numbers in the left column.

His number must be between _____ and _____. How do you know?

_____

_____

❹ **Challenge** Name three numbers between 3.65 and 3.66

Think Math has minimal references to calculators in their materials. However, **figure 2** illustrates one example of how calculators are used in the student materials from Think Math.

The Everyday Mathematics *Teachers' Reference Manual* states, "Special emphasis is placed on using calculators as tools for counting, displaying numbers, developing concepts and skills, and solving problems—especially real-life problems in which numbers are not always 'nice'" (2007, p. 6). The problem in **figure 3** illustrates how fourth graders use calculators in Everyday Mathematics to work with numbers from real-life data.

The Math Trailblazers *Teacher Implementation Guide for Grade 4* (2004) states the following:

> As noted in the NCTM *Principles and Standards*, "...most of the arithmetic and algebraic procedures long viewed as the heart of school mathematics curriculum can now be performed with handheld calculators. Thus more attention can now be given to understanding the number concepts and the modeling procedures used in solving problems" (NCTM 2000). More practical topics—probability and statistics, geometry, measurement, mental computation, and estimation—deserve more attention. (p. 188)

Figure 4 illustrates how Math Trailblazers uses calculators to build understanding for number concepts. **Figures 2–4** show that current elementary mathematics curricula include creative and challenging mathematical tasks that use the calculator as a tool. In these cases, calculators help students investigate and solve problems that would not be possible or would be too time consuming to solve without a calculator. The calculator creates new possibilities for mathematical tasks that were not plausible before the introduction of calculators. Moreover, "the use of realistic data is motivational and helps children see connections between school mathematics and mathematics used in the world" (Charles 1999, p. 11).

Concerns and Potential Consequences

Although current elementary mathematics textbooks cite many creative and potentially effective uses of calculators, some also present problematic aspects of calculator use:

- Calculator activities are "add on" or "optional." They are primarily used in "extra features," such as extension problems, rather than embedded in lessons.
- Calculators are used solely for checking answers.
- Students are "exposed to" calculators rather than having opportunities to consistently and purposefully use them.
- Students pull out the calculator for a single word problem that has more complicated numbers.
- Calculators are used unnecessarily.
- Materials show an example of how to use calculators but then do not provide opportunities for students to use calculators themselves.
- Materials provide limited opportunities for students to develop conceptual understanding; analyze situations; or describe, justify, and reason with calculators.
- Teacher materials do not provide sufficient guidance for teachers to effectively use calculators.

Such aspects of calculator use are problematic in that teaching demands often constrain use of activities that teachers view as optional or unnecessary. When calculators are used rarely or only for one computation within a lesson, it is not worth valuable class time to distribute and manage them. When activities are labeled as optional, teachers will probably skip them so they can address required material. Similarly, when teachers have not been provided sufficient opportunities to learn how to use the calculator's memory system, trial-and-error feature, or fraction keys, they cannot introduce these features to students. Moreover, if students do not engage while using calculators, then teachers will be less likely to use them. Finally, teachers do not have opportunities to realize the calculator's potential as a teaching and learning tool when curriculum materials do not provide opportunities for students to develop conceptual understanding; analyze situations; or describe, justify, and reason with calculators.

Recommendations

The presence of calculators in K–5 classrooms and mathematics textbooks was rare thirty years ago, and resistance to calculator use was strong and widespread. Since then, calculators have become more sophisticated and cost-effective as the resistance to their use in classrooms has decreased (Chval 2008). Today reference is made to calculators in state standards, textbooks, and assessments. All these indicators suggest that calculator use in K–5 classrooms is more common and acceptable. However, as discussed above, indications are that further improvement is warranted to best use the potential of calculators in K–5 mathematics classrooms. The question still applies: "How should we make use of this extraordinary technology to

Figure 3

Sample problem from Everyday Mathematics *Math Masters Grade 4* (2007, p. 288)

In 2001, there were about 2,317,000 marriages in the United States. The table below shows the approximate number of marriages each month.

1. Use a calculator to find the percent of the total number of marriages that occurred each month. Round the answers to the nearest whole-number percent.

2. According to the table, which is the most popular month for a wedding?

   _____

   Which is the least popular month for a wedding?

   _____

3. Describe how you used your calculator to find the percent for each month.

   _____

   _____

   _____

Month	Approximate No. of Marriages	Approximate % of Total Marriages
January	147,000	6%
February	159,000	
March	166,000	
April	166,000	
May	189,000	
June	237,000	
July	244,000	
August	225,000	
September	224,000	
October	217,000	
November	191,000	
December	152,000	

Source: U.S. Department of Health and Human Services

further the mathematics education of our students?" (Usiskin 1999, p. 1).

New curriculum materials should avoid the problematic issues of some current materials on the market, and more purposeful, consistent use of calculators should be incorporated within daily lessons. Additionally, teacher materials must provide sufficient guidance on how to use the calculator and how to use the calculator to teach children. NCTM's 2008 position statement cautions that "teacher education and professional development must continually update practitioners' knowledge of technology and its classroom applications."

As calculator technology changes, teachers need opportunities to learn not only how to use the tools well but also how to teach with them effectively. Regardless of the textbook or calculator model that is used in a specific district, teachers need print resources, professional development, and a supportive environment to facilitate their learning related to the use of calculators—especially as calculator technology improves and K–5 textbooks take more advantage of calculators' potential. School or district leaders must facilitate conversations and professional development related to using calculators to ensure effective schoolwide implementation.

For the purposes of this article, we focused on one specific technological tool, the calculator. However, we believe that the issues we raise apply to other technologies used in elementary mathematics classrooms, such as PDAs and computer software. Technological tools have the potential to positively influence both the teaching and learning of mathematics. Yet this influence requires the availability of these technological tools, teachers who know how to both use and effectively teach with the tools, curriculum materials that facilitate their use, and supportive school and community environments.

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Figure 4

Sample problem from Math Trailblazers *Student Guide Grade 4* (2004, p. 183)

Operation Target

This is a cooperative contest for two or three people. The goal is to use four digits and four operations (+, –, x, and ÷) to make as many different whole numbers as you can. You need paper, a pencil, and a calculator.

You must use each of the four digits exactly once. You can use operations more than once or not at all. (All division operations must give whole numbers. For example, 9 ÷ 2 = 4.5 is not allowed.)

16. Use 9, 5, 2, and 1 and +, –, x, and ÷ to make as many whole numbers as you can. For example, 9 + 5 x 2 – 1 = 18. List the numbers you make and show how you made them.
    A. What is the largest whole number you can make?
    B. What is the smallest whole number you can make?
    C. How many whole numbers less than 10 can you make? Write number sentences for each number.
    D. What whole numbers can you make in more than one way? Show at least two number sentences for each.

17. Pick four different digits. Make as many whole numbers as you can using your four new digits and +, –, x, and ÷. List the numbers you make and show how you make them.

18. Nila used 1, 2, 3, and 4 to make 10. How do you think she did it? Can you think of another way?

19. Luis used 1, 2, 3, and 4 to make 24. How could he have done it?

20. Romesh used 1, 3, 5, and 7 to make 8. How could he have done it?

21. Make up your own problem like Questions 18, 19, and 20.

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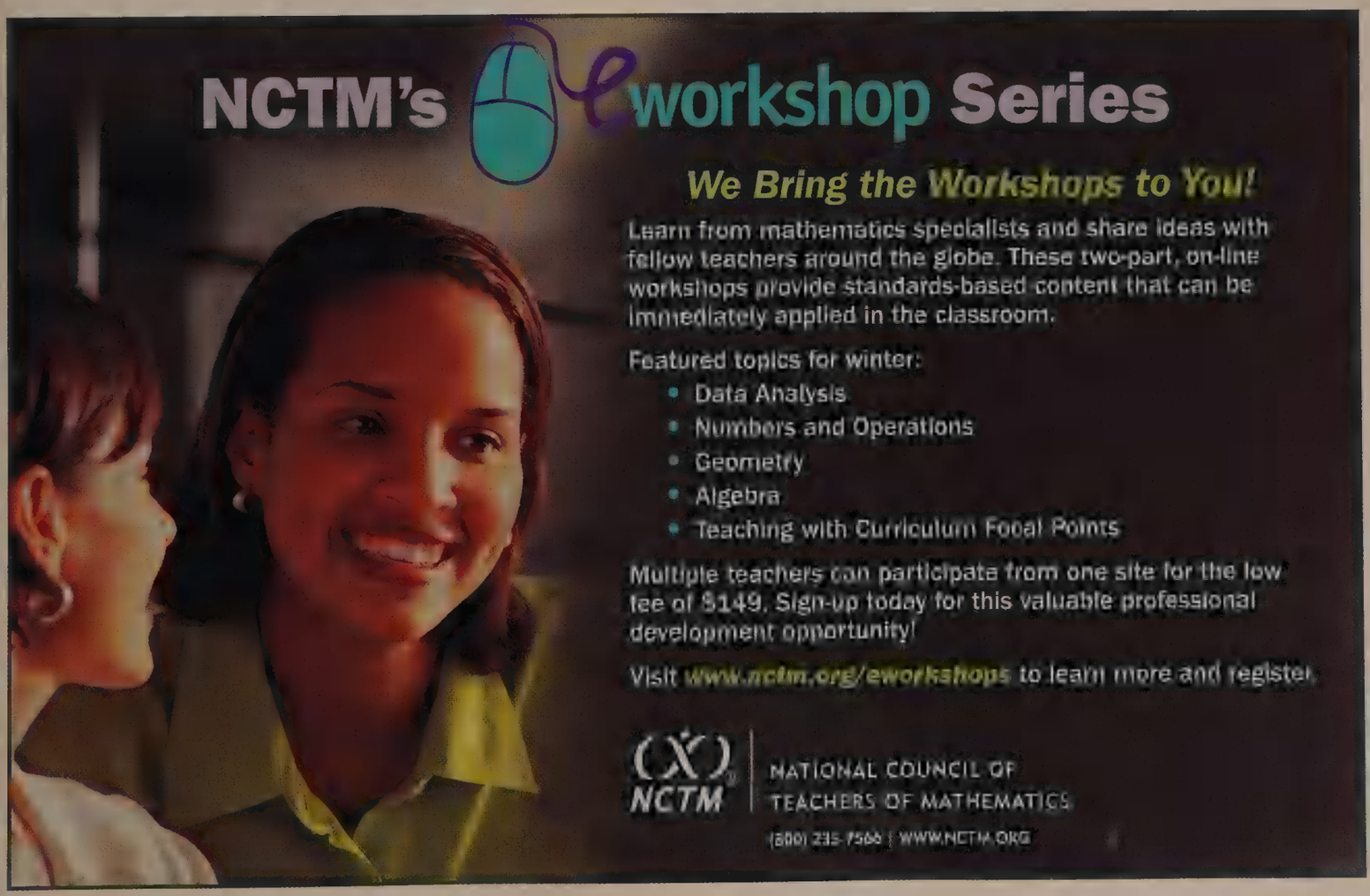
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Nesrin Cengiz and Theresa J. Grant

Children Generate Their Own Representations

"How many teeth have *you* lost?"

Imagine second graders collecting data from their peers about how many teeth each child is missing and then creating their own data representations that mean something to them. Instead of showing her students how to create a bar graph of the data, this teacher asks them to display the information in some way that helps them make sense of their data. Students work in small groups, discuss their ideas with peers, and create their own data representations. Then they share their data with the entire class, discussing differences in their representations and interpretations.

This classroom scene stands in stark contrast to conventional instruction on data analysis, which focuses on teaching students how to create and read particular representations (e.g., pictographs, bar graphs) and on determining the data's mean, median, or mode—but not necessarily contextually connecting these measures (Russell 2006). Conventional instructional goals may encourage students to think about representation features, such as title, axis, and key, and to use algorithms to compute the measures of center; however, little focus may be brought to bear on the data themselves.

Why might children benefit from a different approach to data analysis? Having students generate their own data representations and interpretations better prepares them to make sense of an increasingly data-driven society and thus become "informed citizens and intelligent consumers" (NCTM 2000, p. 48). Furthermore, research suggests that the more active a role that students have in constructing their own knowledge, the more useful that knowledge is. For example, when students are simply *shown* steps for drawing representations, rather than creating their own, they often misinterpret the information in those representations (Economopoulos and Wright 1998; Konold and Higgins 2003; Russell 2006).

We illustrate this perspective by sharing our observations of elementary school students who learned to make sense of data via this new approach. This article describes ways that students represent data when they are encouraged to create their own displays and discuss the issues they grapple with during this process. Additionally, we share ideas on how teachers can use student-generated representations to extend student thinking about representing data in ways that illuminate the data's important features.

Student-Generated Representations

When youngsters are encouraged to create their own data representations, they use a variety of forms. In this section, we describe the most prevalent representations that we have observed in elementary classrooms, along with an indication of how the tasks influenced students. For the lower elementary grades that we observed, instruction was guided by the unit *How Many Pockets? How Many Teeth?* (Economopoulos and Wright 1998), which focuses on helping students learn to analyze numerical data, and more specifically on assisting students with finding ways to count and keep track of data, to create their own representations of data, and to use those representations as a way of communicating information. In the upper elementary grades, students had previous experiences with some aspects of data analysis, such as drawing bar graphs and pie graphs, and were engaged in *The Shape of the Data: Statistics* unit (Russell et al. 1998), which focuses on recording, analyzing, representing, and comparing data sets about real-life situations. We invite you to analyze some examples of students' representations in the paragraphs below.

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Cube towers

How Many Pockets? How Many Teeth? contains lessons in which students physically represent their own data point by creating a cube tower for a given situation, such as the number of teeth they have lost. Students are prompted to line up their towers on the edge of the board, allowing them to see the lowest and highest values and the number of instances of the same value. They are not shown how to represent the data on paper in any particular way but are encouraged to create representations that make sense to them. We observed a common approach: drawing the entire collection of cube towers. **Figures 1** and **2** show representations that are similar to listing data values. For example, the cube towers (see **fig. 1**) correspond with the following list: 5, 7, 5, 2, 5, 12, 4, 12, 13, 1, 7, 0, 14, 13, 0, 5, 7.

Figure 1

Todd's drawing of the entire collection of cube towers is a typical approach.

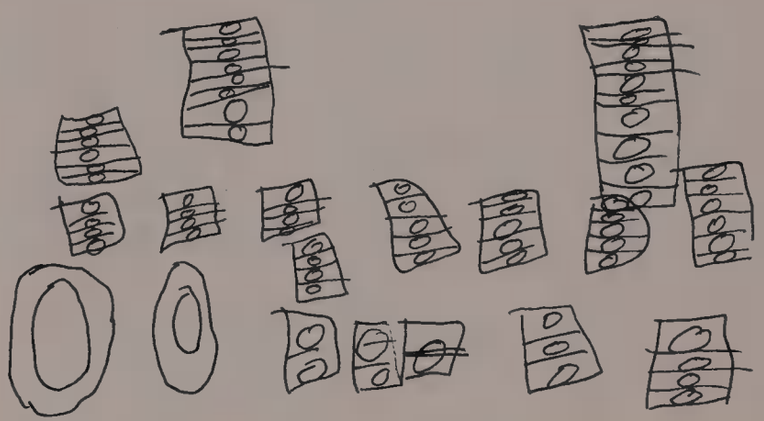
Line plots

A line plot is typically introduced as a quick way to sketch numerical data. In the classrooms that we observed, students were encouraged through whole-group discussions to create informal line plots from cube towers. Students first built cube towers to represent the number of teeth each had lost, wrote that number on a sticky note, and attached the note to the cube tower. For example, a class of seventeen students made seventeen cube towers. Seventeen corresponding sticky notes represented the lost-teeth data from all class members.

Students next arranged the sticky notes—rather than the cube towers. This method of organizing the data spawned displays such as Heidi's (see **fig. 3**), in which the stack of three squares in the center represents the fact that three students had all lost exactly five teeth.

Figure 2

Jane's illustration shows representations that are similar to listing data values.



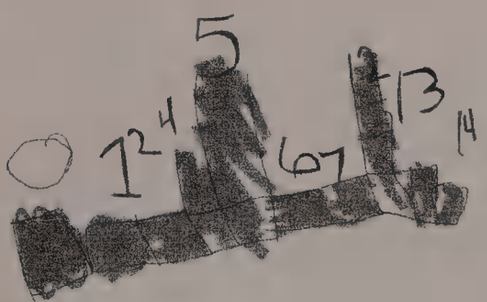
Bar graphs

As mentioned, many upper elementary students had been taught to create bar graphs earlier in the school year. It is therefore not surprising that some of the representations created by these students were variations of a traditional bar graph. **Figures 4a**, **b**, and **c** are representations created by three different groups as they sought to answer the question, "How much taller are sixth graders than second graders?" Some students drew two separate bar graphs, which they placed at opposite ends of their paper (see **fig. 4a**). Others combined the data sets in various ways (see **figs. 4b** and **c**).

Before you continue, reflect on these questions: Which representations might be more useful in making sense of data and answering questions

Figure 3

The stack of three squares in the center of Heidi's drawing represents the fact that three students all lost exactly five teeth.



about data? Which aspects of these representations would you want your students to focus on during class discussions? What kinds of issues might students face? What understanding do you want students to develop as they construct and share their own data representations?

Issues That Arise

When students are encouraged to generate their own data representations, they must make decisions that impact how their data story is told: how to present and organize data, keep track of it, and represent gaps. In the next section, we use student-generated data displays to discuss these closely related issues that we observed in classrooms.

Presenting and organizing data

Student-generated representations clearly look different from the standard pictographs and bar graphs—which we have become accustomed to—and may initially be somewhat disconcerting to teachers. For example, young children often present data in drawings with little regard for scale. Consider Todd's representation of cube towers (see **fig. 1**) depicting the number of missing teeth. Although the heights of Todd's cubes are fairly consistent, particularly for a young child, the resulting display provides false information. For example, what appears to be the largest tower (in the center of the display) is partitioned to show thirteen cubes, whereas the greatest number of lost teeth is actually fourteen (represented by the next tallest stack to the right). Similarly, the last tower in the display represents seven lost teeth, but this tower is considerably smaller than the second tower in the display, which also represents seven teeth. Interestingly, issues related to scale have been shown to persist into the upper elementary and middle school grades as students engage in coordinate graphing involving the scaling of axes (Watson 2006).

At first glance, Jane's cube tower representation (see **fig. 2**) seems no different from Todd's in that they both have issues with scale and both appear to be disorganized. However, closer inspection reveals that Jane has organized her data: The bottom left corner shows the two students who have no missing teeth, followed by three students who have lost two teeth, and so on, moving from left to right, row by row, from the bottom of the page to the top. Jane not only put her data points in order but also accounted for the two students who have not lost any teeth. Each cube in a tower represents a lost tooth, so the physical representation is restrictive in the sense that it does not provide a means for showing "zero teeth." Notice how Jane chose a different shape—circular rather than rectangular—for the zero values. This points to another issue that we observed: keeping track of all the data values.

Keeping track of data

Lower elementary students commonly forget either to represent all data values in their diagrams or to insert extra data values that do not belong in the original data set. Keeping track of the data was an issue for upper elementary students as well. They usually represented each data value in their diagrams but not necessarily with the correct frequency. **Figure 4a**, for example, shows two second graders with a height of fifty-six inches; in fact, only one such student existed. **Figure 4c**, on the other hand, indicates one second grader who was forty-seven inches tall; there were actually two.

Although some of these inaccuracies might be viewed as careless errors, the case of representing values of zero poses a particular challenge for young students, as mentioned above. In **figure 2**, Jane literally represented the number of missing teeth by drawing each cube tower; that is, each rectangle (cube) represents a tooth. Thus, she needed to create a "no tooth" icon to be able to indicate that two children had not lost any teeth. In contrast, Heidi represented the *frequency* of lost teeth (see **fig. 3**), where each rectangle stands for a child, and the number of lost teeth is recorded above each stack. Heidi's informal line plot allowed her to represent any number of lost teeth, even zero. Furthermore, some students were challenged to distinguish the case of five students losing zero teeth from that of nobody losing five teeth—an issue that has been shown to persist in upper grades (Konold and Higgins 2003). This brings us to a third related issue.

Representing data gaps

Students must understand when and why it is important to represent gaps in data; this may not be as trivial an issue as one might think. Consider that physical models, such as cube towers, focus on data that *exist*, rather than data values that do not occur. Thus, when students in the classrooms that we observed displayed their cube towers, they simply lined them up, one next to another. When the towers were organized from least to greatest (see **fig. 5**), gaps in the data were discernable. In this display, the large change in the heights of the towers identify gaps; for example, the jump from

the seven-tower to the twelve-tower indicates that no students lost eight, nine, ten, or eleven teeth. In contrast, the one-cube difference between consecutive data values—such as from four to seven, where the cube towers look like a staircase—denotes no gap between these data points. That is, students lost four, five, six, and seven teeth. However, students might not easily engage in this kind of analysis of the display and thus would not readily recognize gaps in this type of representation.

The conventional line plot, on the other hand, has a location for every data value between the minimum and the maximum data points, regardless of whether or not these data values are observed. So, gaps in the data are obvious at a glance. When students first encounter the concept of a line plot, however, they commonly focus only on *observed* data and thus create representations like Heidi's (see **fig. 3**). As students have more experiences with this kind of representation, they begin to create informal line plots with gaps (see **fig. 6**) and, eventually, more conventional line plots (such as **fig. 7**) with a scaled axis. Unlike **figure 3**, the latter two figures reveal the data's clusters, peaks, and valleys. Such features that define the shape of the data are important to the story portrayed in a data display.

Extending Students' Thinking

The two main tasks discussed thus far require students to specifically examine different ways of representing the same data set and to find meaningful ways to represent two sets of data on one graph. These tasks not only elicit the issues we have discussed but also provide the means for helping students confront those issues. Encouraging students to compare and contrast one another's drawings and to base data descriptions and interpretations on these drawings were effective instructional strategies in the classrooms that we observed.

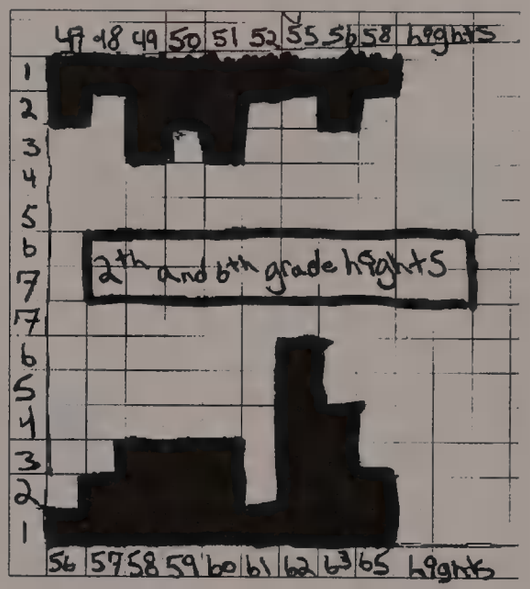
Connections between representations

The young elementary students we observed were exposed to both cube towers and line plots as they learned to collect and analyze data. The shift from cube tower representations to line plots was not trivial for these students, and thus required great attention from teachers (Russell 2006). An effective strategy for facilitating this shift has students verify the displayed information by matching data points in different representations of the same data set and discussing the similarities and differences

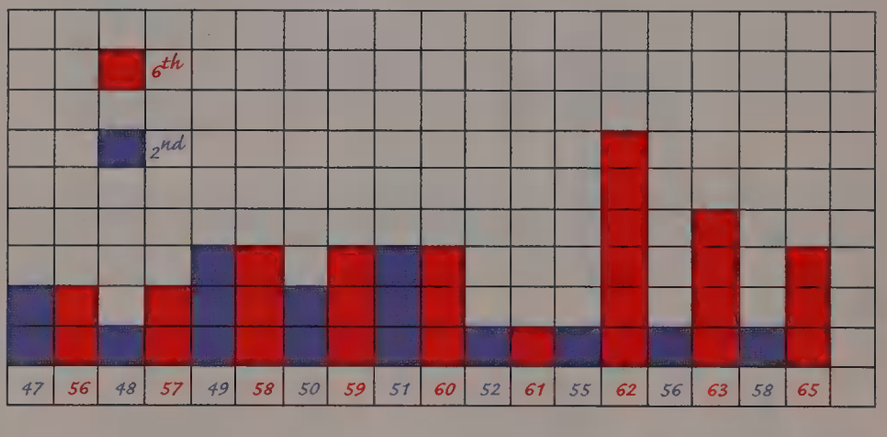
Figure 4

Three different groups sought to answer the question, "How much taller are sixth graders than second graders?"

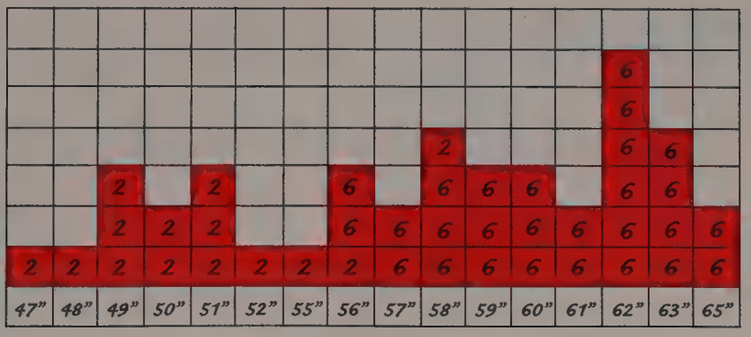
(a) Joe and Hunter placed separate bar graphs at opposite ends of their paper.



(b) Other students combined the data sets. Sally and Jessica alternated data values from two different grade levels.



(c) Melissa and Nick created one bar graph with the data sets combined, labeling each data value to indicate whether it came from a second or sixth grader.



in the various representations (Economopoulos and Wright 1998). For example, in whole-class discussions, teachers prompted students to consider how a tower of seven cubes constructed by a student who lost seven teeth was represented on the line plot by a single sticky note labeled with the number 7.

To make sense of the overall shape of the data, students must recognize the existence of gaps in the data and understand the reasons for representing these gaps in their data displays. Several strategies can help students confront this issue. Research suggests that one source of difficulty with data gaps may stem from the tendency to represent data as they are gathered, rather than organizing and reflecting on the data set before making a plan for creating a representation (Konold and Higgins 2003). Thus, it might be beneficial to explicitly suggest that students take time to look at their data set and make a plan for their representation before starting to draw.

Another effective strategy is to have students discuss the similarities and differences in various representations, particularly when some students highlight gaps in the data and others do not. Finally, encourage students to consider the purpose for representing the data (i.e., what we hope to learn from the display) because different representations can reveal different aspects of data and might be more or less useful in answering the questions at hand.

Figure 5

Ordered cube towers represent the number of lost teeth.

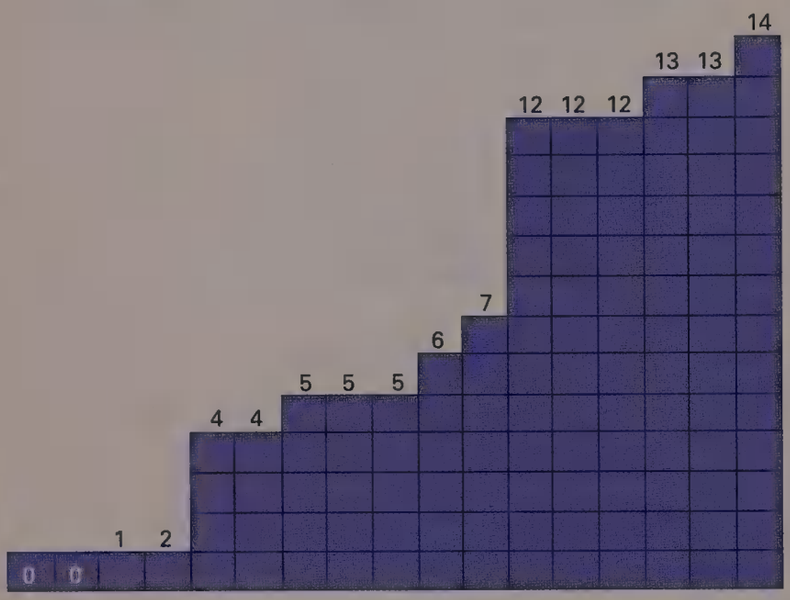
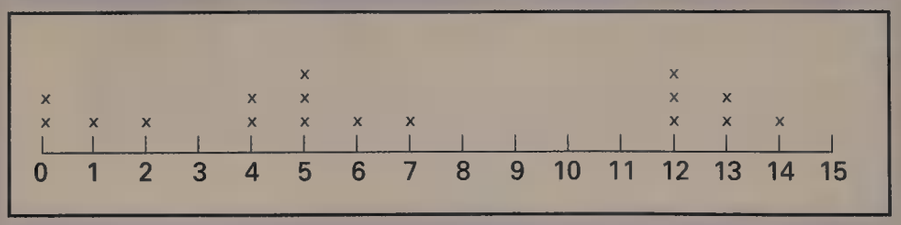


Figure 6

A line plot representation is typically a quick, informal way to sketch numerical data.

Figure 7

A scaled line plot represents the number of missing teeth.



Two data sets, one representation

In addition to comparing and contrasting different representations of a single data set, upper elementary students created representations in order to compare multiple data sets. In one task, they compared a bar graph of their heights with two bar graphs depicting the heights of students in two other classes. In a subsequent task, students created a single representation of two data sets. These tasks gave students an opportunity to extend their understanding of the issues by creating representations in the context of a new challenge: dealing with two data sets at one time. Once again, whole-class settings were critical for providing a forum to discuss these issues. Consider the following whole-group discussion between a teacher (T) and her students (A–J) using student work from the classrooms that we observed.

T: [On the basis of your representations] what can we say about the heights of second graders and your [heights]? [*See* ***figs. 4a****,* ***b****, and* ***c****.*]
A: We are much taller than the second graders.
T: What makes you say that?
A: The shortest person in our class is almost as high as their tallest. Look, the highest second grader is fifty-eight inches, and the shortest sixth grader is fifty-six inches.
T: What do the rest of you think about A's claim?
B: I agree with her. Most of the second graders are shorter than fifty-six inches, and we are as tall as, or taller than, fifty-six inches. This makes us taller.

T: What are other things that you notice about your heights and second graders' heights by looking at these three representations?
C: I see that I am the shortest in our class. I am fifty-six inches tall, and there is a second grader who is taller than I am!
D: Looking at Joe and Hunter's graph [*see* ***fig. 4a***], I can see clumps where most heights are.
T: How about the other two? Can you see where the clumps are in them also?
E: It is harder to see the clumps in Sally and Jessica's [*see* ***fig. 4b***].
T: What makes it harder?
F: Well, they started with the shortest second grader and the shortest sixth grader, and then they kept alternating until they reached the tallest. This makes it harder to see the clumps.
T: How about Melissa and Nick's representation? [*See* ***fig. 4c***.]
G: They labeled the heights according to grade levels, and I can easily see the heights that are in common—like there are students from both classes that are fifty-eight inches tall.
H: It's not as easy to see the common heights in Sally and Jessica's or in Joe and Hunter's.
I: I think Joe and Hunter's feels like two separate graphs. They're not connected, so I don't see how it is different from putting those two graphs on different papers.
T: Are there other things that you notice about the data?
J: There are no people that are fifty-three or fifty-four inches tall, but it's kind of hard to see this.
B: You could easily fix this on Melissa and Nick's graph [*see* ***fig. 4c***] by putting some space between 52 and 55….

Several purposes overlap in this kind of discussion. First, it is critical to direct students to focus on what we can learn from the data by asking questions such as, "What kinds of things do you notice about the height of the second graders and the sixth graders?" After all, the purpose of collecting, representing, and analyzing data is to learn something about the world around us.

Second, by comparing and contrasting student-generated representations, youngsters can focus on how particular features of different representations illuminate or obscure certain aspects of the data. This, then, provides students with a rationale, for example, for showing gaps in the data; that is, comparing representations helps illuminate interesting data features.

What Teachers Can Do

Analyze the activities you currently use for data analysis:

- Do students have opportunities to create their own representations and to discuss the features of those representations? (If not, try some of the tasks we describe. Help students identify an interesting question or topic to explore.)
- Do students always have a purpose for analyzing data? For instance, the data answer a question or tell us more about something that interests us.

With younger students, focus on a question involving a single data set:

- How many teeth have all the second graders lost?
- How many pockets are in our classroom?

With older students, consider a question requiring a comparison:

- Which fourth-grade class is taller?
- Can fifth graders jump farther than sixth graders?

At all levels, allow students to create and use their own representations to describe and analyze data. Consider the issues raised in this article as you facilitate whole-group discussions about students' representations. Add to this list as you observe your own students' efforts at constructing data displays.

Conclusions and Suggestions

Teaching data analysis by encouraging students to generate their own data representations is a challenging undertaking that requires teachers to be knowledgeable about the types of representations that students create on their own, the issues that students face during the process of creating those representations, and how to facilitate discussions to help students make sense of their representations. Because many teachers have learned to represent data by following step-by-step procedures, they may have difficulty considering a different way of teaching this topic. What, then, makes it worthwhile to have students generate their own data representations?

When we show students how to generate a particular type of representation by providing them with a step-by-step procedure for creating it, we essentially deprive them of opportunities for learning *why* certain graphing conventions are followed. For example, when teachers demonstrate the method to create a bar graph, we make assumptions about how to order and group the data. In our experience, many elementary students do not initially recognize the

need to organize or group data. Therefore, allowing them to generate and analyze their own representations provides the opportunity to arrive at these assumptions in their own time and in a meaningful way and thus develop a solid understanding of the role of representations in analyzing and interpreting data. Furthermore, allowing students to generate their own representations helps them recognize that more than one way exists to solve problems and represent situations. Students realize that their own thinking is important and valued. In short, the classroom environment supports teaching that is built on student thinking, where students learn data analysis with understanding.

By allowing students to generate their own data representations, you will learn more about your students' thinking and be better equipped to capitalize on that thinking to further their understanding and use of data.

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The *TCM* Editorial Panel uses reader feedback to improve the overall quality of this journal. We want *Teaching Children Mathematics* to be your most useful teaching resource. That goal cannot be realized unless we hear from you.

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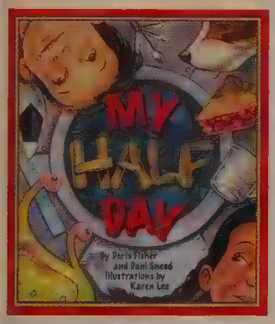
New Resources Are Blooming

Books

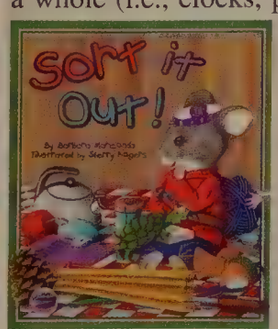
For Students

My Half Day, *Doris Fisher and Dani Sneed, 2008. ISBN 978-1-934359-14-3.* **Sort It Out!** *Barbara Mariconda, 2008. ISBN 978-1-934359-11-2. 32 pp., $16.95 cloth ea. Ages 4–8. Sylvan Dell Publishing; (877) 958-2600; www.SylvanDellPublishing.com.*



My Half Day revolves around fractions of all types. A young boy has an exciting, oddly eventful day at Fraction Camp, where all activities relate to parts of a whole. The day culminates in an out-of-this-world soccer game with fractional scoring.

The book's theme is confusing, especially for primary-aged children, for whom the authors recommend the book. The title is misleading; the book is actually about encounters with a variety of fractions during an entire day. Although the author points out many ways that we can show parts of a whole, the larger number examples are too difficult for young children. The book is better suited for students in grades 4 and up, who study a wider variety of fractions and decimals. Many of the fractions are present merely for decoration and do not add to the story's theme. However, the illustrations are engaging—showing examples of parts of a whole (i.e., clocks, phases of the moon, musical notes, money, etc.)—and the story is clever.



Sort It Out! is an attribute story told in rhyme and geared to students in grades K–3. A packrat sorts his collection of treasures and then notices that the quantities are shrinking (thanks to his sister). Children who are studying patterns and sorting will find this book interesting, especially because the packrat has so many different ways to sort objects in addition to using color, size, and shape. Some ways are not necessarily obvious. Engaging illustrations, especially of everyday items from a "rat's-eye" perspective, draw children into the story. The author also cites resources that complement the book and includes several pages of ideas to further develop concepts.

Youngsters will need a strong foundation in basic sorting techniques before working with this book because the categories are not necessarily related to one another; some are determined through higher-level thinking strategies. I did not use the book with kindergarteners. First and second graders would find the ideas challenging, but older students would enjoy all the sorting options.—*Patty Corwin, math coach K–5, Forest Glen Elementary School/District 41, Glen Ellyn, IL 60137.*

For Teachers

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JRME Monograph Number 14, Journal for Research in Mathematics Education: A Study of Teaching Multiple Lenses, Multiple Views, *Alan H. Schoenfeld, volume editor, and Neil Pateman, series editor, 2008. vi + 200 pp., $29.95. ISBN 978-0-87353-603-5. Stock no. 13245. National Council of Teachers of Mathematics; (800) 235-7566; www.nctm.org.*

Prices on software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

This recent research monograph provides teachers, researchers, and other mathematics education leaders a collection of articles about the complexity of Standards-based mathematics teaching. The monograph illustrates instructional practices that are highlighted in *Principles and Standards* (NCTM 2000) and other recommendations about how to improve mathematics teaching and learning (RAND 2003; U.S. Department of Education 2008). The lessons and associated commentaries are based on the belief that students learn most effectively when they engage in worthwhile tasks and are allowed to explore, discuss, and unpack the mathematics in each task with support from their teacher. The entire monograph focuses on a six-minute video excerpt from a third-grade classroom. In the excerpt, students revisit their personal definitions of odd and even numbers.

This monograph will benefit various populations: Classroom teachers will benefit from the example of Standards-based teaching. Specifically, teachers will note Ball's facilitation of the class discussion, in which she creates and maintains an atmosphere that is comfortable enough for students to voice their opinions and disagree with their classmates' comments. Teacher educators will benefit from the characteristics of Standards-based mathematics instruction that they can model and integrate into teacher education programs. Last, researchers will benefit from the various ways to analyze and study this short excerpt of teaching.—*Drew Polly, assistant professor, Mathematics Education, Instructional Technology, Deptartment of Reading and Elementary Education, University of North Carolina at Charlotte, NC 28223.*

Mathematics Education in the United States 2008, *John Dossey, Katherine Halvorsen, and Sharon McCrone, 2008. v + 73 pp., $19.95 paper. ISBN 978-0-87353-610-3. Stock no. 13415. National Council of Teachers of Mathematics; (800) 235-7566; www.nctm.org.*

Written for the Eleventh International Congress on Mathematical Education to bring together available data about mathematics education in the United States, this document begins with general information about U.S. education; includes a special focus on developments in statistical education; and deals with programs for high-achieving students, teacher education, and higher-mathematics education.

Researchers, mathematics department heads, and decision makers at the state and national levels will find valuable information in this document. The document could be used when decisions are being made about current requirements for mathematics education or to compare one teacher education program with others across the country.

In their preface, the authors acknowledge that so many programs are available that they were unable to characterize them adequately in such a brief document. The authors also acknowledge that the information they present will quickly become dated. However, this document is a starting point for anyone desiring more data about mathematics today or assessing or revising current programs.—*Emily Hendricks, Lafayette School Corporation, Lafayette, IN 47904.*

From Other Publishers

Algebra in the Early Grades, *James J. Kaput, David W. Carraher, and Maria L. Blanton, editors; 2008. 522 pp., $55.00 paper. ISBN 0-8058-5473-8. Taylor & Francis Group; (800) 634-7064; www.taylorandfrancis.com.*

This book sequentially and thoroughly covers the research and foundations of early algebra and gives reasons for exploring algebra's complex concepts in the early grades. Thinking of growth as a factor is wonderful for educators to embrace and promote in their planning. The chapter titled "Children's Reasoning about Change over Time" discusses common mathematical themes that provide a familiar foundation for young children. The notion of varying events over time underlies stories that children tell from an early age. This relates to one variable being represented in a table, graph, and groups of additive changes. The authors illustrate multiple classroom episodes as well as cite several resources. Their abundant evidence attests to their dedicated research. With over 500 pages, the book leaves very little out. This resource is packed with multiple levels of information for an audience that is interested in research or thesis development.—*Monica Cavender, Quinnipiac University, Hamden, CT 06460.*

Culturally Specific Pedagogy in the Mathematics Classroom: Strategies for Teachers and Students, *Jacqueline Leonard, 2008. 207 pp.,*

$39.95 paper. ISBN 0-8058-6105-X. Routledge Taylor & Francis Group; (800) 634-7064; www.routledge.com.

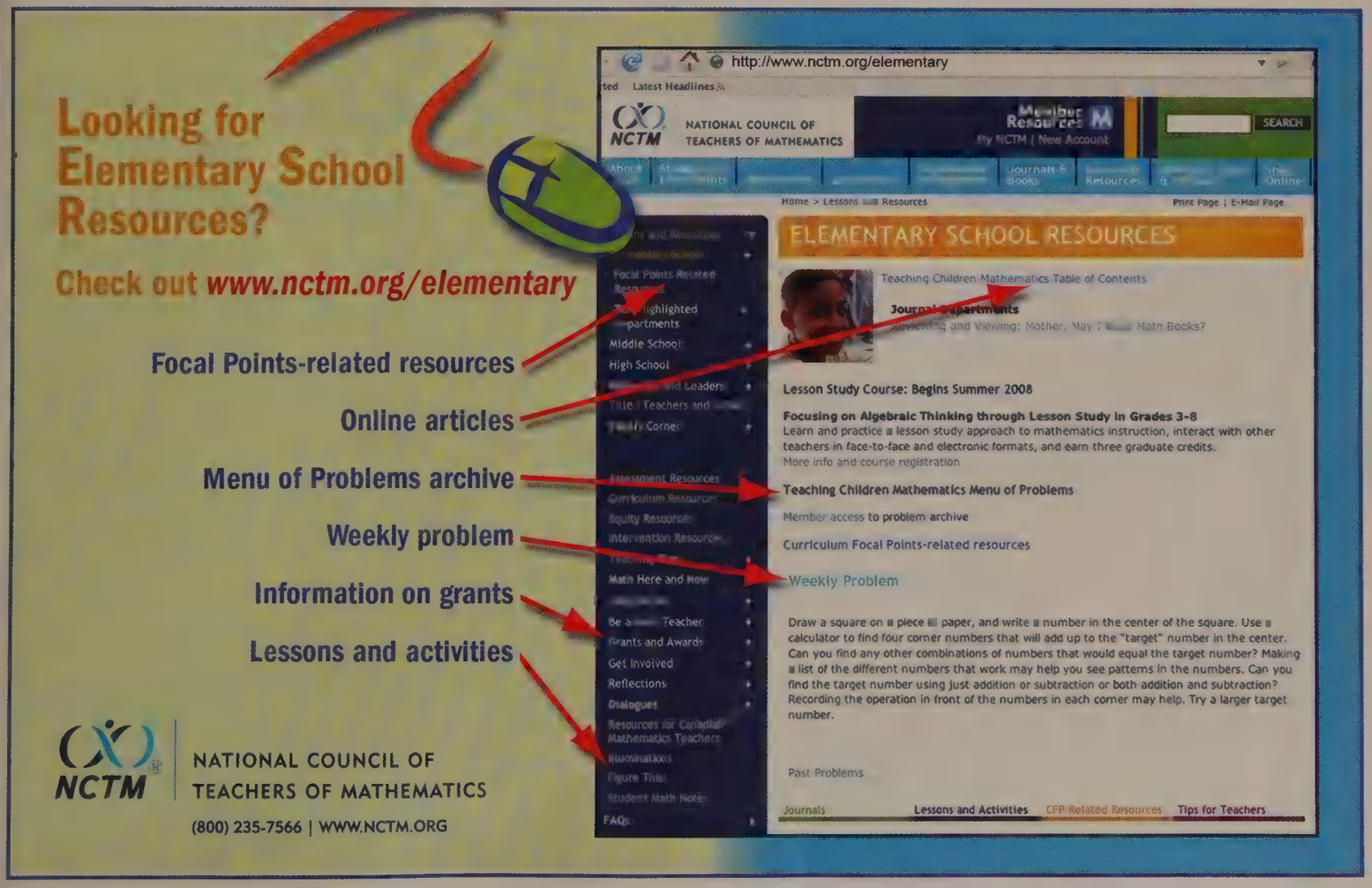
Today's classrooms include heterogeneously grouped students with various ethnic backgrounds. Teaching to one type of learner is not an option. Quoting the famous educational researcher Rousseau, Leonard states that "one size does not fit all." Instead teachers must teach toward integrated learning styles and intelligences to give all learners equal opportunity to succeed. Leonard discusses the importance of understanding that our students need culturally specific pedagogy in order to meet their needs. She continues, "While there are common characteristics and traits among people who share the same race or ethnicity, no culture is monolithic" (p. 9). Linking mathematics for these diverse learners is important.

Gender bias has lessened at the elementary and secondary levels; however, Leonard cites statistics to support the notion that female enrollments in mathematics and science-related fields are still underrepresented and that females are still not taking active roles in the math and science world.

This is an informative book on cultural and gender bias perspectives in mathematics. I have recommended the book to my colleagues in a postmasters supervision certification program because of its practical applications for making connections. The book's only weak point is that the title refers to culturally specific pedagogy, but a majority of the references are to African American students. I would have liked to have read more information on other cultures as well.—*Erin McGeough, Mathematics Specialist, Pt. Pleasant, NJ 08742.*

Dice Activities for Math, Grades K–3, *Mary Saltus, Karen Moore, Diane Neison, Marcia Fitzgerald and Chet Delani, 2008. viii + 94 pp., $11.95 paper. Didax; (800) 458-0024; www.didax.com.*

This book covers a wide range of skills and activities from using one die to using a handful of dice, and from matching the pattern on the die to finding half the sum of three dice. Activities include adding or subtracting 1, 2, or 10 to the total on the dice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and writing equations to represent the mathematics. I especially like the activities involving doubling, finding half, and practicing with even and odd numbers. Place-value practice requires having two colors of dice, one representing ones and the other, tens. One activity is to find groups of dice to make 10 from a handful of thrown dice.

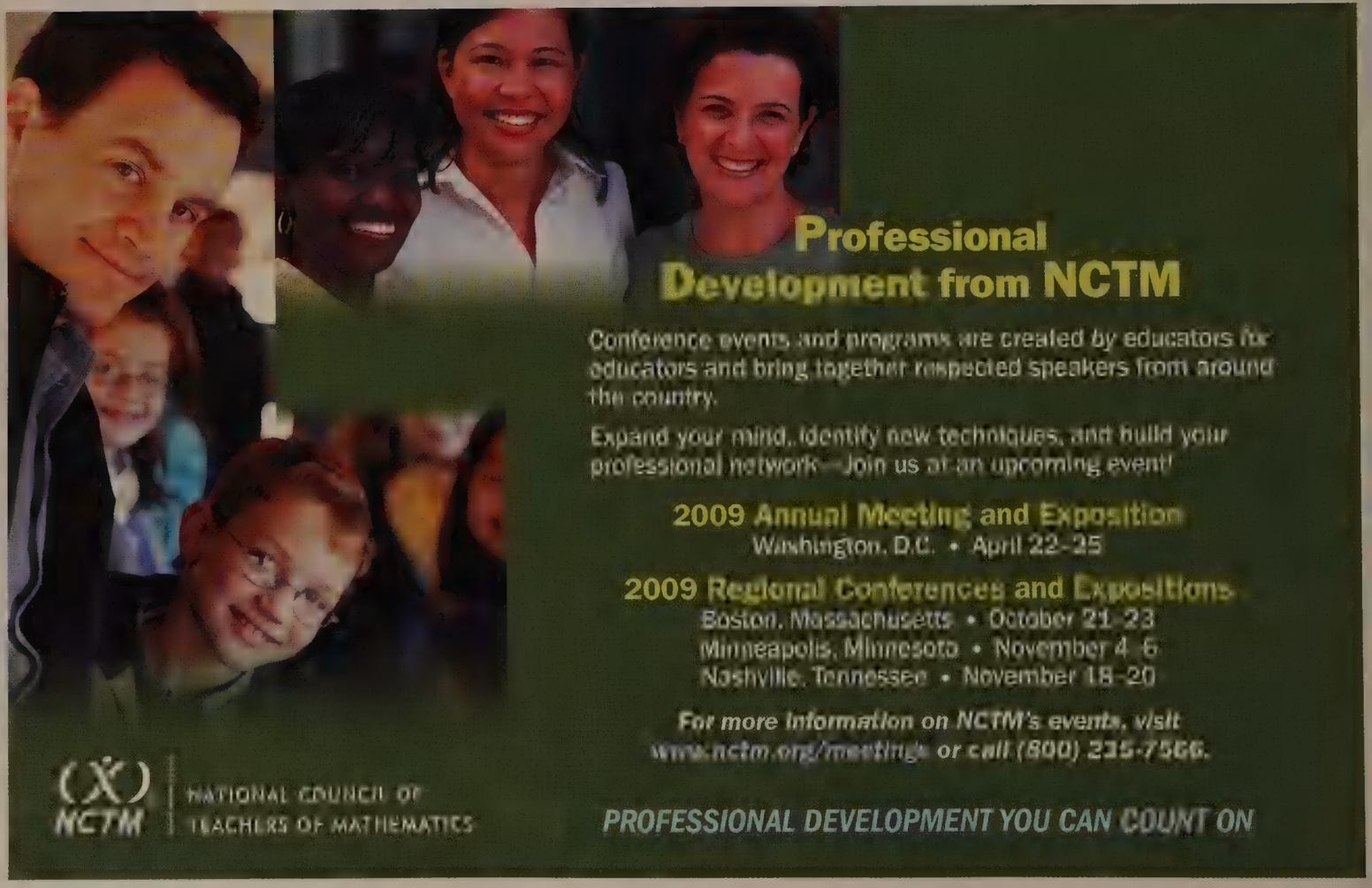
Simple directions allow the activities to be used with a whole class, a small group, or as take-home practice to share with the family. Their game-like nature permits students to have fun while learning valuable skills. The target audience is students in grades K–3, but I would use the activities with older students who still need to master such skills.

As a former Title 1 mathematics teacher for grades 2–6, I would have definitely found this book useful. The simple, fun activities on basic skills and concepts that my students so frequently lacked would make it a valuable resource.—*Sharon Summar, retired elementary teacher, Pekin, IL 61554.* ▲

THE SQUARE ROOT OF PI

This year's observance of Pi Day and Albert Einstein's birthday on March 14 (3.14) ought to include the recognition of Square Root Day, which is celebrated on dates where both the day and the month are the square root of the last two digits of the current year, such as 3.3.09. The last Square Root Day was February 2, 2004.

Although fruit pies and pizza pies are the traditional foods of choice for Pi Day, the menu this year might include root vegetables—cut into squares, of course.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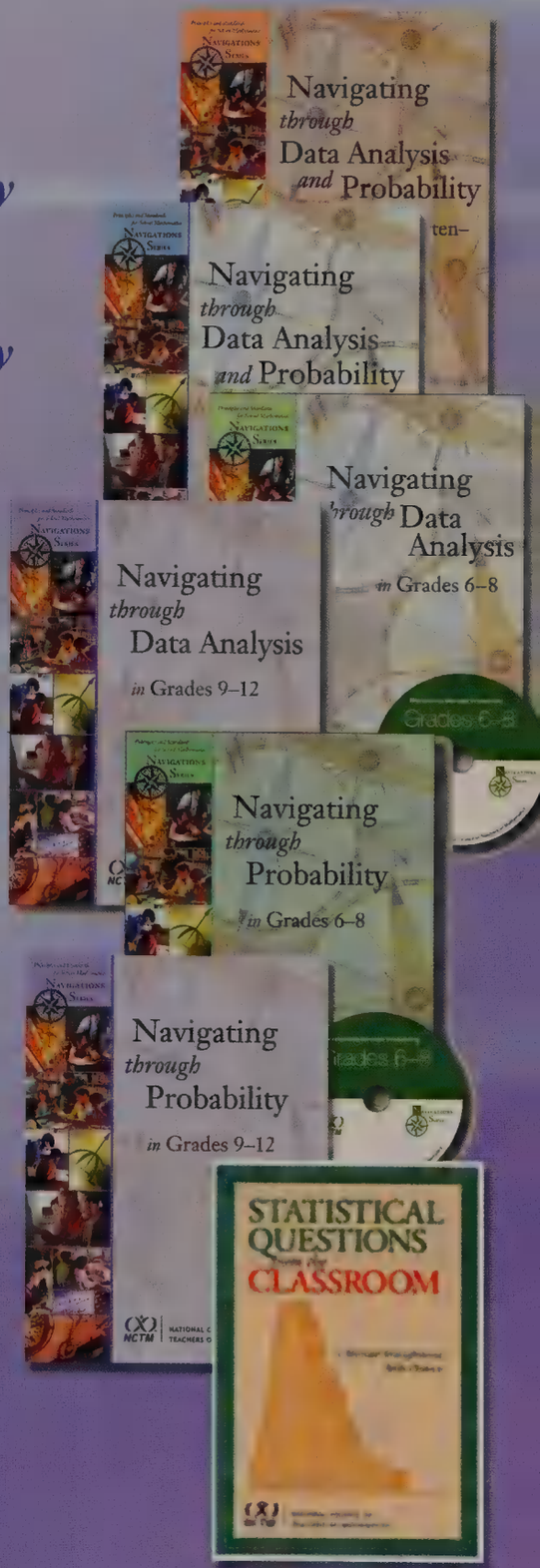
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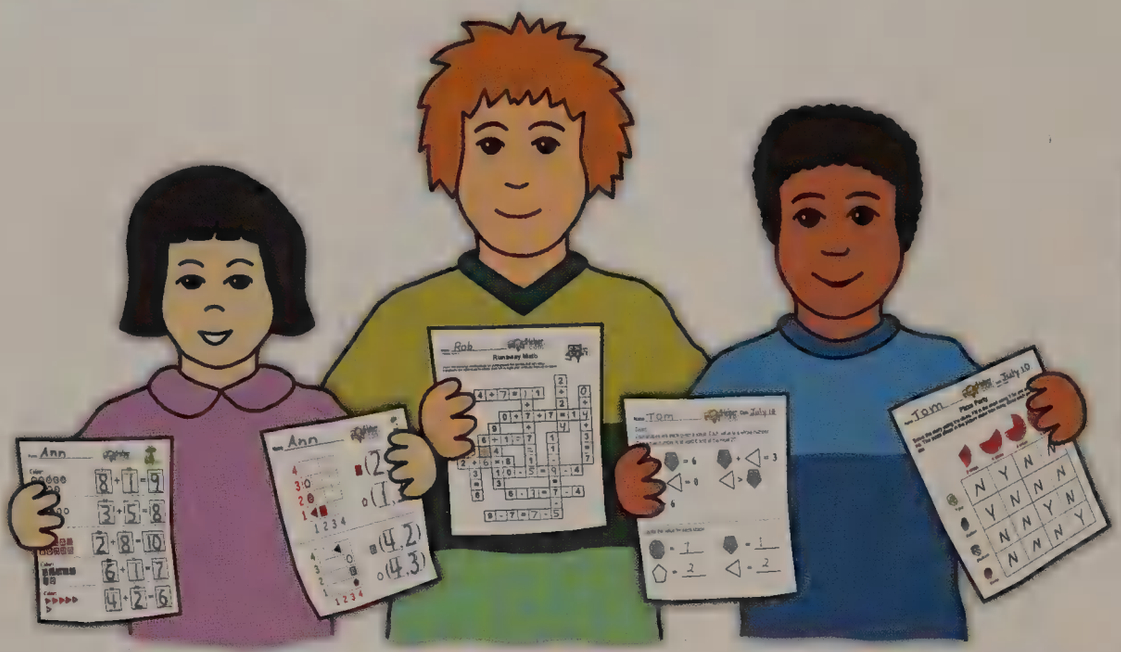

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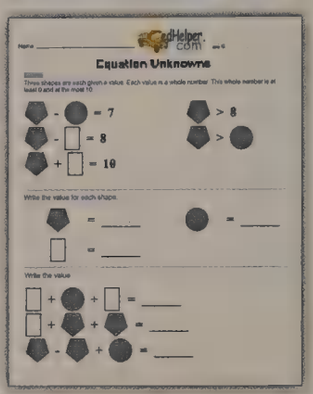

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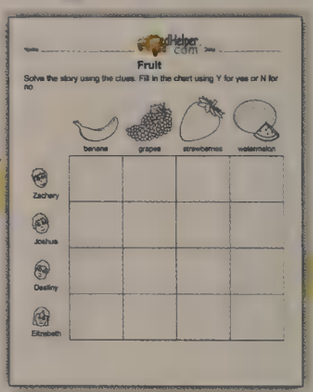

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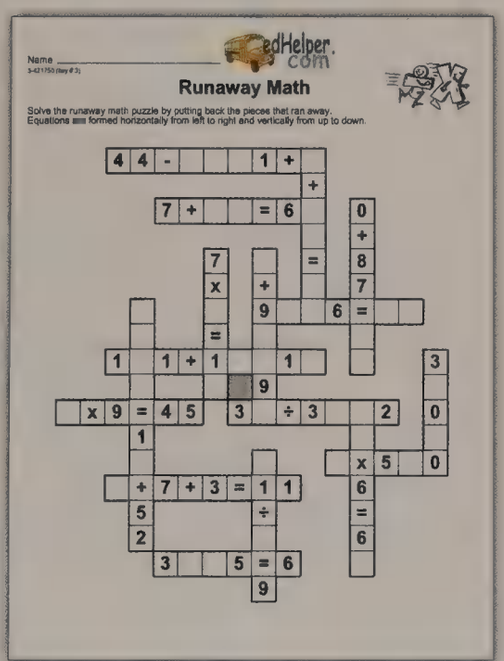
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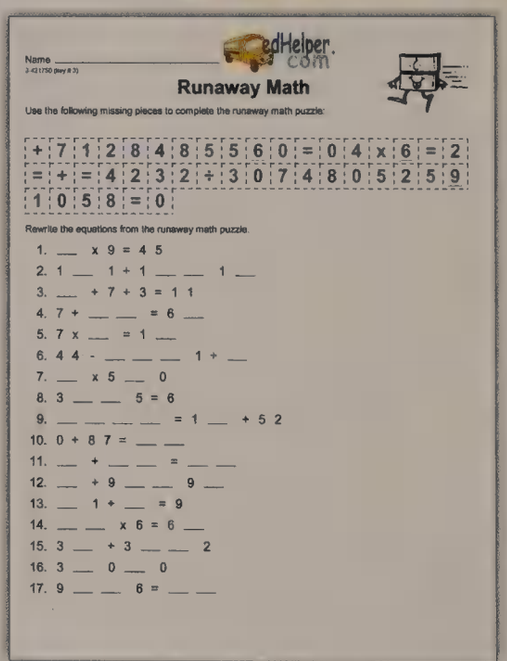
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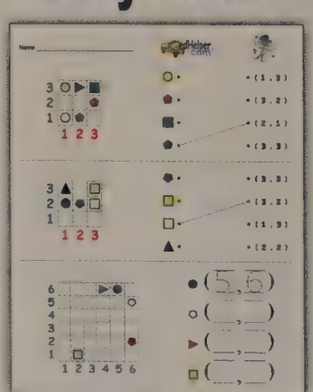

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Exploring Geometric Shapes

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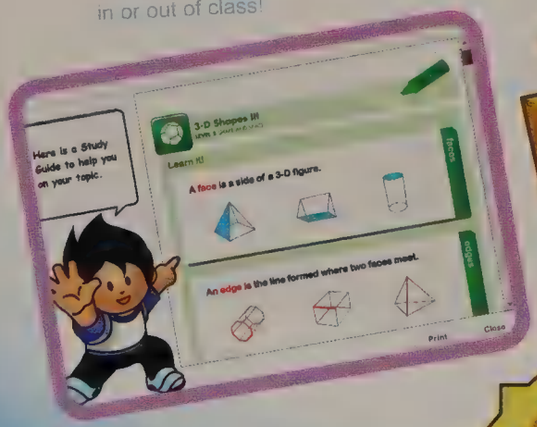
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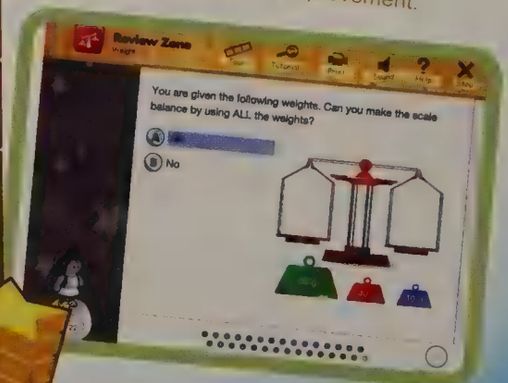

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TEACHING CHILDREN
Mathematics

Volume 15, Number 8,
April 2009

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ON THE COVER

David J. Whitin and Phyllis Whitin examine how fourth graders explore important geometric concepts embedded in the shape and design of natural and manufactured objects. Their article "Why Are Things Shaped the Way They Are?" offers a range of activities to demonstrate how inquiring about shape can effectively integrate science and mathematics.

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TEACHING CHILDREN MATHEMATICS

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Information is available from the Headquarters Office regarding the three **other official journals**, the *Mathematics Teacher*, *Mathematics Teaching in the Middle School*, and the *Journal for Research in Mathematics Education*. Dues support the development, coordination, and delivery of the Council's services. Dues for individual membership are $78 (U.S.), which includes $33 for each *Teaching Children Mathematics* subscription, $16 for a nine-issue *NCTM News Bulletin* subscription, and *ON-Math*. Each additional school journal (*Mathematics Teacher* and *Mathematics Teaching in the Middle School*) subscription is $33. Each additional subscription to the *Journal for Research in Mathematics Education* is $60. **Foreign subscribers** add $18 (U.S.) postage for the first journal and $4 (U.S.) postage for each additional journal. Special rates for students, institutions, bulk subscribers, and emeritus members are available from the Headquarters Office.

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More Ideas to Support ELLs

Lynda Weist's article, "Problem-Solving Support for English Language Learners," in the April 2008 issue of *TCM* is full of practical ideas on how to work with a fourth-grade class that contains many ELL students.

The author clearly understands that guess and check is a good way to approach the [Pigs and Chicken] problem, and this is mentioned in the article. However, the article gives no indication that there was any instruction (1) on how to decide when guess and check might be a useful strategy in solving a mathematics problem, or (2) on high-road transfer of learning of the guess-and-check strategy to problem solving in other disciplines.

Thus, two very important ideas that could have been included in the lesson are missing.

Dave Moursund
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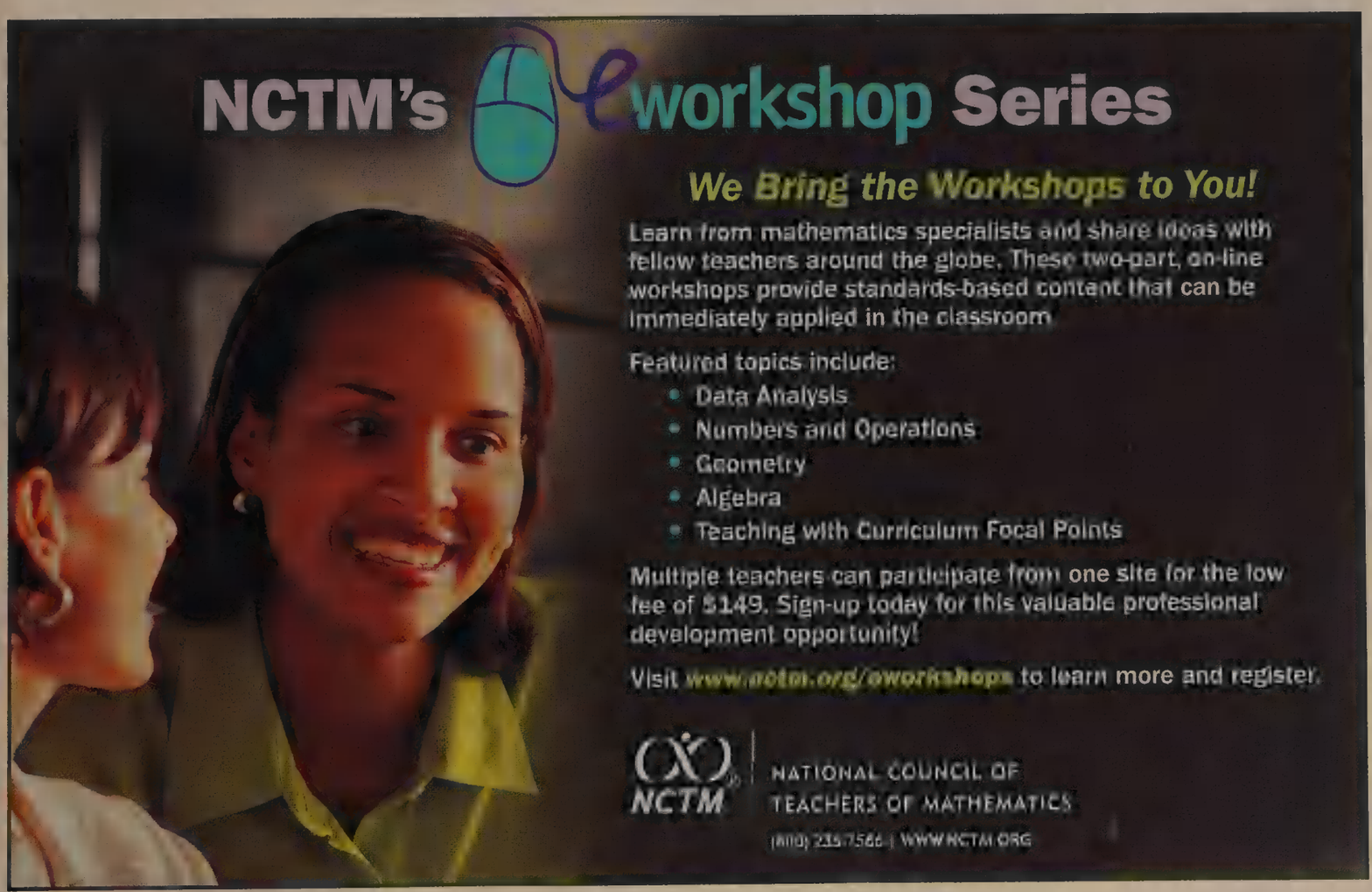
The Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Letters may be submitted to Teaching Children Mathematics *at* **tcm@nctm.org**. *Please include "Readers Exchange" in the subject line. Letters and rejoinders from authors beyond the 250-word limit are subject to abridgment and also are edited for style and content.*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Where Is the Center?

Problem

This month our problem-solving quest takes us to the small community of Lebanon, Kansas. Just north of town, a monument marks the spot that the Coast and Geodetic Survey determined to be the geographic center of the continental (i.e., excluding Alaska and Hawaii) United States (USGS 2001). Whether this is the *exact* geographic center is up for debate.

Studying the monument prompts us to wonder what the term *center* means and how this particular location was determined to be the exact center of the contiguous states. How could one determine the center of such irregular shapes as countries and states?

Using the map, mark the location that you believe is the geographic center of the United States. Do you agree that the center is indeed in Kansas? Show your work on the map and use a separate sheet of paper to describe how you completed the problem. Explain why the point you marked is the center and why you made the measurements you did, or describe how you manipulated the map to determine its center.

Extensions

1. Locate the center of the state in which you live. Which city in your state is the closest to the center? Did you solve this problem the same way that you solved the continental United States problem?
2. How might you find the city that is closest to the center of an unusually shaped state like Michigan, Alaska, Florida, or Oklahoma?
3. Would you use a method similar to what you used for the maps to find the center of your classroom? Try to use at least two different methods to locate the center of the classroom.

This problem presents a conundrum that students enjoy exploring because it can be argued that determining the *exact* geographic center of the continental United States is impossible, and students relish a challenge. According to Wikipedia, in 1918 the Coast and Geodetic Survey "found this 'center' by balancing on a point a cardboard cutout shaped like the U.S. Thus, measured in this manner, the actual 'center' of the U.S. could be located twenty or more miles from this point [in Lebanon, Kansas], but still might not pinpoint the true 'center' at all as an official measurement would have to be defined by an agreed upon standard which does not exist."

Classroom Setup

With students in the lower elementary grades, begin by talking about the word *center*. What does it mean? Use examples from everyday life to talk about the center of items, for instance, a pizza, a room, a table, a flat piece of paper, and so on. Discuss ways to find the center of these objects. You might even bring in enough small items to put something in each child's hand. Then ask students to talk about their objects' centers and how they would go about finding them.

Some younger children may have difficulty finding the center of such an irregular shape as the continental United States; it may be useful to first

have them work together to determine the center of a simpler shape, such as their classroom floor. Ask upper elementary students for a description of how to find the center of various geometric shapes such as squares, equilateral triangles, or circles. Encourage them to discuss how to find the center of an irregular shape.

Show a map of the continental United States and talk about its center. Point to the approximate location of Lebanon, Kansas, and explain that some people have determined that this town is the U.S. geographic center. Show a photograph of the monument in Lebanon (www.kansasphototour.com/center.htm).

Make sure that students understand the task before you distribute the problem. Allow them to work in small groups. Encourage multiple methods of finding the center. Ask them if they agree that it is in Lebanon and have them explain why or why not. Older students may work with measuring instruments such as rulers and compasses.

As an extension, have students find the center of your particular state or another state that you believe would be interesting or challenging. Discuss with the children whether the same methods that they use for the U.S. map will also work for state maps.

Where's the Math?

The problem this month concerns the idea of *center*. Mathematics involves many centers. For example, the *mean*, *median*, and *mode* are each measures of central tendency in statistics because they represent a data set's center in some measurable way. A data set's mean, for instance, represents the data's center of balance, and children find similar centers of balance whenever they play on a playground teeter-totter.

In geometry, the center of a one-dimensional line segment is called its *midpoint*, and it is relatively easy to find and to construct. Centers of two-dimensional figures are sometimes more challenging to find. If the flat figure is regular, its center can be found through folding. For example, folding a rectangular sheet of paper along its horizontal and vertical axes of symmetry will result in two intersecting lines that meet at the rectangle's center. For a rectangle, this center is equidistant from its four vertices. Other symmetry-related procedures could be used to determine the centers of equilateral triangles, other regular polygons, and circles. Some of your students may attempt to fold the U.S. map in various ways to obtain its center. Others may try measuring and equating distances or other values. Although such methods are more difficult with the irregularly shaped map, students may be able to justify their folds or measurements through approximations or other reasoning.

Center of gravity is another way to interpret the idea of center. Like a data set's mean, this center of balance takes into account the weighted distribution of an object. A flat shape should balance at its center of gravity. For an irregular shape like the continental United States, trial and error might be an effective method for determining the center of gravity. Creating a cardboard replica of the map, for example, would allow students to place a pencil under various points of the map in an effort to balance the irregular structure. Getting that cardboard model of the United States to balance on the tip of a pencil is a challenge but would succeed in locating the approximate center of gravity. This modeling method was used in 1918 to determine the geographic center of the United States and was endorsed by the Coast and Geodetic Survey (Wikipedia 2008).

The geographical center of the United States can be approximated by trial and error, and its location can be estimated by observation. But its precise location cannot be exactly determined and is open to many interpretations. Nevertheless, the idea of a center is important and useful in many areas of mathematics. For students working on this problem, finding the precise location is probably less important than understanding the properties associated with center and with learning methods to approximate, measure, and determine centers. Such learning and understanding can be developed by balancing regular shapes at their center point, finding their lines of symmetry, and then moving on to irregular shapes such as lakes, states, or countries.

Edited by **Joseph Georgeson**, *jgeorgeson@usmk12.org, middle school mathematics department chair and teacher of eighth-grade students at the University School of Milwaukee in Wisconsin;* **Sarah Bunten**, *sbunten@gmail.chelsea.k12.mi.us, a third-grade teacher at Pierce Lake Elementary School in Chelsea, Michigan; and* **Brian Schad**, *schad@aaps.k12.mi.us, a fifth-grade teacher at Lawton Elementary School in Ann Arbor.* *Each month this section of the "Problem Solvers" department features a new problem for students. Readers are encouraged to submit problems to the editors to be considered for future "Problem Solvers" columns. Receipt of problems will not be acknowledged; however, problems selected for publication will be credited to the author.*

In the Classroom

Monitor students as they work on this problem and provide them with the appropriate guidance, tools, and encouragement. Note the methods that they use to determine their results and listen closely as they discuss their findings. After the children have determined the geographic centers, have them explain to you and their classmates how they determined the center and whether they agree with one another about the Kansas location. Encourage students to justify their methods and answers and to extend their reasoning beyond the visual justification of "because it looks like the geographical center."

Share Your Students' Work

Please try this problem in your classroom. We are interested in how your students responded to the problem, what problem-solving strategies they used, and how they explained or justified their reasoning. Include information about how you posed the problem, samples of student work, and photographs.

E-mail your thoughts, reflections, scanned student work, and photographs to **Joseph Georgeson**, jgeorgeson@usmk12.org, or you may mail them to him at 2100 W. Fairy Chasm Road, Milwaukee, WI 53217. Submit your results with your name, grade level, and school's name by **June 1, 2009**. Selected submissions will be published in a subsequent issue of *Teaching Children Mathematics* and acknowledged by name, grade level, and school.

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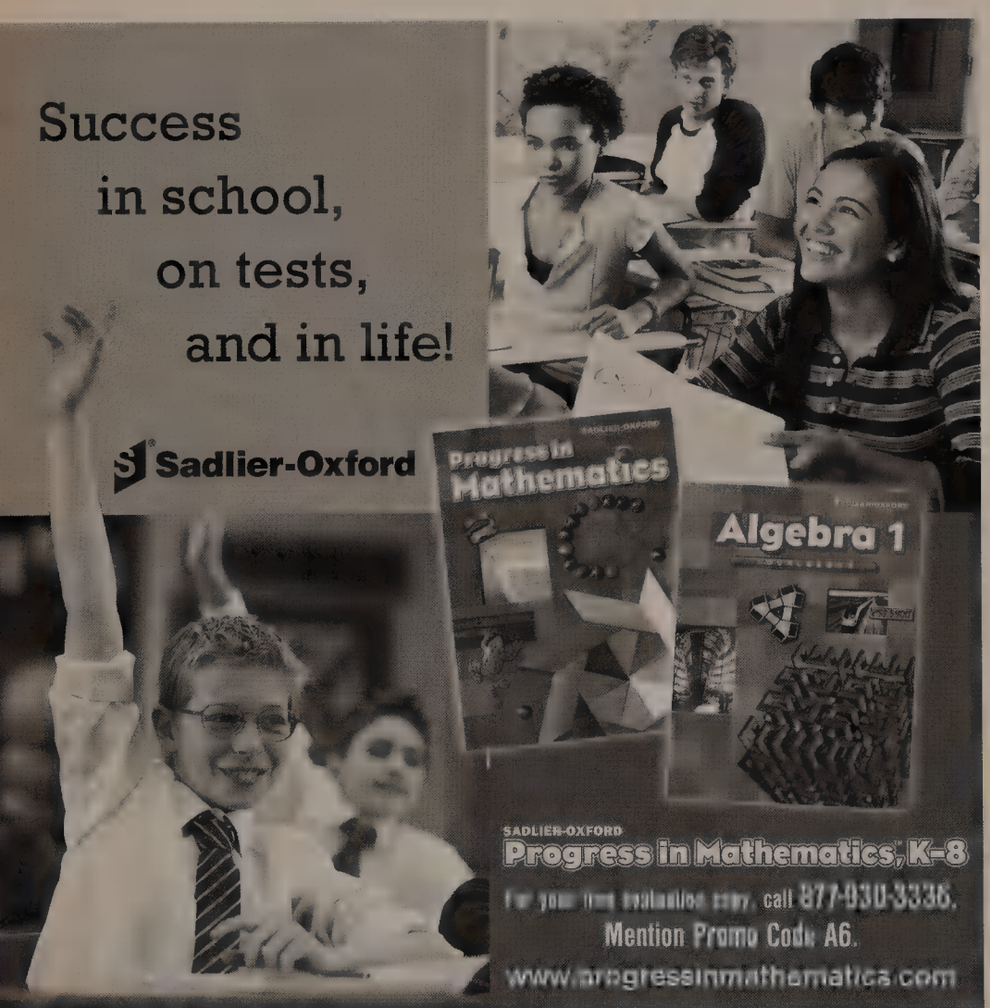
U.S. Geological Survey (USGS). U.S. Department of the Interior. *Elevations and Distances in the United States*. "Geographic Centers." Online edition, 2001. egsc.usgs.gov/isb/pubs/booklets/elvadist/elvadist.html#Geographic%20Centers.

Wikipedia. "Geographic Center of the Contiguous United States." en.wikipedia.org/wiki/Geographic_Center_of_the_Contiguous_United_States. ▲

(Solutions to a previous problem begin on the next page.)



Show off Your Students!

Each month, the "Problem Solvers" department features a challenging mathematics problem for you to try with your students. Take notes as your students tackle the problem, and then share with us any insights you gain from observing their efforts. The department editors compile teacher reflections and student results; a short report appears in the "Problem Solvers" solution section of a subsequent issue of the journal.

Sharing your students' work makes a valuable contribution to the journal, and it is fun to see your input in print. Details about how to contribute appear under the heading Share Your Students' Work in each "Problem Solver" problem.

Solutions to the Coloring Maps Problem

The Coloring Maps problem is a modified version of the classic Four-Color theorem. The problem in the April 2008 "Problem Solvers" section was stated as follows:

Problem

Emy was known as the best map colorer in Zildor. Everyone loved her simple yet elegant way of coloring maps. As she became more and more famous, she was asked to make more and more maps. Finally, Emy realized she would need an assistant—that's you!

1. Help Emy color the maps (see **figs. 1–3**) using the rules below. For each map, explain how you figured out the fewest number of colors needed.

Figure 1

Straight City Street Map

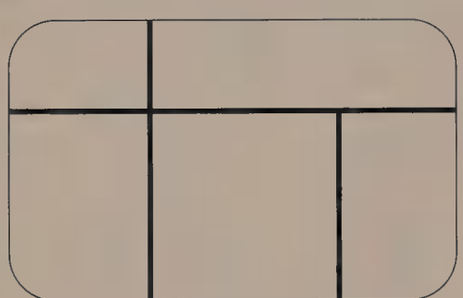
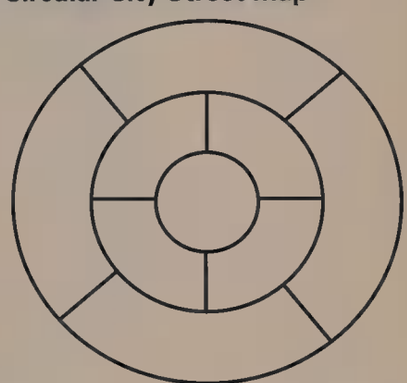


Figure 2

Circular City Street Map



- When regions share a border, they must have different colors.
- Regions that share only a corner may be the same color.
- Each map must be colored using the fewest number of colors possible.

Figure 3

Tricky Town Map

2. Now Emy wants to see your map-making skills. Create an original map and write the minimum number of colors it requires. Then challenge a classmate to figure out the least number of colors needed to color your map. (This activity is similar to a two-player game created in the late 1800s by Lewis Carroll of *Alice in Wonderland* fame!)

Variation

Rather than having younger students explain how they figured out the fewest number of colors needed for each map, ask them to try to color each map in two different ways (using the fewest number of colors each time).

Extensions

The problem can be extended in the following ways:

- After students have the chance to make their own maps, challenge them to explain (a) how maps can be made more difficult to color and (b) how they know the fewest number of colors needed for each map.
- Give students an unlabeled map of the United States and challenge them to color it using the fewest number of colors possible. Ask them to determine how many colors are required and explain why.
- Direct the students to try to create a map that requires the use of five colors.
- Ask students to examine several maps and figure out the characteristics that exist when four colors are needed.

Figure 4

Students realized that four colors must be divided between the two rings.

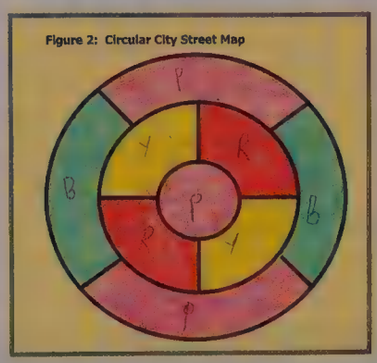
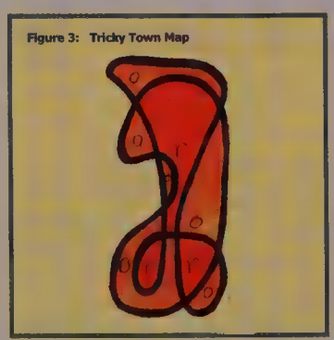


Figure 5

To Schnitzer's amazement, a student pointed out that the Tricky Town map could be colored using only two colors.



Jennifer Schnitzer is a numeracy coach in the Bridgeport, Connecticut, school district, where she co-teaches and models mathematics lessons at Webster Elementary School and Park City Magnet School. Schnitzer was interested in using this lesson with students in both schools to see if they would approach it differently. She presented the lesson in three different classrooms.

Schnitzer first worked with Maquel Skowronski's third-grade students at Webster. To start the lesson, students were to look at a U.S. map and determine how many different colors were used to color the states. They discussed the criteria for the Coloring Maps problem and concluded that the U.S. map met the same criteria. Afterward, Schnitzer distributed the "Problem Solvers" Straight City Street map to the students. She recounts her observations:

> Some students used a labeling strategy to plan their colored blocks while other students just jumped right into coloring. I saw students suggesting to each other that it would be best to label each box according to what color they were going to use in certain squares. This way they could just erase the letters and rearrange them until they met all the criteria. This method seemed to work the best, and the whole class adopted this strategy for future maps. Students in this class understood that they needed to make sure the first two criteria were met, but the third criterion (of coloring the map using the fewest number of colors possible) required them to use even more critical thinking skills and proved to be difficult at first.
>
> As students correctly completed each map, I let them move on at their own pace. Their excitement grew as they saw how crazy the maps were getting. The labeling strategy proved successful with the second map because many changes were needed as I went around the room checking. This map allowed me to discuss key vocabulary such as *line symmetry*, *border*, and *edge* with the children, which gave many students the extra support needed when they were stuck.
>
> On the circular map, a few students used four different colors on the outside ring and then duplicated them in the middle ring. This approach took a lot of problem solving to make sure those colors did not touch. When the students got to the center circle, they realized that they had to use a fifth color because that middle circle touches all four sections of the middle ring, making the last criterion invalid. That is when these students realized that the four colors needed to be divided between the two rings (see **fig. 4**). The final map was a challenge for my students and me, but it was the most enjoyable for the students. They were so excited [about] the challenge of the Tricky Town map. It wasn't until I repeated the problem at Park City Magnet that a student pointed out that the map could be done using only two colors (see **fig. 5**). I was amazed!

Schnitzer also used the problem at Park City Magnet School in Albina Bennett's third-grade classroom and Lisa Joy's second-grade classroom. Following the same instructional format that she

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had in the Webster classroom, Schnitzer incorporated some technology, displaying the problem examples and student work with a digital document camera and LCD projector. Able to view one another's work in an enlarged format with the document camera system, students effectively shared their work and brought the lesson to a close.

As a problem extension, students were directed to create their own Tricky Town maps at home and color them using the same three criteria (see **fig. 6**). Schnitzer was thrilled with students' excitement about using what they had learned from map coloring to create their own maps. A student said to her, "I can't wait to get home to do my math homework."

"This problem opened up many new doors for me as a coach," said Schnitzer. "It gave me the opportunity to see how different grade levels process the same information. It also allowed me to give students an opportunity to enjoy mathematics. Students were excited to work both in groups and individually to complete challenging work problems while mastering critical skills."

Meanwhile, Amy Pacifico used the Coloring Maps problem with her fourth graders at Pleasantdale School in West Orange, New Jersey. She also began class by having students examine and discuss why the maps in their atlases have multiple colors. Pacifico next read the story "The Young Map Colorer" from the Mega Math Web site (www.c3.lanl.gov/mega-math/workbk/map/mpprstory.html) and then presented the students with the maps from the "Problem Solvers" problem and from the Mega Math Web site. She noted a variety of approaches among her students.

Students used different strategies for coloring. Joshua explained his strategy: "I would look at the regions to determine how many colors I will need. I usually start at a corner of the map and work my way around." Some students labeled first with a pencil and then started coloring. However, when they noticed that two colors touched, they had to start all over again.

After students had colored several maps, Pacifico introduced the Four-Color theorem. Some students then developed a different map-coloring strategy. One student made the following comments:

> After I learned the Four-Color theorem, I started my map coloring with four crayons. I take out four different colors and start with one color first. I color all possible spaces with the first color. Then I move to the next color. This doesn't always work because at the end sometimes I realize that I can use fewer than four colors. It is hard for me to just look at a map and know how many colors I will need. I need to think about it and plan it out on the map before I do anything.

Photograph by Jennifer Schnitzer; all rights reserved

The math coach allowed students to move ahead at their own pace.

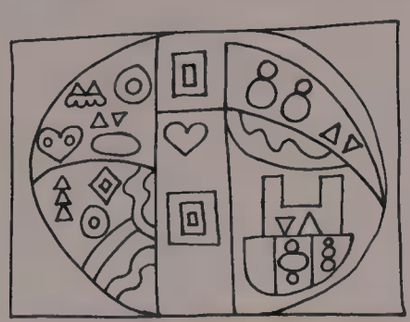
The fourth graders experimented with drawing maps that start and end at the same point. Students quickly realized that maps drawn using this method are always two-color maps. One student, Tyree, stated, "I know it is a two-color map if the line starts and ends at the same point. Usually maps need three or four colors if they have a lot of shapes."

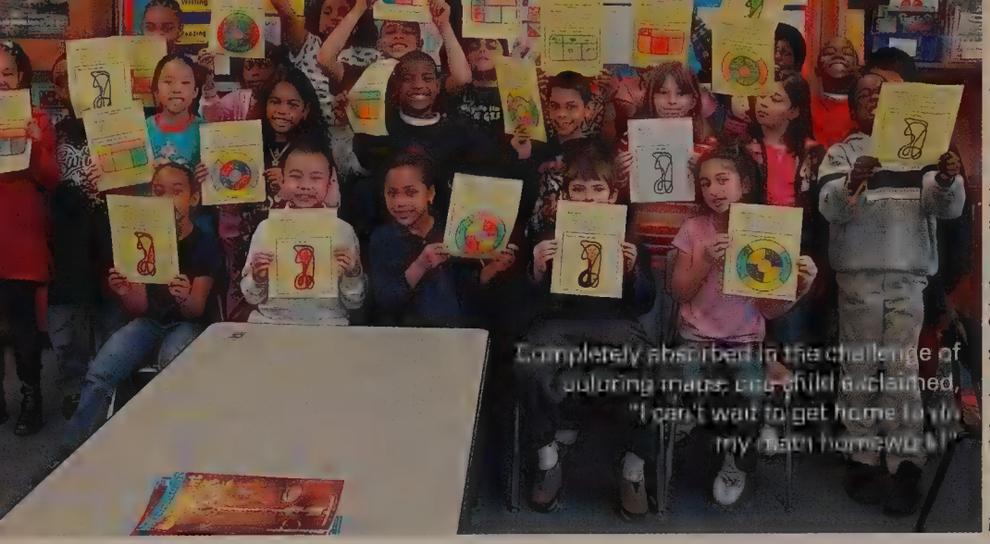
Pacifico noticed that this problem helped students work on organizational skills as they strategized on coloring their maps in the most efficient manner:

> Many students took the time to plan before they stated coloring. They would take a pencil and label the regions with letters, check their map for accuracy, and then begin coloring with crayons. I feel that this demonstrated students' drive for accuracy. I was thrilled that they took the time to plan before jumping in and making mistakes.

Figure 6

Without an answer key, Pacifico's fourth grader Tatiana could not recall how many colors she had used on her own Shapeville map.





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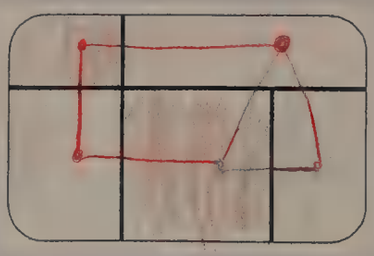
The next day, Pacifico's students created maps requiring at least three colors and explained their map-making plans using concepts they had learned while coloring maps:

> When I first made my map, I made an oval and put in different shapes. I labeled the colors with a pencil first and tried to determine if any colors touched. After I colored, I realized I messed up once because two colors were touching. So then I did it again, and I looked at the mistake map and started over. I made an answer sheet so I could remember how many colors it was (see **fig. 6**).
>
> When I made my map, I looked at the maps that we worked on in school, and took ideas from them. I made the shapes touching each other to try to make it trickier. When I was done making the cubic shape, I started putting in more shapes of my own. If I need to make a map with four colors, I would try to make a map with lots of shapes and then color it in to check to be sure. If it didn't work, I would just try it again!

Pacifico was amazed by how much her students had learned from coloring maps. Her fourth-grade students were challenged to develop their spatial awareness, logical reasoning, persistence, and drive for accuracy. Equally important, students had great fun solving the problem.

Figure 7

Varley showed her fourth graders how to try the triangle strategy on their maps.



Mary Kay Varley's fourth graders at Fort Worth Country Day School in Texas had just finished studying the western region of the United States in their geography class when Varley noticed the Coloring Maps problem. She realized that it offered a mathematical link to their social studies unit and also provided an opportunity for her students to engage in problem solving as they explored the different maps. Varley led the class to an interesting discovery to help find the minimum number of colors needed for each map. Her account of the classroom discovery follows:

> After the class agreed that the map for **figure 1** required three colors, I placed vertices in the center of each piece on my overhead map. I then connected the vertices with line segments. I asked the class to look for a polygon with three sides and three angles. When they discovered that the triangle made up the part of the map that required the third color, they thought there might be a connection. They noticed that each of the boxes containing these three vertices touched each of the other two and therefore could not be the same color (see **fig. 7**). I told them that when they colored another map, they might want to look for those triangles to see if three colors would be required.

Varley presented the circular Tricky Town map and a U.S. map to her students for coloring. After a brief discussion of their strategies, students decided that they must search for those areas on their map that seem "congested" and determine if these areas require the use of four colors.

Varley extended the problem by having students work on coloring three-dimensional shapes using the same criteria as the Coloring Maps problem. Varley brought out Geofix® plastic equilateral triangles for her students and had them create two-dimensional nets that can turn into three-dimensional shapes. She asked her students to create a net for the tetrahedron using Emy's rules for coloring from the "Problem Solvers" problem. Students discovered that they needed only two colors for the net (as in a checkerboard) because like colors touched only at the corners. But when Varley asked her students to fold the net into a tetrahedron, students were surprised to find that every face had an edge that touched an edge to another face. The class members realized that they would need four different colors for the three-dimensional tetrahedron.

As the "Where's the Math?" portion of this problem mentioned last April, map coloring is easy

to understand and demonstrate; however, a simple mathematical method has not yet been established for determining the minimum number of colors required. As students explore different maps and try to find the minimum number of colors needed for each, surprising things happen. Before students try to color, have them think through what might happen and ask them to create an informal argument to justify the number of colors needed for each map. Watching them discover map-coloring strategies can be fascinating. Students will use inductive reasoning as they discover patterns. They will use deductive reasoning as they establish knowledge with concepts and ideas they already know to be true. Explaining their claims for the validity of each step gives them experience in completing an informal mathematical proof.

This problem can be adapted to any age level. As Schnitzer showed, the Coloring Maps problem can be solved by early elementary school students. Varley and Pacifico's fourth graders made more sophisticated conjectures and discoveries about map coloring. The more time that students invest in the problem, the more interesting it becomes.

In an extension, students discovered that two-dimensional tetrahedron nets need only two colors but their three-dimensional shapes require four colors

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Albina Bennet and her third-grade students and Lisa Joy and her second-grade students at Park City Magnet School in Bridgeport, Connecticut;
Amy Pacifico and her fourth-grade students at Pleasantdale Elementary School in West Orange, New Jersey;
and Mary Kay Varley and her fourth-grade students at Fort Worth Country Day School in Texas.

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Jennifer Taylor-Cox

Teaching with Blocks

Blocks are powerful mathematical tools when used to teach young children early concepts in measurement, number sense, computation, geometry, data analysis, and algebra. For nearly one hundred years, blocks have played a role in early childhood classrooms (Smith 2001). Yet not everyone understands the mathematics concept-building power associated with blocks. From sorting to patterning, young children can build a strong mathematical foundation one block at a time. This article examines how blocks can be especially useful in engaging children in activities that address algebraic, geometric, and spatial thinking.

Early Algebra

Algebraic thinking is one branch of mathematics that can be fortified by using blocks. In the early primary grades, children need opportunities to develop habits of mind that will lead to formal algebraic thought. Experiences in patterning and sorting are crucial to such development.

Patterns

Young students can construct repeating patterns and growing patterns with blocks. Sometimes repeating patterns involve colors; other times they include size or position. A prekindergarten student once showed me a repeating pattern that she created using blocks of all the same size and color. How she positioned the blocks was what characterized the pattern. As she pointed to each block, her description of the repeating pattern in the block positions was imaginative: "sleeping, awake, sleeping, awake, sleeping, awake, sleeping, awake, ..."

Not all patterns repeat. Patterns may increase or decrease. With young students, we often refer to these as growing patterns. A basic growing pattern may increase by one (e.g., 1, 2, 3, 4, 5, ...). More complex growing patterns may increase or decrease by other amounts (e.g., 18, 15, 12, 9, ...). When youngsters construct growing patterns with towers of blocks, they often describe these patterns as *steps* or *stairs*. On the basis of the growing pattern, they can predict how many blocks are needed to make the next tower. They can also read the pattern by reporting the number of blocks in each term.

Young students can also explore algebraic concepts of equality and balance as they make designs and structures with blocks. To encourage work with some important early algebraic concepts, teachers can ask students the following questions:

- Is the structure balanced?
- How do you know it is balanced?
- How many more blocks do we need to make these block buildings equal?
- What happens if a block building is unbalanced?
- Are the block towers equal?

Many activities help students explore the algebraic concept of balance. One such activity involves building a balanced structure with blocks. Students take turns deconstructing the edifice by removing two blocks at a time. The children focus on maintaining balance and talk about which blocks should be removed to help the structure remain balanced. The mathematics discourse also includes predictions about what will likely happen if specific blocks are removed. These same ideas can also be used to build balanced structures with blocks by adding two blocks and predicting the outcome on the basis of potential placement.

Sorting

Sorting and classifying are also among the many mathematics concepts explored and fortified through the use of blocks. The ability to sort and classify is connected to numerous NCTM Standards (2000). Sorting is a critical component of algebra, geometry, and data analysis. Without knowledge of sorting, children have difficulty recognizing algebraic patterns. Likewise, they must know how to sort in order to compare and contrast the geometric characteristics and attributes of shapes.

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Furthermore, organizing and displaying data is difficult when they do not know how to sort. Even measurement and number concepts are enhanced by the ability to sort, such as understanding number relationships and units of measure through sorting. Sorting is such an important part of early childhood mathematics that young children should be engaged in sorting experiences on a regular basis. Block play offers the perfect opportunity.

When young children sort blocks, they create groupings and classifications. They learn that a particular block may be part of several different groups because its attributes can be described in various ways: *large*, *flat*, *smooth*, *rectangular*, and so on. Teachers ought to encourage students to form groups with the blocks. As students sort, a teacher can ask questions that encourage thinking and talking about some of the classification systems associated with mathematics:

- Why are all these blocks together?
- What is the name of your group?
- What were you thinking about when you put these blocks together?
- Why are some blocks not in the group?
- Could any other blocks go in this group?

Several years ago I was working with a group of prekindergartners who were sorting blocks. I pointed to a group of blocks that one youngster had clearly sorted, and I asked why the blocks were together. In response, he shrugged his shoulders. I then asked the name of his group. This second question prompted a smile. He replied, "It is my group."

I asked, "How will I know if a block belongs in your group?"

He smiled again. "I only like the big blocks."

"Yes!" I exclaimed, "Now I know which blocks will go in your group."

Without missing a beat he responded, "I will call it the *big blocks group*."

This kind of exchange is a helpful way to encourage young students to use words to describe how they have sorted specific blocks. Sometimes one question is not enough. The teacher must probe more deeply to find out what students are thinking as they sort. We do not want sorting to be a silent or lonely task; we want students to engage in mathematics talk and ask one another questions about the sorting.

In Ms. Popper's kindergarten classroom, students sort pattern blocks in various ways. When her students finished sorting one day, Ms. Popper invited them to take a gallery walk around the classroom to view how others had sorted the pattern blocks. Many groups had sorted by color. Group labels included yellow, red, blue, green, orange, and tan. Ms. Popper asked students to explain why certain blocks belonged to specific groups. Students described the sorting as matching blocks of the same color and shape. "You know it goes there because of how it looks," one student explained.

Another replied, "I can even close my eyes."

Ms. Popper asked, "What do you mean?"

The student demonstrated. He closed his eyes, felt around the edges of a block, and announced, "I know where it goes because I can match it."

Ms. Popper invited students to describe the shapes, and then she introduced some new shape names.

One group sorted the pattern blocks by position. Students used blocks in different positions to make two groups. When Ms. Popper asked, "What are the names of your groups?" a student replied, "This is the *up group,* and this is the *down group*."

Another student responded, "I get it, but you could make them all go down."

Ms. Popper asked, "How do you know which blocks go in the *down group*?"

Students explained that they dropped blocks on the floor and sorted them by how each one landed.

Another group sorted the pattern blocks in a way that made Ms. Popper initially unsure if the

Young students actively engage in mathematics when they manipulate building blocks.

students understood how to sort. She asked, "How did you decide which group to put the blocks in?"

A student answered, "We just know."

Ms. Popper persisted, "*How* did you know?"

Another student replied, "By how it fits."

Still uncertain, Ms. Popper asked, "What are the names of your groups?"

The children responded, "This is the *sandwich group,* and this is the *not sandwich group*."

Even more confused, Ms. Popper asked, "Which blocks go in the *sandwich group*?"

Pointing to the first group, students chorused in unison, "These."

Chuckling, Ms. Popper asked them to show her what they meant by *sandwich*. Students stacked two red trapezoids on a yellow hexagon. Then they stacked three blue rhombus blocks on top and added six green triangles, followed by a yellow hexagon.

"See? If it makes a sandwich, it goes in the *sandwich group*," explained one student. Now everyone understood how the students had skillfully sorted the blocks.

Sorting with blocks is a powerful way to build strong foundations in mathematical thinking. Teachers must ask questions and promote mathematics discourse during sorting activities. Sometimes what appears to be random is actually quite clever.

Measurement

Measurement is another important topic associated with using blocks. Youngsters can order blocks by size, a concept referred to as *seriation* by early childhood educators. In early measurement experiences, students compare and order blocks (tall, taller, tallest) and come to understand size in relation to other sizes. A block may be small compared to another block, but the same block could be large compared to a third block. Because size is relative, comparison is foundational to understanding measurement. Educators should encourage youngsters to compare other size categories as well:

- How heavy is the block?
- Which block is lightest?
- How would you order these blocks by weight?

Teachers can also ask questions and guide students' mathematical understanding of capacity:

- How many blocks fit inside this container?
- How can we pack these small blocks?
- Which container holds the most or fewest blocks?

After many chances to compare sizes, students can begin using blocks as nonstandard units:

- How many blocks will we need to make the building as tall as you are?
- How many blocks long is the road?
- How many blocks wide is the castle?

Using blocks as nonstandard units of measurement includes understanding one-to-one correspondence and using number sense. To measure table height, for example, students must use one-to-one correspondence as they stack the blocks one on top of another. They use number sense as they count the blocks. They must keep track of which blocks have been counted and which blocks are yet to be. Students use cardinality to name the total number of blocks in the stack. Using blocks as units of measure invites children to think about the size of units and spaces.

In Ms. Viola's prekindergarten class, youngsters use blocks to measure. In one particular lesson, Ms. Viola asked students to work in groups to measure her shoe with blocks. One group decided that her shoe was "five blocks." Another group declared that her shoe was "two blocks." A third group called the shoe "four blocks," and the final group said Ms. Viola's shoe was "a lot of blocks."

Ms. Viola brought the class together to share the conflicting results. She asked, "Why did each group come up with something different?" The groups showed their classmates how they had measured Ms. Viola's shoe. The group that found the shoe to be five blocks long used medium-sized cubes lined up beside one another. The group that declared the shoe two blocks long used large rectangular prism blocks. The group that called the shoe four blocks long used medium-sized cubes, but they were not lined up one beside the other. The group that said Ms. Viola's shoe was a lot of blocks actually placed many small cubes *inside* the shoe.

Using the opportunity to teach students about unit size and spacing, Ms. Viola also prompted a discussion about the difference between measuring length and capacity. Students' experiences helped them build a stronger understanding of measurement.

Geometry

Blocks also provide the chance for children to engage in early geometry as they "recognize, name, build, draw, compare, and sort two- and three-dimensional shapes" (NCTM 2000, p. 96). Children can analyze shape characteristics and properties as they construct

buildings and roads with blocks. I find it useful to save students' photographs and drawings of their structures. We record the creations in ways that allow us to revisit the activity using mathematical conversations:

- Which blocks did you use?
- How many rectangular prisms are in the structure?
- Where did you place the cylinders?

The last question brings into play other crucial components of early geometry—position and order. Using words to describe relative block positions (*above*, *below*, *beside*, *under*) is beneficial. Likewise, using words to express order (*before*, *after*, *between*) builds understanding of spatial relationships.

Of course, area and perimeter concepts are part of geometry and measurement. Even young children can work with these notions as they explore how many blocks outline the perimeter or comprise the area of a structure. According to one contributor to the prominent *Block Book*, "Because of the regular properties of the blocks, ... opportunity for recurring experiences ... lead toward conservation of area" (Lee-Lundberg 1984, p. 41). Through repeated experiences with area and perimeter, students' visual and spatial reasoning skills are enriched.

In one classroom, I noticed a group of prekindergartners using blocks to make a fence for a plastic horse. When I asked how many blocks they used for the fence, they counted the blocks on each side of the rectangular structure. When I asked how many blocks would fit inside the fence, they placed blocks inside to find out. Using great precision, they learned that their fence used ten blocks for its perimeter and that six blocks fit inside the area. Because they were so comfortable with the blocks and the situation's context, students could work with complex mathematical ideas with relative ease. However, they did decide to remove the blocks from inside because of concern that the horse would use them to jump over the fence.

Blocks Enliven Early Math

Although safety is always a consideration, no particular blocks are better than any other type. Wooden block sets, attribute blocks, pattern blocks, snap cubes, and cardboard blocks serve as excellent mathematics manipulatives for young children to use. The block area can include paper and crayons for recording the creations. Tape and other fasteners can be used to hold some structures together. Even everyday items such as soup cans, cereal boxes, and other containers add to the wealth of the block experience. Every early childhood classroom needs blocks.

In the name of "academic focus," some early childhood programs are considering removing blocks from classrooms. Proponents of such ideas clearly fail to understand how blocks build foundations in mathematics for young students and provide opportunities for the "high quality, challenging, and accessible mathematics" that every child deserves (NAEYC and NCTM 2002). Through free exploration and teacher-directed mathematical experiences with blocks, youngsters begin to develop deep understanding of measurement, geometry, and algebra.

We do not simply wait for students to find the mathematics; the point is that students actively engage in mathematics as they manipulate blocks. Early childhood mathematics educators ought to be familiar with and focus on the specific mathematical content embedded within block play. Because blocks contain so much mathematical potential, thinking that "academics" should replace blocks is oxymoronic. Block play in the early childhood classroom make mathematics come alive.

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Why Are Things Shaped the Way They Are?

Why do corn kernels grow in staggered rows? Why are manhole covers always round and honeycomb cells hexagonal? Important geometric concepts are embedded in the shape and design of natural and manufactured objects.

Why are manhole covers round? They are so much a part of our daily landscape that we hardly notice them. However, the fact that they are circles is no accident (NCTM 2004). This article examines how children can explore important geometric concepts that are embedded in the shape and design of both natural and manufactured objects. Posing and investigating the question of why objects are shaped the way they are is consistent with NCTM's *Principles and Standards for School Mathematics* (2000), which advocates the application of geometry in real-world situations. Such a question can be a valuable central thread in the geometry curriculum of the elementary grades. For example, looking closely at the everyday manhole cover can help children identify unique attributes of a circle.

One fourth-grade class investigated the manhole cover question by experimenting with models. As part of their mathematics curriculum, the children had to be able to identify the attributes of regular polygons. The following activity afforded them the opportunity to compare these shapes in terms of sides, angles, number of diagonals, and symmetry. The teacher also chose this activity because it placed learning these skills into a real-world context. She prepared materials for the lesson by using a utility knife to cut convex shapes from card stock, leaving the resulting hole intact as a frame. Each group of children received a set of shapes consisting of a circle and nine polygons: four quadrilaterals (a trapezoid, a rectangle, a square, and a rhombus), two triangles (an isosceles and an equilateral), an octagon, a hexagon, and a regular pentagon (see **fig. 1**). Students were instructed to record whether or not each shape could pass through its frame without bending and without touching the frame, to describe the differences among the shapes, and to defend why manhole covers are circular.

The children identified the circle as the only shape that could not pass through its own frame. All the polygons that the children tried could be rotated so that a side passed through one of the frame's diagonals. The children observed that the circle's center is equidistant from all its edges. Because all diameters

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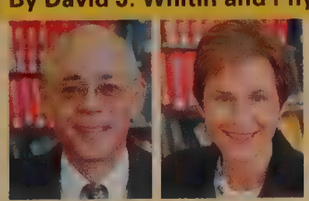

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Figure 1

Students received a set of ten shapes—a circle and nine polygons—for the Manhole Cover investigation.

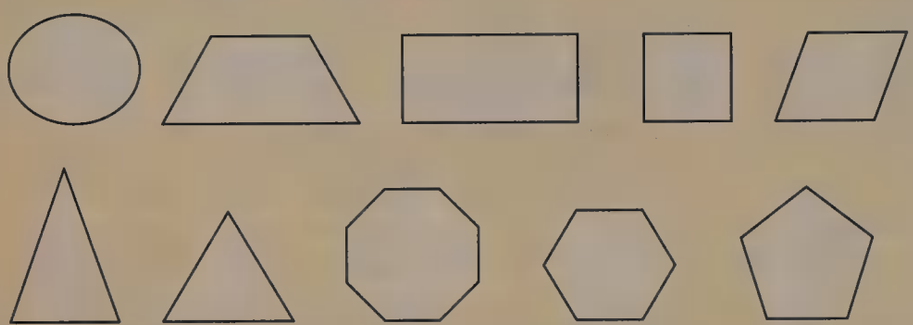
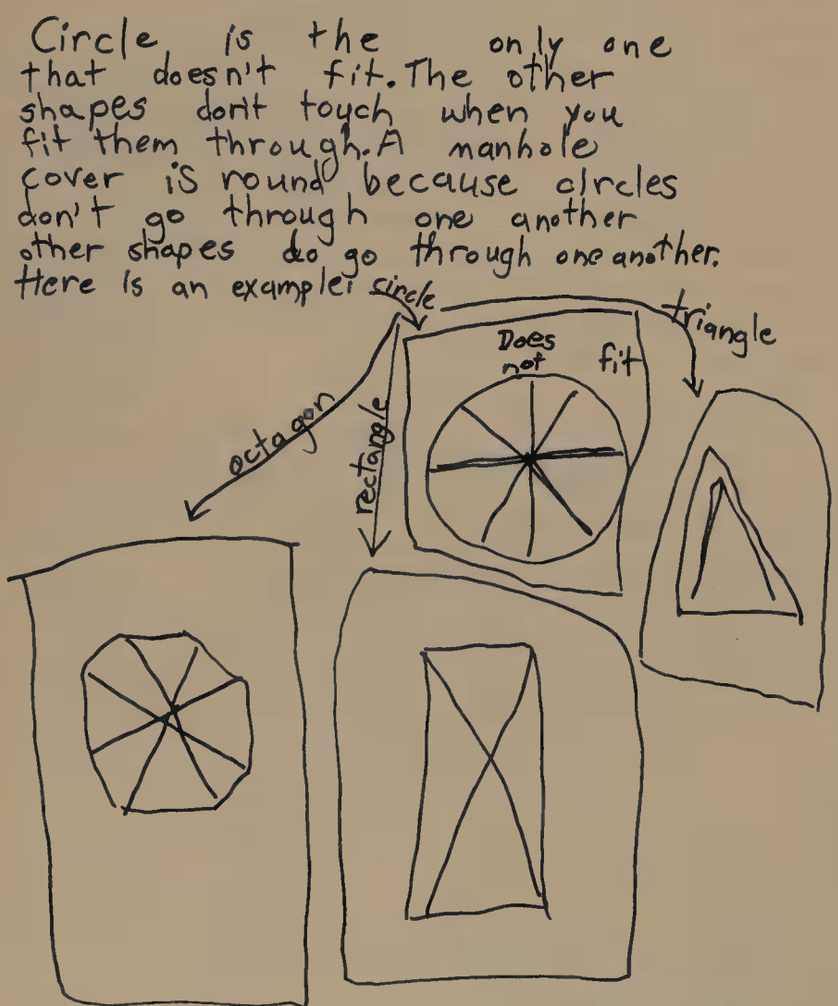


Figure 2

Teresa's response to the Manhole Cover investigation shows that "a manhole cover is round because circles don't go through one another."


Circle is the only one that doesn't fit. The other shapes don't touch when you fit them through. A manhole cover is round because circles don't go through one another other shapes do go through one another. Here is an example! circle
triangle
Does not fit
octagon
rectangle

are equal, the circle cannot fall in on itself. Such observations underscored for the children how these equal-length diameters of a circle make it uniquely different from other polygons. Teresa described her discoveries in this way: "A circle is the only one that doesn't fit. The other shapes don't touch when you fit them through. A manhole cover is round because circles don't go through one another. Other shapes do go through one another." Teresa's drawings (see **fig. 2**) emphasized the way that different shapes can be turned to slip through the openings. The rectangle and octagon can be tilted on their diagonals, so she drew lines from vertex to vertex on those shapes. Her drawing of the isosceles triangle, however, reflects the twisting motion required to fit that shape through its opening. For the circle, she drew several diameters to signify her repeated attempts and wrote, "Does not fit."

Other students described their findings in other ways. Ashley succinctly concluded, "A circle has no angles or no straight sides, so you can't fit a circle through."

Matthew emphasized the functional application of the investigation: "I think the manhole lid is a circle because if it got sideways, the manhole [lid] wouldn't fall through and hit the person on the head. Most of the time, the other shapes went through [their frame] by angles. A circle has no angles."

After testing these polygons, the children also enjoyed rotating each shape to fit exactly inside its cutout. Such actions give them practice in recognizing the rotational symmetry of each polygon.

The children's descriptions and manipulations of the shapes demonstrated that a polygon's sides are always shorter than its diagonal. (Some exceptions to this observation exist if we consider concave shapes.) However, because a circle does not possess this attribute, it becomes the safest choice, and therefore the most appropriate shape, for a manhole cover.

The children logged all these observations in their mathematics journals. After sharing their findings during a whole-class meeting, students recorded their results on a large chart that summarized their ongoing geometry insights.

This particular activity was only one of several that the teacher planned for her students. We share it here because it highlights the unique relationship between form and function of shapes in the world. To pursue this idea further, we devote the remainder of the article to a range of explorations that demonstrate how inquiring about shape can integrate

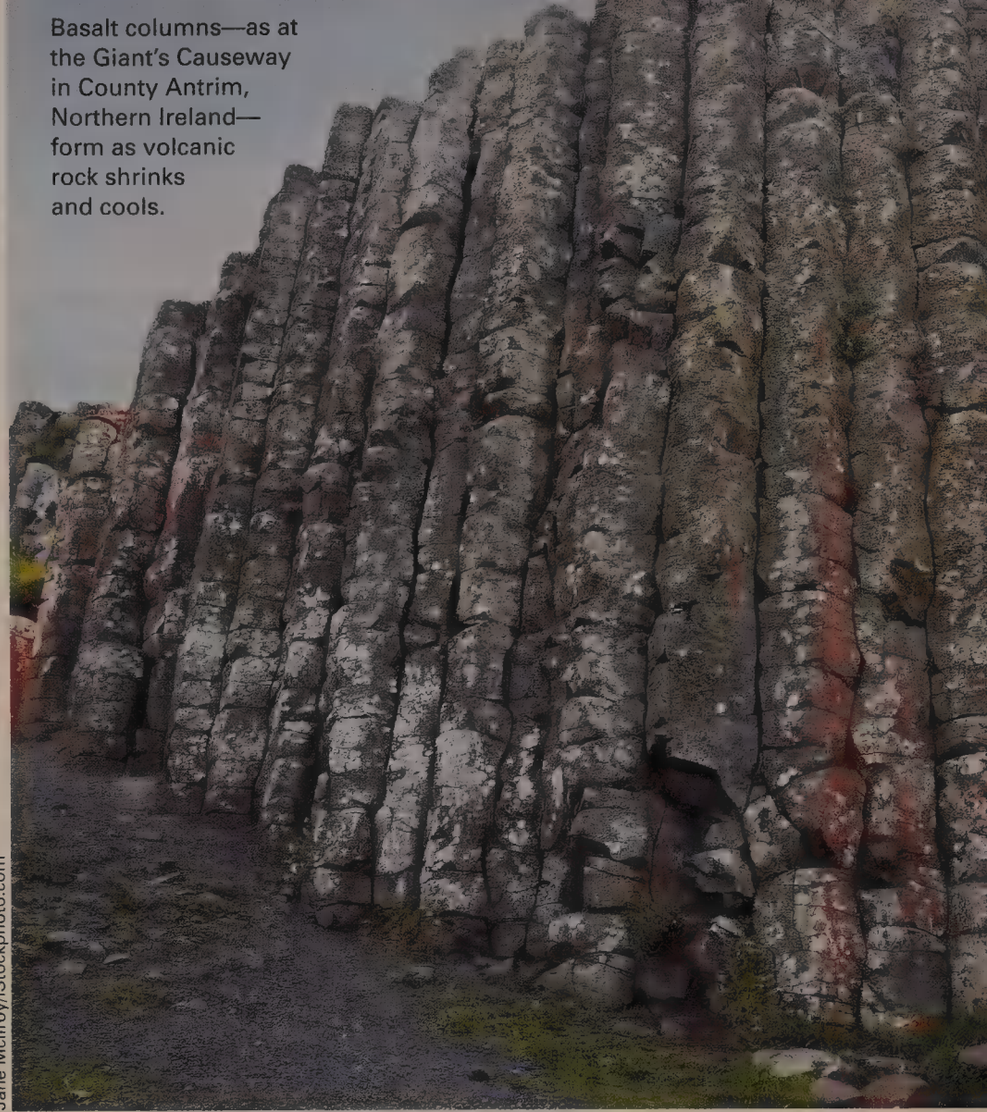
mathematics and science in interesting ways. The examples reflect the functionality of shape in both constructed and natural contexts. Fostering a spirit of inquiry is an essential habit of mind that builds a classroom of thinking, conjecturing students. Teachers can engender this attitude by demonstrating the reasons for shapes through various activities that set the tone for discovering the "why" behind shapes and designs. Such activities can spark children's interest in posing this question themselves in other contexts throughout the curriculum. Just as the manhole cover example tied to the study of polygons, the following activities incorporate such concepts and skills as tessellations, angle measurement, radial and bilateral symmetry, and ratio.

Shapes in Nature: Efficiency and Survival

In nature, shape is generally determined by the principle of efficiency: taking the shortest path, conserving energy, or making the tightest fit (Murphy 1993). For example, soap bubbles are spherical because trapped air exerts pressure equally in all directions on the soap-and-water mixture, causing it to expand. The film resists being stretched, so it is energy-efficient to assume a spherical shape, which has the least amount of surface area for a given volume (Wick 1997). Children can inquire into this phenomenon by dipping polygon-shaped wires into a soap solution. The film assumes the polygonal shape as it adheres to the wire, but if blown gently and released from the frame, the film reshapes into a sphere.

Entomology

Investigating why honeycomb cells are hexagonal can build children's understanding of the attributes of a hexagon and its relationship to a circle. Lots of energy is required for bees to make wax. When freshly made, each cell of wax is cylindrical; its rounded shape uses the least amount of wax to form the greatest possible area. As the cells align with one another, the weight of the wax pressing equally on all sides compresses the center cells into hexagons (see **fig. 3**). Teachers can challenge children to create this transformation themselves: Have them make a short cylinder out of clay, and then ask, "What shape is the end of that cylinder? How can you reshape the end so it is a regular hexagon and not a circle?" Once they have accomplished this task, invite them to create the hexagon in another way: "Now let's see what happens when we place



Basalt columns—as at the Giant's Causeway in County Antrim, Northern Ireland—form as volcanic rock shrinks and cools.

Jane McIlroy/iStockphoto.com

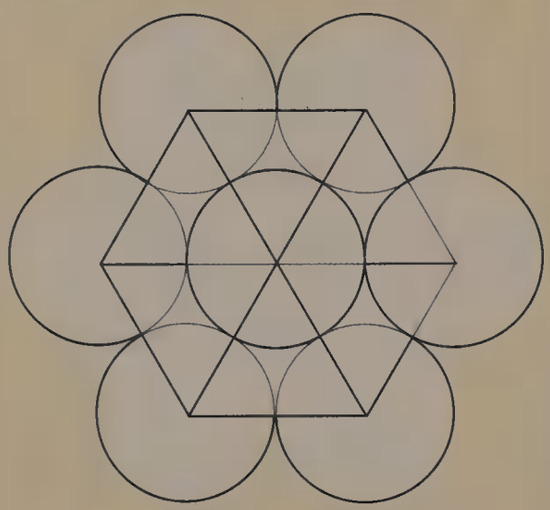
lots of cylinder pieces next to one another." Have the children make a clay cylinder about the size of a pencil's diameter and then cut it into seven pieces, each about one-half inch long. Stand the cylinders on their ends, surrounding one central cylinder with the other six. Next, roll the entire bundle, pushing evenly but forcefully. Ask the children what they notice. The pressure of the surrounding cylinders will reshape the center one into a hexagon. This simple experiment demonstrates how bees conserve

Figure 3

The weight of fresh wax pressing equally on all sides of cylindrical honeycomb cells compresses the center cells into hexagons.

Figure 4

Using tracings of pennies, students can demonstrate why honeycomb cells are regular hexagons.



energy by using the least possible amount of wax (a circle encloses the greatest amount of area with a fixed amount of material), thus increasing their likelihood of survival. Pressing the circles together is what remolds the wax into hexagons. Children can measure the 120-degree angles formed at the meeting points and observe that three of these angles equal a complete rotation of 360 degrees. Younger children can build honeycomb models with pattern blocks and discuss how the cells fit together with no spaces (tessellate), thus conserving space and making the hive strong.

A related activity shows children why the honeycomb cells are hexagons. Trace around a penny, and then surround that circle with six other penny-sized circles. Each outer penny touches the inner one at only one point, so there are six tangential points at equal intervals around the central penny. Mark the center of each circle and draw lines to connect the centers of adjacent pennies in the outer ring. The children should notice that the result is a regular hexagon. The centers of the outer circles can also be connected to the center of the central penny, thus dividing the hexagon into six congruent equilateral triangles (see **fig. 4**). This demonstration preserves the original circular shapes so to better compare them to the hexagon. Children can defend why the angles in the triangles are each 60 degrees and why the triangles will always be equilateral, even when other circular objects are traced. Such discussions highlight that equal-sided polygons have equal angles. Children might explore this relationship further by using different-sized circles. Their discoveries can help show that all circles are similar, or proportionate.

Botany

Kernels of corn are a good example of the principle of efficiency in botany (Murphy 1993). Corn kernels grow in staggered, interlocking rows, thus conserving energy by being tightly packed. Children can examine ears of corn to find that a kernel in one row touches two kernels in the row below. The junctures form angles of approximately 120 degrees. Looking at the single kernel as a center, children will see that six kernels surround it (two below, two above, and one on either side), an arrangement similar to the penny demonstration. Such hexagonal packing can also be seen in the commercial world, described later in this article.

Earth Science

Hexagonal formations can be found in earth science as well. Columnar basalt, an igneous rock, shrinks and cracks as it cools. The cracks form angles close to 120 degrees, creating hexagonal patterns. Devil's Postpile National Monument in California is a stunning example of this kind of formation (see **fig. 5** and the phtotogallery on the NPS 2006 Web site).

Rocks (and cement) also crack under circumstances other than cooling, and the resulting shapes show different patterns. Rocks begin to crack when they are under pressure, perhaps as a result of water seeping into tiny crevices and then freezing and expanding. Once begun, the crack follows the path

Figure 5

Columnar basalt at Devil's Postpile National Monument shrank and cracked simultaneously as it cooled, forming hexagonal patterns.

Photograph courtesy of Devil's Postpile National Monument; all rights reserved

of least resistance, which is generally a straight line (Murphy 1993). Now the greatest pressure on the rock on either side of the crack is at the midpoint of that line. Over time, a second crack will begin at that point. Because equal pressure is exerted on each side of the midpoint, the new crack forms perpendicular to the first crack. Perpendicular fissures in rocks, then, are the result of sequential cracking, rather than the simultaneous cooling of basalt.

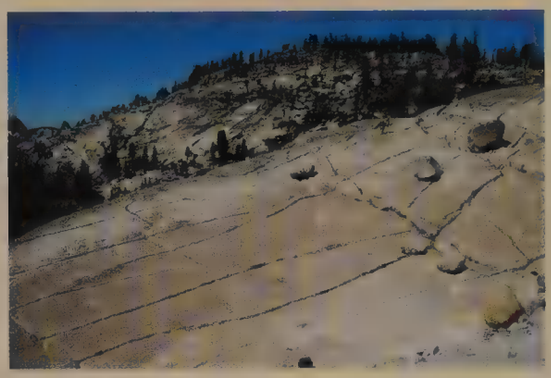
Children can observe these differing mathematical cracking patterns to help explain geological histories. They might explore and photograph examples of rock, pavement, or drying mud in their own schoolyard or community or find images on the Internet (see **fig. 6**). Exploring the cracking and splitting apart of surfaces can prompt children to pose larger questions about the weathering effects of erosion: Where else do we see rocks cracking? How do these cracks occur? Why are some mountain ranges different shapes and sizes? How did the shape of steppes and buttes come to be? Introductory activities promote this inquisitive spirit.

Biology

In the field of life science, adaptation for survival is another key concept. How symmetry relates to survival comes into play when children investigate the shapes of creatures, particularly with regard to their appendages. Teachers can read aloud *One Is a Snail, Ten Is a Crab* (Sayre and Sayre 2003), which tells the story of different creatures' legs added together to show the numbers one through ten and the multiples of ten to one hundred. The book often prompts a discussion about the number of legs found on different creatures. Teachers can invite children to make a list of their findings (see **table 1**), which may cause children to wonder why no creatures correlate to many of the odd numbers. Teachers can encourage their students to offer some hypotheses to explain this phenomenon. Related discussions highlight the need for creatures to balance their weight and to move about in an efficient manner. These observations can lead to a discussion about the body symmetry of humans and many other animals.

Figure 6

Rocks with approximately perpendicular cracks are the result of sequential cracking.



Teachers can then ask children what they notice about the symmetry of trees and plants. Children will notice that these shapes have many lines of symmetry. Continue to challenge them by asking, "What benefit is it for plants to have this radial symmetry?" Students' observations should highlight the fact that trees and plants lack locomotion and must remain fixed in one spot. To reach light, water, and nutrients, they send out roots and branches in all directions from their centers. So, radial symmetry serves the survival needs of trees and flowers. In every case, creatures' and plants' shapes are tied to their mobility and need for survival (Ross 1996).

Constructed Shapes

From bridges to coat hangers, from silos to umbrellas—humans employ geometric principles in design. Looking at everyday objects and posing the question, "Why are things shaped as they are?" can engender an inquisitive stance about the most mundane objects. For instance, why is a coat hanger a triangle shape and not square, rectangular, or hexagonal? Furthermore, why is it an isosceles and not a right or a scalene triangle? Such questions invite children to think about how an object's shape is determined by its function. An isosceles triangle best mirrors the shape of a human's upper torso.

Table 1

Number of Legs on Creatures

1	2	3	4	5	6	7	8	9	10
worm	person		dog cat	starfish	insect		octopus spider		crab

The angles from the triangle's apex approximate the slope of our shoulders, and the two equal sides match the bilateral symmetry of our bodies.

Another interesting application of circles from the commercial world involves packing. Ask students to imagine that they have to pack some very expensive, fragile glasses into a crate. Invite them to arrange a set of discs into a flat box of any shape (Mold 1967). An interesting conversation can occur around two possible arrangements. To prevent the glasses from breaking, where must students place some stiff cardboard separators? Have them draw circles to represent the glasses and lines between the glasses to show where to place the cardboard. The lines correspond to those places where the glasses are tangent. **Figure 7a** shows a square pattern of cardboard inserts, and **figure 7b** illustrates a hexagonal pattern for packing (as illustrated previously by the kernels of corn). Students could examine each packing solution and determine which way uses less space. Clearly, the hexagonal packing conserves the most space because the circles can be packed together more tightly.

Have the children discuss why square packing is the better way to ship fragile objects (fewer points of contact), whereas hexagonal packing is more useful and efficient when no separators are needed (Benenson and Neujahr 2002). Insights about shape arrangement can give children a new perspective for viewing the world and can inspire them to do further research. Can they find other examples of such packing? For instance, hexagonal packing examples might include timber on a logging truck (see **fig. 8**), straws in a box, firewood in a woodpile, or marbles in a jar. Examples of square packing could include boxes of chalk or light bulbs as well as egg cartons.

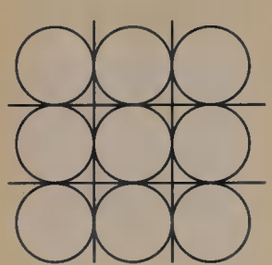
Another widespread use of a constructed shape is the *golden rectangle*. Thought to be the most eye-pleasing rectangle because of its ratio of length to width (approximately 1.618), it looks neither too thin nor too wide. Students can hunt for golden rectangles in their school and home environments. Ask them to first predict if a particular rectangle is golden, then measure the dimensions, and finally divide the length by the width to see if they obtain the ratio of 1.6. They will find that many picture frames are golden rectangles, such as a three-by-five-inch and a five-by-eight-inch ($5 \div 3 = 1.67$, and $8 \div 5 = 1.6$). Teachers might share examples from the art world, such as the Parthenon in Greece and some of Piet Mondrian's abstract works (Queen's University 2006).

Parts of the human body are said to reflect this golden ratio, too. One example is a person's overall height compared to the distance from the floor to the person's navel. Dividing the distance to the navel into one's total height obtains the ratio of approximately 1.6. Observe, too, how companies use this well-proportioned rectangle in their packaging. Credit cards and some food packaging reflect this ratio. One of the more interesting locations for this rectangle is on the faces of analog clocks and watches (Garland 1987). When these time pieces are advertised, they are usually set at 8:20 or 10:10, suggesting that these times are the most visually appealing because our eyes tend to construct a golden rectangle by using the hands of the clock as partial diagonals (see **fig. 9**). Invite students to find examples of watches on display and then create such rectangles to see if their findings confirm this idea. Such investigation helps children see how shapes with this special ratio are purposely designed for commercial reasons.

Figure 7

Packing drinking glasses shows an interesting application of circles from the commercial world.

(a) Square packing is a safer solution for fragile round objects.



(b) Hexagonal packing, however, uses less space.

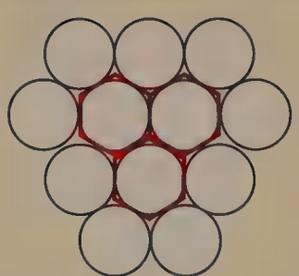
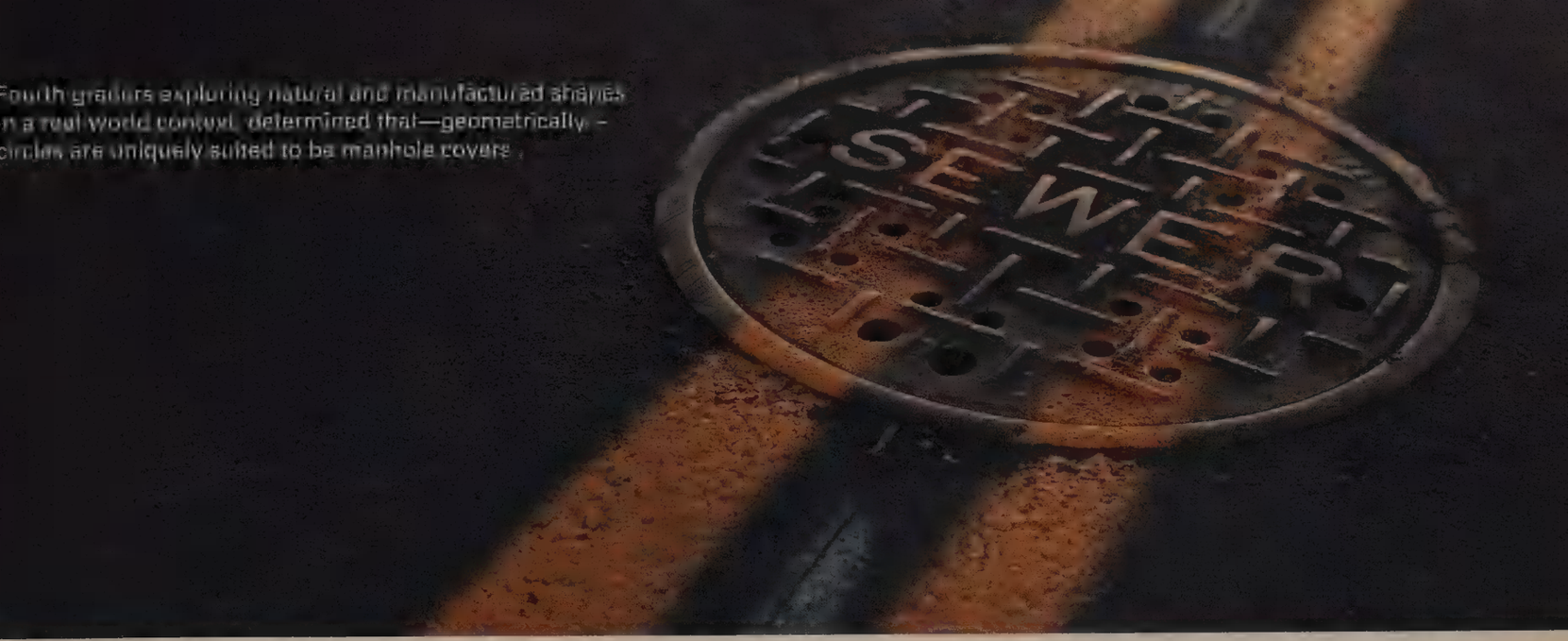


Figure 8

Logs on a truck are another example of hexagonal packing.



Fourth graders exploring natural and manufactured shapes in a real-world context determined that—geometrically—circles are uniquely suited to be manhole covers.

Reasons for shapes also abound in the field of architecture. Children can gain perspective about the shape of a house by engaging in an experiment that highlights the relationship between perimeter and area (Whitin 1993; Zaslavsky 1989). Set the challenge by saying, "Pretend you live in a village in which you have limited resources, and you must make almost everything yourself. Let's pretend that this piece of string (32 cm in length) represents the amount of material you have available to build a house. How can you use this string to encompass the greatest possible area?" The children can outline their shapes on a piece of centimeter paper and then count the squares to compute the area. They will notice that the area becomes the greatest as the shape begins to approximate a circle. This important relationship can lead them to further inquire about constructing and using round homes or buildings (Grifalconi 1986). Some examples include igloos, tepees, yurts, and sports arenas.

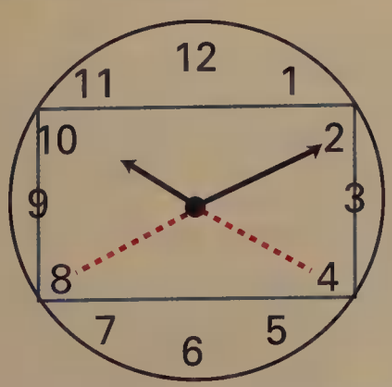
The relationship of perimeter to area can be extended into three dimensions as well. Spheres yield the greatest possible volume for a fixed amount of surface area (as the soap bubbles illustrated earlier). Given this relationship, why aren't soup containers spherical? Even children realize how cumbersome it would be to have spheres rolling around the kitchen shelves! However, the next best shape is a cylinder, which yields the greatest volume among prisms with the same base perimeter. To test this idea, children can bend a piece of cardstock into a cylinder and fill it with rice or sand and weigh the amount as a measure of its volume. They can fold another piece of cardstock of the same size into a rectangular solid or a prism and compute those volumes as well.

Once children have proven that the cylinder shape yields the greatest volume for a given surface area, they can look for everyday applications of this principle. For instance, various canned goods are packaged in cylindrical containers. Children can also draw on their knowledge of circles to justify the shape of other examples. Because a circle is defined as a set of points equidistant from a center point, children can reason that a cylinder is a shape that is least likely to burst because the pressure on its contents is equally distributed. This same principle explains why the hatches on a submarine are circular (Gates 1995). Silos are another example; more grain can be stored in this shape, and the continuously curved side allows the grain to fill more completely, minimizing the potential for a pocket of air to develop and allow bacteria to grow.

These examples demonstrate how objects from nature and manufacturing can provide interesting cross-disciplinary investigations. **Table 2** summarizes some main mathematical content embedded in these various real-world applications.

Figure 9

The hands of a clock set at 10:10 or 8:20 suggest a golden rectangle.



Why Explore?

The activities that we describe suggest many benefits to understanding the reasons for shapes:

- Subject fields become integrated. The examples cited in this article demonstrate that children learn not only important mathematical principles during such investigations, such as symmetry, angle measurement, ratio, and area, but also foundational concepts in other disciplines. For instance, children learn such scientific concepts as adaptation and economic principles of cost-benefit considerations involved with packaging goods and designing products. Interdisciplinary investigations allow children to make better connections in their learning.
- Learners become more critical consumers. When students realize that corporate advertising personnel use shape to influence a buyer's purchasing decisions, they become more astute in analyzing the decisions they make as consumers. For instance, which minivan really has the most usable storage space? Which shapes in the grocery store give the illusion of holding more? (Executives are reluctant to change product prices but will often alter the shape, size, and volume of a company's container instead.)
- Students appreciate and understand geometric elements of design in everyday manufactured objects. There are reasons why manhole covers are round and coat hangers are isosceles triangles. Examining the reasons for shapes sparks a curiosity about the commonplace and fosters a spirit of reflection and inquiry about our natural environment. Why are beach pebbles smooth? Why are bird eggs ovoid? Vivian Gornick's words echo this inquisitive stance toward the world:

> The natural biologist walks through a city park, across a suburban lawn, past an open shopping mall and is half-consciously wondering: Why two leaves instead of three? Why pink flowers instead of white? Why does the plant turn this way instead of that way? Such ruminations go on without end in the scientist's mind, a continuous accompaniment to the rhythm of daily life. (1983, pp. 38–39)

In summary, the question, "Why are things shaped the way they are?" provides an exciting, rewarding avenue to explore how mathematical concepts intersect with the work of scientists, engineers, artists, and others in the world beyond the classroom.

Table 2

Mathematical Content of Real-World Shapes

Real-World Applications	Mathematical Content
Manhole covers, umbrellas, submarine doors	Attributes of a circle
Soap bubbles, soup cans	Surface area, volume
Honeycombs	Tangents, similarity, angles, tessellation
Cracking	Angles; horizontal, vertical, and perpendicular lines
Body and plant shapes	Bilateral and radial symmetry
Coat hangers	Bilateral symmetry, angles
Packaging	Tangents, area
Golden rectangle	Ratio
Architecture	Perimeter, area, polygons

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CALL FOR MANUSCRIPTS FOR THE OCTOBER 2010 FOCUS ISSUE

Teaching Mathematics in a Flat World

Author Thomas Friedman's "flat world" phrase points to our new century's globally connected nature, exponential technical advances, and instantaneous communication across the planet. Former U.S. Secretary of Education Richard Riley keenly observed, "We are preparing students for jobs that do not exist, using technologies that have not been invented, in order to solve problems we cannot even imagine at this time."

Teaching Tomorrow's Citizens Today

As we hurtle through this rapidly advancing flat world, what does it mean to "know," "do," and "teach" math? Attempting to prepare students for a dynamically changing environment, what mathematical knowledge and essential understanding do elementary teachers need today to help them better educate the workforce and leaders of tomorrow?

The *Teaching Children Mathematics* Editorial Panel is seeking articles to address themes related to the mathematical knowledge of these teachers. We strongly encourage submissions that reflect the value of teacher knowledge directly within the classroom and its influence on student learning and achievement. Moreover, we welcome articles related to preparing preservice teachers as well as practicing teachers. The following topics are to guide—not limit—authors in addressing one or more such issues.

Teachers' Essential Mathematical Knowledge

- Identify the skills, concepts, and dispositions elementary teachers must know to promote both the procedural fluency and the conceptual understanding of their students.
- Provide evidence regarding content areas or specific topics that enable teachers to become more confident and competent in teaching mathematics.
- Describe the *essential* mathematical understandings necessary for elementary teachers and how these may differ from typical mathematics requirements.
- Explain how these essential concepts allow teachers to empower *all* their students with the mathematical skills and innovative problem solving needed in today's flat world.
- Demonstrate the connection between teachers' essential knowledge and the elementary curriculum in state or national standards or within NCTM's *Curriculum Focal Points*.

How to Best Learn

- Provide examples of innovative courses or professional development programs that have been successful in increasing teacher content knowledge and understanding.
- Describe effective, collaborative efforts—among school districts, institutes of higher education, and other agencies—for preparing elementary teachers to teach mathematics.
- Explain obstacles to increasing teachers' mathematical knowledge and how they have been effectively addressed.
- Clarify the importance of "mathematical habits of mind" and pedagogy in relation to learning and teaching mathematics in a flat world.
- Describe technology tools and other resources that support teachers in their work and in their acquisition of essential knowledge.

Impacting the Classroom

- Explicate the effects of an increase in a teacher's mathematical knowledge on the teacher's curriculum selection, teaching practices, and assessment choices.
- Discuss the connections between teachers' mathematical knowledge and student learning and achievement in mathematics.
- Describe specific classroom examples or experiences that demonstrate the importance of teacher knowledge in learning a particular mathematical concept or idea.
- Explore beliefs that teachers hold with regard to the value and type of mathematical knowledge needed in their classrooms.
- Explore the roles of math coaches and elementary math specialists in advancing the understanding of mathematics for both teachers and students.
- Examine the role of teacher collaboration in aiding teachers to reflect on and improve their mathematical knowledge and classroom practice.
- Discuss the level of mathematical understanding required for teachers to successfully address the needs of diverse learners and differentiated instruction.

Manuscripts should not exceed 2500 words; include figures and photographs at the end. Submit completed manuscripts to *Teaching Children Mathematics* by accessing **tcm.msubmit.net** by July 31, 2009. On the cover page, please state clearly that the manuscript is being submitted for the October 2010 *TCM* Focus issue. Author identification should appear only on the cover page. For manuscript preparation guidelines, visit my.nctm.org/eresources/submission_tcm.asp.

MORE THAN

Naturally curious about shapes, preschoolers explore materials, engage in activities, and work in collaboration with peers and teachers in early learning environments that support geometric thinking.

By Carmen S. Brown

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JUST NUMBER

In the mathematics center of their prekindergarten classroom, four-year-olds Joey and Corey examine squares and nonsquares drawn on paper. Joey asks Corey to point to a square on the paper. Corey points to all the shapes that are on the paper. Joey uses a child-sized pointer and aims it at a large square on the paper. He asks Corey, "Is this a square?"

When she replies yes, he asks, "Why?"

She answers, "Because it's a big one."

Joey continues with his questions and asks, "And why else?" Corey does not reply but points to another square on the paper.

Again Joey uses the pointer and traces the sides of the square. He tells Corey, "The four lines are connected." He puts his finger on a tilted square and says that the shape is a diamond that looks like a square. He has a questioning look on his face as he points to another square on the paper. He says, "This is a square." He repositions the paper 180 degrees, apparently to get a better view of the tilted square. He uses the pointer to trace the sides and verbally affirms that it is a square. Joey points to a shape that has three connected lines and two right angles and tells Corey that this is *not* a square because it is not connected. He physically uses the pointer to draw in the missing line that would make it a square.

Developing Geometric Knowledge

Children begin learning mathematics well before they enter elementary school (NRC 2001). Starting from infancy and continuing throughout the preschool period, they develop a base of skills, concepts, and misconceptions about numbers and mathematics. The state of children's mathematical development as they begin school both determines what they must learn to achieve mathematical proficiency and points toward how that proficiency can be acquired (p. 157). Most children enter preschool with some informal knowledge of geometry. Young children are exposed to geometry in the daily context of their natural environment. The rectangular bedroom door, the oval dining room table, and the kitchen clock's circular face are some examples of objects that children see in a typical home environment. Youngsters begin developing concepts of shape before formal schooling. They explore geometric ideas as they notice shapes and compare them to familiar objects during play and other daily activities. The earliest sign that children understand shape is their ability to match basic shapes of the same size and orientation and then compare basic shapes to real-world objects.

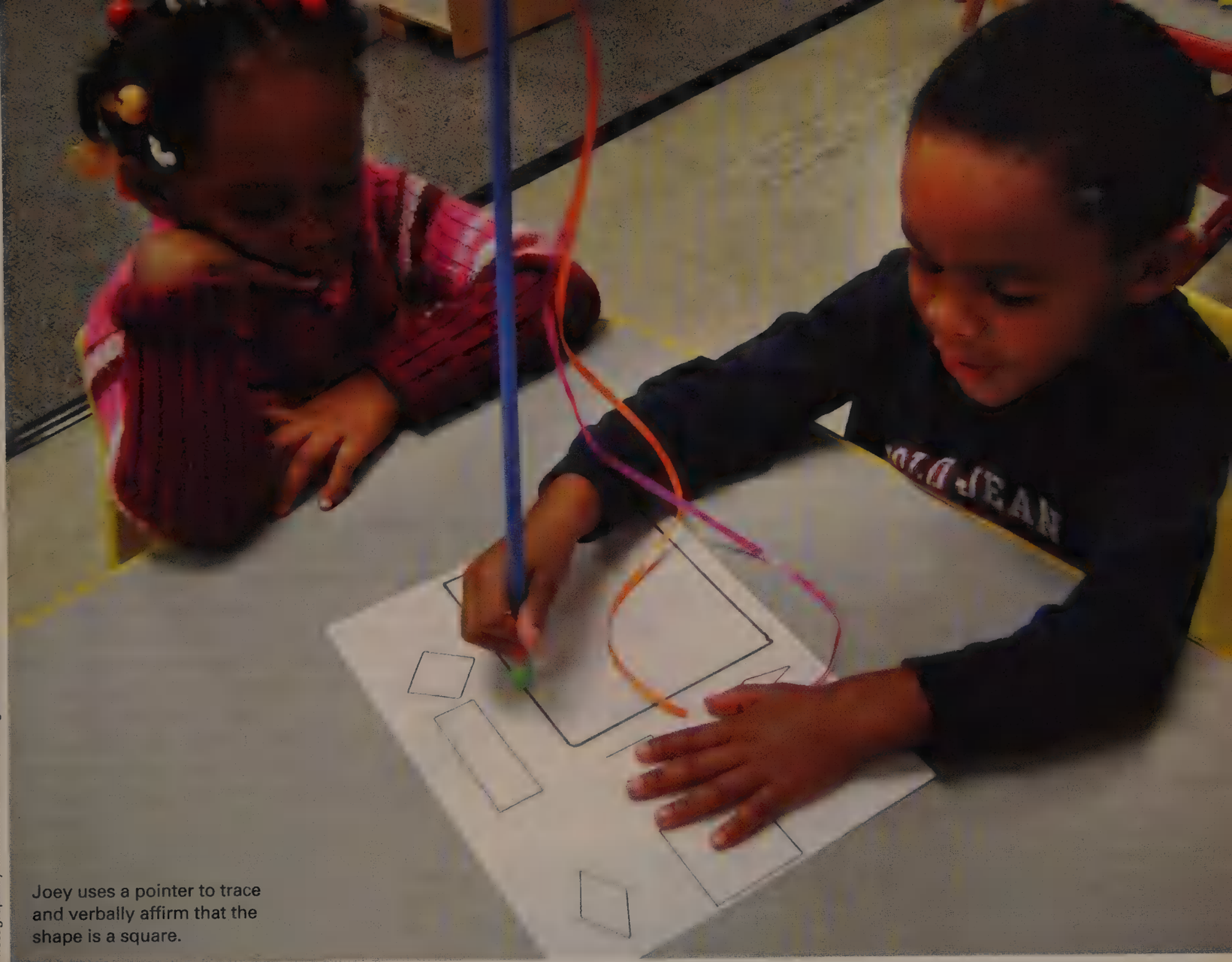
For children, geometry begins with play. Building blocks offer one illustration of play's significance for geometric learning. As children build with blocks, they gain experiences with the various ways that objects can be related. Such experiences become the foundation for a multitude of geometric concepts far beyond simply matching and sorting. Standard unit blocks and other building materials, such as connecting or foam blocks, give children entry into a world where objects have predictable similarities and relationships (Ginsburg, Inoue, and Seo 1999). With such materials, children reproduce objects and structures from their daily lives and create abstract designs by manipulating pattern, symmetry, and other elements (Edwards, Gandino, and Forman 1993). Children perceive geometric notions inherent in the blocks (such as two square blocks as the equivalent of one rectangular unit block) and the structures they build with them (such as symmetrical buildings with parallel sides). Over time, children can be guided from an intuitive to a more explicit conceptual understanding of geometric concepts (Shane 1999). Through continuous exploration of their environment, children constantly experiment and begin to make sense of previously unknown concepts. A learning environment with opportunities—to make choices; voice authentic questions and concerns; connect prior knowledge to acquired information; and be successful as they explore, experiment, and discover—allows students to develop ideas in an appealing and engaging atmosphere.

The Environment's Role

Children are naturally interested in shapes and spatial ideas. Geometric learning builds on their inquisitiveness and enthusiasm and challenges them to explore ideas about shapes: their characteristics, spatial relationships, and symmetry. Early childhood environments should provide opportunities for children to explore materials, engage in activities, and work in collaboration with peers and teachers to construct their own knowledge of the world around them. Teachers can organize the environment to encourage students to explore shapes and their characteristics by providing pictures of various typical and atypical shapes in different sizes and orientations throughout the classroom. Labeling pictures—including everyday objects such as an octagonal stop sign and square floor tiles—with the names of specific shapes facilitates sight word recognition. Quality instruction occurs in environments that feature the following three characteristics:

1. Use of rich geometric language. Using words such as *lines*, *corners*, and *angles* to describe shapes gives children the opportunity to apply geometric terms in their play. Geometric phrases allow children to actively participate in conversations when they ask and answer questions, therefore increasing their language and literacy development and enhancing their self-esteem. Children have the opportunity to play with words and make up stories as they begin to comprehend the meanings of the new terms.

The opening vignette depicted Joey's confidence in describing shapes, his ability to verbalize, and, more important, his explanations of why the shapes on the page were or were not squares. This activity during free-choice time allowed Joey and Corey to play with their newly acquired vocabulary in a setting that was neither intimidating nor teacher directed. As a cooperative learning experience, the two children worked together to determine what was or was not a square, thus building their social skills.



Joey uses a pointer to trace and verbally affirm that the shape is a square.

Photograph by Carmen S. Brown; all rights reserved

2. Support of children's thinking. Playing games such as shape bingo or lotto with children allows them opportunities to match, sort, and compare as they begin to apply knowledge to new theories. Children explore cause-and-effect concepts when they compose and decompose shapes. Investigating and experimenting with shapes and shape puzzles also gives children many experiences using problem-solving techniques.

While working in the mathematics center, Sherice and Michael explore pattern blocks. Sherice takes out one pattern block at a time, naming each shape as she removes it from the container. As she places them on the table, she appears to be exploring and familiarizing herself with the names of the shapes. There is no evidence that she is attempting to construct a pattern or picture.

In contrast, Michael takes the pattern blocks out of the container and quietly places triangles, trapezoids, and hexagons in a simple arrangement. What he has created is unclear, but he walks over to the teacher assistant and points to his newly constructed picture. When she asks him what he has made, he replies, "I made a robot."

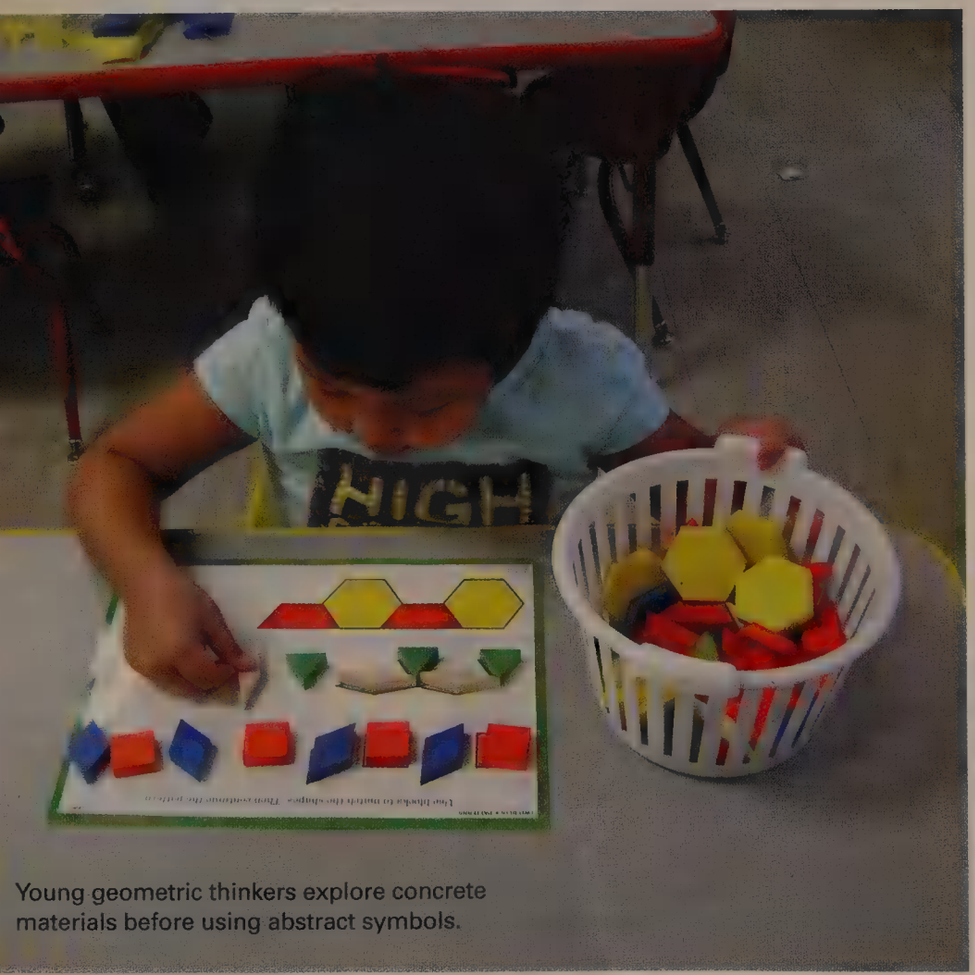
To become geometric thinkers, children need to explore, manipulate, and experiment with concrete materials before they can be expected to use abstract symbols. Children also explore part-whole understanding by using pattern blocks to cover predetermined shapes.

3. Nurture of children's explorations and ideas. Providing opportunities and authentic activities encourages children to experiment and share ideas. Environments that support children's curiosities allow for risk taking in a nonthreatening atmosphere. Such environments support children's individuality and their need to interact with peers.

For example, taking a shape walk allows children to look for shapes in both the natural and

designed world. Youngsters can see the shapes that they are learning about in the classroom as they explore the world around them. Transferring knowledge to authentic activities also supports their cognitive development. Shape walks give children opportunities to discover and develop concepts and relationships that have a real-world connection and are relevant to them. During shape walks, children can point out, name, and discuss the shapes they see in the objects they encounter. This supports their language development by permitting them to use their new language in a familiar context. As children search for shapes embedded in everyday objects, they practice following directions and displaying observational skills.

Developing such a rich environment within the preschool classroom requires careful attention to the types of materials available to students. Following are examples of materials that foster geometric learning and development in the early childhood classroom.



Young geometric thinkers explore concrete materials before using abstract symbols.

Photograph by Carmen S. Brown; all rights reserved

Manipulative centers

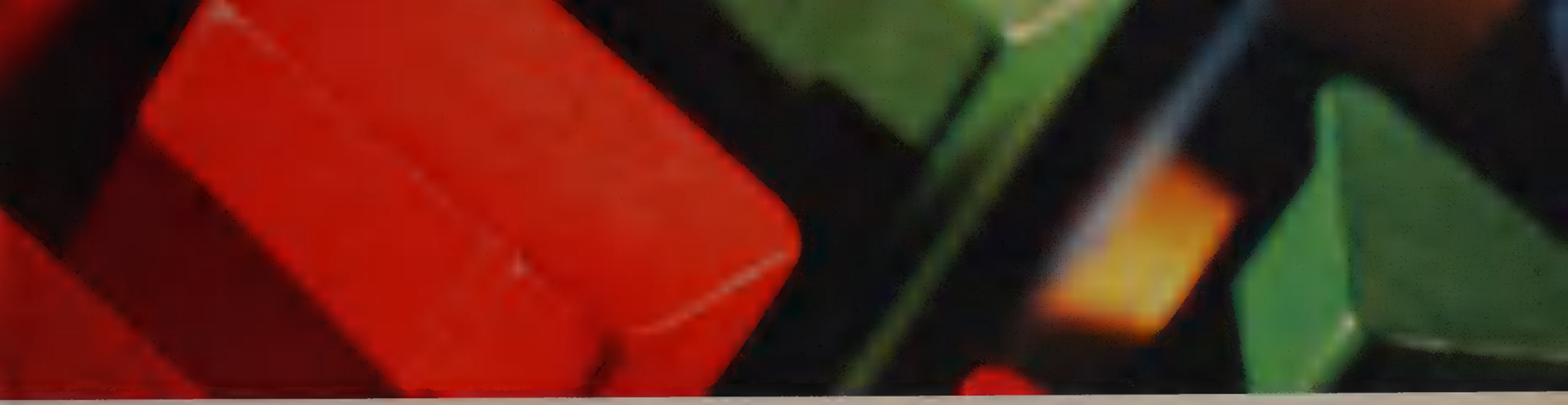
Pattern blocks, puzzles, and tangrams challenge children's inquisitive minds, tap into their spatial and reasoning skills, teach shape recognition and spatial relations, develop fine motor skills, and open a whole world of possibility. Putting puzzles together is a fun, engaging way for children to explore how the pieces are related to one another and to determine how many different combinations of the shapes in the puzzle set can make another given shape. Pattern blocks in a classroom's manipulative area allow children to use the six common geometric shapes (square, triangle, hexagon, trapezoid, and two different-sized rhombi) to build patterns and solve problems. Tangrams give youngsters the chance to explore shape characteristics as they combine the seven Chinese puzzle pieces into various pictures.

Art centers

Providing children with a variety of shapes that they can trace, copy, cut, and paste allows them to become familiar with shapes and also enhances their fine motor skills and eye-hand coordination. When children draw, paint, and make collages, they experiment with color, line, shape, and size. Art also fosters vocabulary development as children learn and use related geometric vocabulary.

Block centers

To facilitate matching and naming shapes, provide a variety of unit and foam blocks. Children learn much through their own spontaneous, independent block play. Playing with blocks encourages children to create and construct objects. For example, the process of matching and sorting blocks builds geometric thinking. As children build, they become aware of height, length, volume, and weight. In their play, they also begin to use comparison terms such as *longer than*, *shorter than*, and *the same*



length as. To facilitate transitions during clean-up time, trace and cut out shapes from colored contact paper and place them on the shelves where students can match the blocks as they put them away.

Teaching geometry concepts in preschool can be fun and inspiring for teachers as well. Early childhood instructors can support children's geometric learning through scaffolding, by providing exciting new challenges, and by valuing children's investigations and explorations (Hawthorne 1992). As young students begin developing geometric skills and knowledge, teachers may discover additional creative ways to start laying the foundation for early geometric learning while ensuring that each lesson and activity is both enjoyable and educational.

Conclusion

Children's mathematical thinking in geometry allows them to make a connection to number (NAEYC 2002). When children say that a square has four sides, encourage them to verbally count (counting and number names), to physically show the four corners (one-to-one correspondence), and to tell how many (cardinality). As children begin to recognize shapes, specific shape characteristics, position, and transformations, they develop spatial reasoning. Give students the opportunity to count how many scoops of water it takes to fill a cup. Encourage them to guess how many steps it takes to get from the cubby area to the door. The knowledge they acquire when they search for the answers to these questions supports the foundation for understanding not only their spatial world but also other topics in mathematics, such as patterning and seriating.

Some children's competence with geometric and spatial concepts exceeds their number skills (Clements et al. 1999). Expanding on these strengths nurtures enthusiasm for mathematics and provides a context in which to develop number and other mathematical concepts. Providing an early childhood environment in support of geometric learning aids children's language and cognitive development. They can express themselves using their newly acquired language skills and become more competent in their sentence structure. They approach problems with more flexibility and persistence, attempting more than one solution to a problem. Cooperating and communicating with their peers enhances their self-esteem. Early childhood environments that support and nurture geometric thinking in these ways provide a foundation for young children's future academic success.

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Christine Gallego, Dora Saldamando, Grace Tapia-Beltrán, Karolyn Williams, and Thomas C. Hoopingarner

Watch It Grow

"Math by the Month" activities are designed to engage students to think mathematically. Students may work on the activities individually or in small groups, or the whole class may use these as problems of the week. Because no solutions are suggested, students will look to themselves for mathematical justification, thereby developing the confidence to validate their work.

The following activities are designed to help you and your students investigate growth through mathematical investigations involving number sense and operations, statistics, data collection, communication, and representation.

WEEKLY ACTIVITIES

WATCH IT GROW K–2

APRIL 2009

6 **What makes plants grow?** Ask your teacher for radish seeds, soil, and a cup with a small hole in the bottom for water drainage. Plant your seeds according to package directions. What does your plant need in order to grow? Measure, draw, and write every day about what you notice about your plant's growth.

13 **Plant trees on special days.** Imagine that 3 birthday trees were planted in celebration on the day that you were born. Every year on your birthday, your family plants 1 new tree. How many trees had been planted in your honor when you were one year old? When you turned 2 years old? How many trees will there be after your 15th birthday? Use numbers, words, pictures, and manipulatives to show your thinking.

20 **Let's pretend** that weather data you found about high temperatures for a week are missing some information. (Look at the chart below.) You notice a pattern, though. Describe the pattern to your partner.

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
82° F (high)	85° F (high)	88° F (high)				

Now use the pattern to fill in the missing data. Then make a line graph of the temperatures. What do you notice about your graph? Show your graph to your partner and talk about what you both noticed.

27 **A starfish** has 5 "arms" that give it a shape resembling a 5-point star. Counted together, 2 starfish have 10 points. How many points do 5 starfish have? 8 starfish? How can you predict the number of points for any number of starfish? Show your thinking through writing, drawing, numbers, and manipulatives.

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Edited by **Dana Islas**, *who teaches kindergarten at Pueblo Gardens Elementary School in Tucson, Arizona. Readers are encouraged to submit problems to be considered for future "Math by the Month" columns to the department editor at dana.islas@tusd1.org. Receipt of the problems will not be acknowledged; however, those selected for publication will be credited to the author. Readers are also invited to submit creative solutions and adapted problems to* **tcm@nctm.org** *for potential publication in "Readers Exchange." Please include "Readers Exchange" in the subject line. Because of space limitations, responses beyond the 250-word limit are subject to abridgment. Submissions are also edited for style and content.*

WATCH IT GROW 3–4

APRIL 2009

What grows down when it grows up? In early spring (mid-March to early April in the Northern Hemisphere) plant 1/3–1/2 of a seed packet of any 2 root vegetables (carrots, turnips, beets, or radishes) in prepared garden soil according to package information. Keep a weekly journal, drawing and measuring your sprouts until you pull them out of the ground. At about 5 weeks, pull out 1 of each type. Compare the length of the plants' tops to each other and record the 2 measurements and their difference. Now compare the root lengths. Continue pulling samples, comparing, and recording on a weekly basis. Compare your findings from week to week. Write about some of your observations. What do you predict would happen if you left the plants in the ground longer? Compare your plants' looks, taste, size, and color to store-bought vegetables.

Grow your money. Olivia thinks that pennies are not worth very much, but Andrea suggests that she save them in her piggy bank. On the first day, Olivia deposits 1 penny; on the second day, she deposits 2 pennies; on the third day, 3 pennies. If she continues this growth pattern, how much money will she save in 10 days, in 20 days, and in 100 days? Organize the data in a t-chart.

How fast do webs grow? You notice an orb spider just beginning to construct a web. You decide to time the spider to learn how long she takes. In 3 minutes she finishes the first small circle of her web. In 5 minutes she finishes her second complete circle. And in 8 minutes she has 3 circles in her web. If she continues this pattern, how many circles will the spider complete in 12 minutes? How much time will it take for her to finish 7 circles? Show your work. What patterns do you notice? About how many circles will the spider complete in 1 hour? How do you know? Share with 2 friends how you got your answers and see if they have the same information. How much time will it take the spider to make 25 complete circles in her web?

A spring garden has 9 different colors of blooming flowers. Using exactly 3 colors and 3 different types of flowers, you want to make unique bouquets for each of your 25 classmates. Can you make bouquets for everyone? Show how you know. How many can you make? What pattern do you notice? Would the pattern work using 12 flowers with 3 colors in each bouquet? How do you know? What about 4 different colors in each bouquet? Make up a similar problem for your team.

WEEKLY ACTIVITIES

WATCH IT GROW 5–6

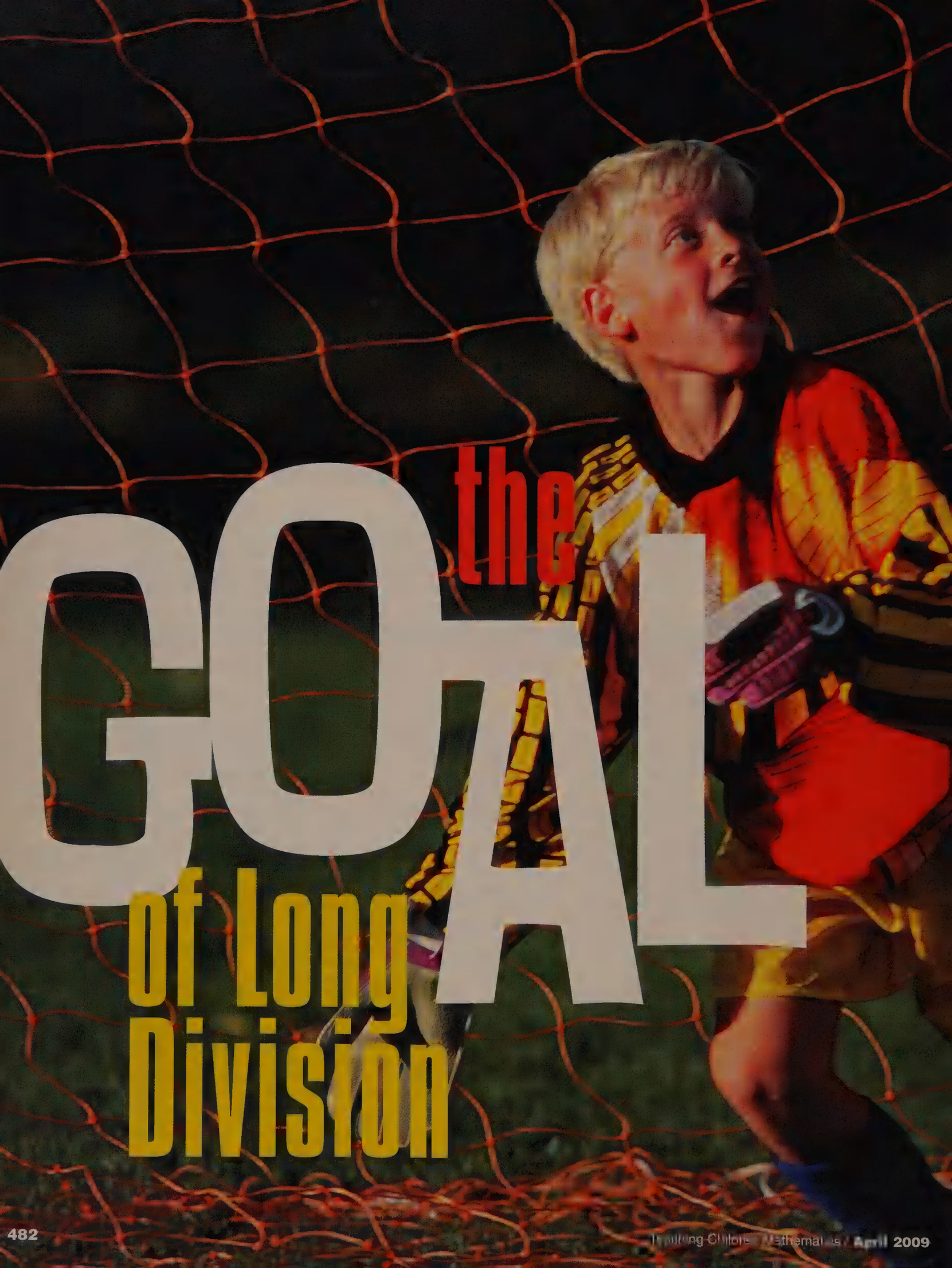
APRIL 2009

Choose your own allowance: Allowance plan A is $5 every day of the month. Allowance plan B is 1 penny on the first day of the month, 2 pennies on the second day of the month, 4 pennies on the third day of the month, and the amount keeps doubling every day until the month is over. Which plan would you want? Explain your choice.

Grow deposits. Your class is collecting money in a jar so that you can have a celebration. On the first day, a student deposits 1 quarter in the jar, and your teacher deposits 1 nickel. On the second day, another student deposits 1 quarter, and your teacher deposits 2 more nickels. Calculate how much money your class collects on the first day. If this pattern continues, how much money will you collect by the second, third, tenth, twentieth, and hundredth day? Create a t-chart to show your thinking. What is the growth pattern? Which numbers stay the same (are constant)? Which numbers change (are variable)? How should your class spend the money for the party?

A magical plant seed arrives in your mailbox with a note: "This is a money plant that will sprout a $1 bill for every 1/3 inch that it grows until it is full grown at 3 feet." You take the seed outside and plant it in a pot of soil. After a week of watering it, you notice that it has grown 1 1/2 inches. If it continues to grow at the same rate, how many weeks will it take to reach 3 feet high? How many $1 bills will it sprout? Use pictures, charts, numbers, and words to explain your answer.

Growing animals can be fun. Your family just bought a farm that has 20 pigs and 30 chickens. Every month you buy 20 more pigs and 30 more chickens. After several months, you count a total of 1120 animal legs on the farm. How many pigs and chickens have you added to the farm? How many months have passed? How many months will it take to have 2240 animal legs? How many of each animal will you have when you have 2240 animal legs? Use pictures, charts, numbers, and words to explain your answer. (Hint: Remember that the farm had 20 pigs and 30 chickens when you bought it.)



the GOAL of Long Division

Two key ideas help students develop the thinking that facilitates conceptual understanding of division: the role of place value in the quotient and the power of multiples of ten in determining the quotient.

For more than a quarter century, leaders in the mathematics education community have called for mathematics teaching to focus on conceptual understanding. Published materials, such as those of the Nufield Foundation and the Madison Project, provide early examples of this effort. A search through NCTM publications from the last two decades reveals continuing effort to focus instruction on conceptual understanding as the basis of computational skills. Yet many lessons and curricular materials published for primary and intermediate students continue to emphasize rote mechanical skills to produce answers. For many students, such arithmetic curricula contain major challenges, including the division algorithm. The instructional strategy for this algorithm frequently falls into a "goes into" mode and totally ignores the long hours spent developing students' prior knowledge and understanding of the powerful concepts of grouping and place value.

The advance of technology has caused many educators to question the time and energy expended for students to master the pencil-and-paper computation skills embodied in the long-division algorithm. In today's world, this mastery is truly a questionable goal. But understanding the conceptual infrastructure of the algorithm will add to students' technological proficiency.

The purpose of this article is to help readers reflect on the conceptual meaning buried within the division algorithm and to see how this understanding connects to students' prior knowledge.

Many primary-level materials and primary classroom teachers spend important time and effort helping students develop number sense. A vital part of such activities involves counting and grouping objects. Grouping activities may begin with observing qualitative characteristic (e.g., color, shape, or size), but such activities easily move toward using quantitative properties. These quantitative grouping activities contribute to developing a sense of place value and the concept of division. Having students determine the cardinality of a collection of discrete objects by forming groups of ten is an appropriate step toward understanding the structure of place-value numeration. Computational algorithms in our numeration system rely heavily on this conceptual structure of place value, so we should have it at the forefront when teaching computational processes.

Before addressing the long-division algorithm, let us examine the concept of division. Most primary mathematics programs include grouping activities involving conditions other than creating groups of tens, and they do not make specific mention of division. Students may be given a collection of objects—for example, twelve hearts (see **fig. 1**)—and asked to do any number of exercises. All these exercises, as well as a follow-up discussion of answers and procedures, lay the foundation and are critical prior knowledge for a solid understanding of division. If we reflect back to how knowledge of multiplication develops, exercises

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such as $2 \times 6 = 12$ are interpreted as "two sets of six objects equal a total of twelve objects." Notice how this gives a quantity (number of parts, 2), and each part has a quality (measure, 6). Connecting the aforementioned grouping activities to the quantity-quality interpretation of multiplication helps students realize the relationship between multiplication and division.

Students must grasp two distinct concepts of division: (1) the idea of partitioning a collection of objects or equal sharing of them (thus the names *partitive* and *fair sharing*), and (2) separating a quantity into sets of a specified size or measure (hence the names *measurement* and *repeated subtraction*). We cannot determine the specific division interpretation for a specific exercise such as $27 \div 6$ or $6\overline{)27}$. The situation's context determines which concept of division is involved.

On the basis of more than thirty-five years of teaching experience, including many lessons with intermediate-grade students, I can attest that when students are asked to model a division exercise apart from a specified context, some students will use the measurement approach and others will use the partitive approach. This difference suggests to me that students create a mental context for division problems. Because the goal of this article is for readers to reflect on the concepts embodied in the long-division algorithm, we will take a moment to reflect on examples of the two concepts. Examine the following situations:

Figure 1

Most primary mathematics programs include grouping activities involving conditions other than creating groups of tens.

Choose one of these problems to complete:

1. Draw three nonoverlapping loops around the hearts so that each loop contains the same number of hearts.
2. Draw loops that do not overlap around sets of hearts so that each loop contains four hearts.
3. Can the hearts be shared equally among three children?
4. Can the hearts be shared equally among five children?

Situation 1: Ted has a collection of 15 matchbox cars. He places the cars into sets, each containing 3 cars. How many sets of 3 cars can he make?

Situation 2: Janis has a box of 18 cookies. She separates the cookies into 3 equal-sized sets of cookies. How many cookies are in each of these sets?

The quantity, or total number of objects, in both situations is known. After this, the situations differ. In situation 1, we know how many objects each part has, but in situation 2, we know the number of equal parts needed. Hence, the unknown components of the two situations are different, and role playing with the models will require different actions. Both situations use the same three components: total number of objects, number of equivalent sets created, and number of objects in each equivalent set. This information is summarized in **table 1**, where yes indicates what we know.

Although the multiplication language of *multiplier*, *multiplicand*, and *product* has essentially been replaced with *factor*, *factor*, and *product*, the division terms *divisor*, *dividend*, and *quotient* remain in many textbooks. To help students make the connections between old and new vocabulary, we could use *factor*, *missing factor*, and *product* for division:

$$\begin{array}{r} \text{quotient} \\ \text{divisor}\overline{)\text{dividend}} \end{array} \quad \text{or} \quad \begin{array}{r} \text{missing factor} \\ \text{factor}\overline{)\text{product}} \end{array}$$

If the language of division is related to the information in the table, then for any division exercise, the total number of objects is the *dividend* or *product*. The number of sets is the *divisor* or *factor*, which makes the number in each set the *quotient*; or the number in each set is the *divisor* or *factor*, which makes the number of sets the *quotient*. Consider the following division exercise, which includes no context:

$$5\overline{)135}$$

If we focus our attention on the results of this computation, we can ask two possible questions, being sure that we do not overlook place value in the answers:

Situation 1: How many groups of 5 are there in 135?

Situation 2: How many objects are there per group when we separate 135 into 5 equivalent groups?

The next section provides sample questions that teachers can use while working toward students' conceptual understanding. After each question is an example of the type of information teachers should strive to obtain from students. Students' background knowledge may require that teachers ask additional probing questions in order for students to demonstrate the desirable thinking.

Conceptual Understanding

Working with situation 1 (a measurement model), the following exchange would be appropriate:

Teacher: Can 100 groups of 5 be created from 135 objects?
Student: No, that would require 500 objects.
Teacher: What does this mean with regard to the answer?
Student: The quotient will have no digit in the hundreds position.
Teacher: Can you get 10 groups of 5 when you have 135 objects?
Student: Yes, you need 50 objects to get 10 groups of 5, and 135 is more than 50 objects. There could be another 10 groups of 5. A total of 20 groups of 5 can be formed using 100 of the 135 objects.
Teacher: Could you create 30 groups of 5?
Student: No, that would require 150 objects.
Teacher: Good. What digit will we place in the tens position of the answer?
Student: Put a 2 in the tens position.

Now we must symbolically record the discussion as it relates to the algorithm. We represent the idea that 100 of the 135 objects have been placed in groups of 5 each and turn our attention to the objects *not* placed into groups of 5. After 100 objects have been placed into groups of 5, 35 objects remain:

$$\begin{array}{r} 2 \\ 5\overline{)135} \\ 100 \\ \hline 35 \end{array}$$

Teacher: What can we do with the 35 objects that remain?
Student: We can place them into 7 groups of 5.

Table 1

Total Number of Objects and Sets

	Total No. of Objects	No. of Sets	No. in Each Set
Situation 1	Yes	?	Yes
Situation 2	Yes	Yes	?

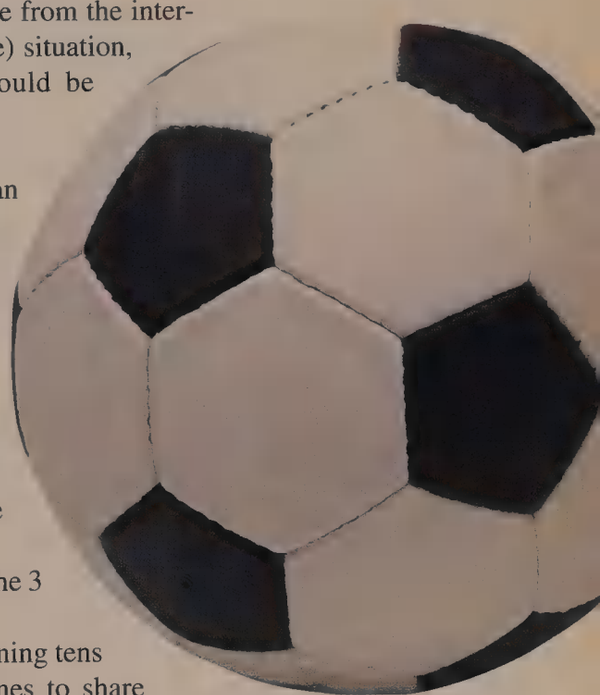
Teacher: How will we show this in the algorithm?
Student: The units place will have a 7 to denote the 7 groups of 5.

So, we created 27 groups of 5 from 135 objects:

$$\begin{array}{r} 27 \\ 5\overline{)135} \\ 100 \\ \hline 35 \\ 35 \\ \hline \end{array}$$

Have your students repeat the meaning of the answer in relation to the computation; that is, "There are 27 groups of 5 in 135 objects."

Considering the same exercise from the interpretation of the second (partitive) situation, the set of questions below would be appropriate:



Teacher: How many hundreds can we place in each of the 5 sets?
Student: None, because that would require 5 hundreds and there is only 1 hundred.
Teacher: How many tens can we place in each of the 5 sets?
Student: We can place 2 tens in each of the 5 sets. This will use 10 of the 13 tens in the dividend.
Teacher: What will we do with the 3 remaining tens?
Student: We rename the 3 remaining tens as 30 ones, making 35 total ones to share equally among 5 sets:

$$\begin{array}{r} 2 \\ 5\overline{)135} \\ 100 \\ \hline 35 \end{array}$$

Teacher: How many ones will each of the 5 sets have?

Student: There will be 7 ones for each of the 5 sets:

$$\begin{array}{r} 27 \\ 5\overline{)135} \\ \underline{100} \\ 35 \\ \underline{35} \end{array}$$

Regardless of which concept of division we use, focusing students' attention on the place values in the quotient and on their number sense can prove useful in any division exercise. Such focus also strengthens the student's estimation skills and ability to determine an answer's reasonableness—effective skills when using technology to solve division problems.

Now consider the following exercise:

> Will the quotient of $17\overline{)10{,}353}$ have a nonzero entry in the ten-thousands position or in the thousands position?

Figure 2

Charts help students understand division problems.

$17\overline{)10{,}353}$	1 ten-thousand is	170,000
	2 ten-thousands is	340,000
	4 ten-thousands is	680,000
	6 ten-thousands is	1,020,000
$\begin{array}{r} 6 \\ 17\overline{)10{,}353} \\ \underline{10{,}200} \end{array}$	1 hundred is	1,700
	2 hundreds is	3,400
	4 hundreds is	6,800
	6 hundreds is	10,200
$\begin{array}{r} 60 \\ 17\overline{)10{,}353} \\ \underline{10{,}200} \\ 153 \end{array}$	1 ten is	170
	2 tens is	340
	4 tens is	680
	6 tens is	1,020
$\begin{array}{r} 609 \\ 17\overline{)10{,}353} \\ \underline{10{,}200} \\ 153 \\ \underline{153} \end{array}$	1 one is	17
	2 ones is	34
	4 ones is	68
	6 ones is	102

Students who reason as demonstrated above should respond with a no and support their response by explaining that placing a 1 in the ten-thousands place requires 17 ten-thousands in the dividend, or 170,000; but the dividend has only 1 ten-thousands. Likewise, placing a 1 in the one-thousands place requires 17 thousands in the dividend, or 17,000; but the dividend has only 10 thousands:

$$17\overline{)10{,}353}$$

Using a sidebar chart (see **fig. 2**) is extremely helpful when tackling a division problem. Discussing how to construct the chart and how to use it to get the place-value digits of the quotient reinforces place value and number sense and makes determining the digits in the quotient more than guesswork; the digits' positions are clearly identified. As we determine successive place-value digits, we use the same chart but drop a zero from the right of the result and adjust the place-value label:

Teacher: Will there be a nonzero digit in the hundreds position of the quotient?
Student: Yes! The dividend has 103 hundreds, or 10,300 in it. Looking at the chart, 6 hundreds would require 10,200. So, the digit in the hundreds position is 6. This leaves 1 hundred remaining, not enough for another hundred, which requires 1700:

$$\begin{array}{r} 6 \\ 17\overline{)10{,}353} \\ \underline{10{,}200} \end{array}$$

Teacher: What becomes of the 100 that remains?
Student: The 1 hundred will be renamed as tens, and the exercise has 15 tens remaining.
Teacher: What digit goes in the quotient's tens position?
Student: By looking at the chart, we can see that a 1 in the tens position requires 17 tens, or 170. So, we record a zero to show that no tens can be recorded in the quotient. We rename the 15 tens as 150 ones along with the 3 ones, yielding 153 ones:

$$\begin{array}{r} 60 \\ 17\overline{)10{,}353} \\ \underline{10{,}200} \\ 153 \end{array}$$

Teacher: What digits go in the ones position?
Student: Referring back to the chart, a 6 in the ones position requires 102 ones. An 8 requires 102 plus 34, or 136. It requires more than 8 but less than 10,

so it must be 9. A 9 in the ones position requires 136 plus 17, or 153 ones:

$$\begin{array}{r} 609 \\ 17\overline{)10{,}353} \\ 10{,}200 \\ \hline 153 \\ 153 \\ \hline \end{array}$$

Reflection

Regardless of whether division questions are structured to reflect the measurement or partitive model, approaching questions in this manner builds on and reinforces students' prior knowledge of number sense, place-value concept, multiples of ten, multiplication and division connections, the basic concept of division, and mental computational skills, each of which contributes to effective technology use. Constructing and effectively using sidebar charts eliminates students' need for the guess-and-check strategy to identify partial quotients (missing factors) in division exercises. If students learn to use the 1, 2, and 4 multiples of the divisor (the given factor), they can easily construct all other single-digit multiples using mental computations. The skills to compute mentally and construct multiples of 10 for place-value adjustments provide the power to master the division algorithm. Such knowledge is extremely useful when students must estimate computation results on their calculators.

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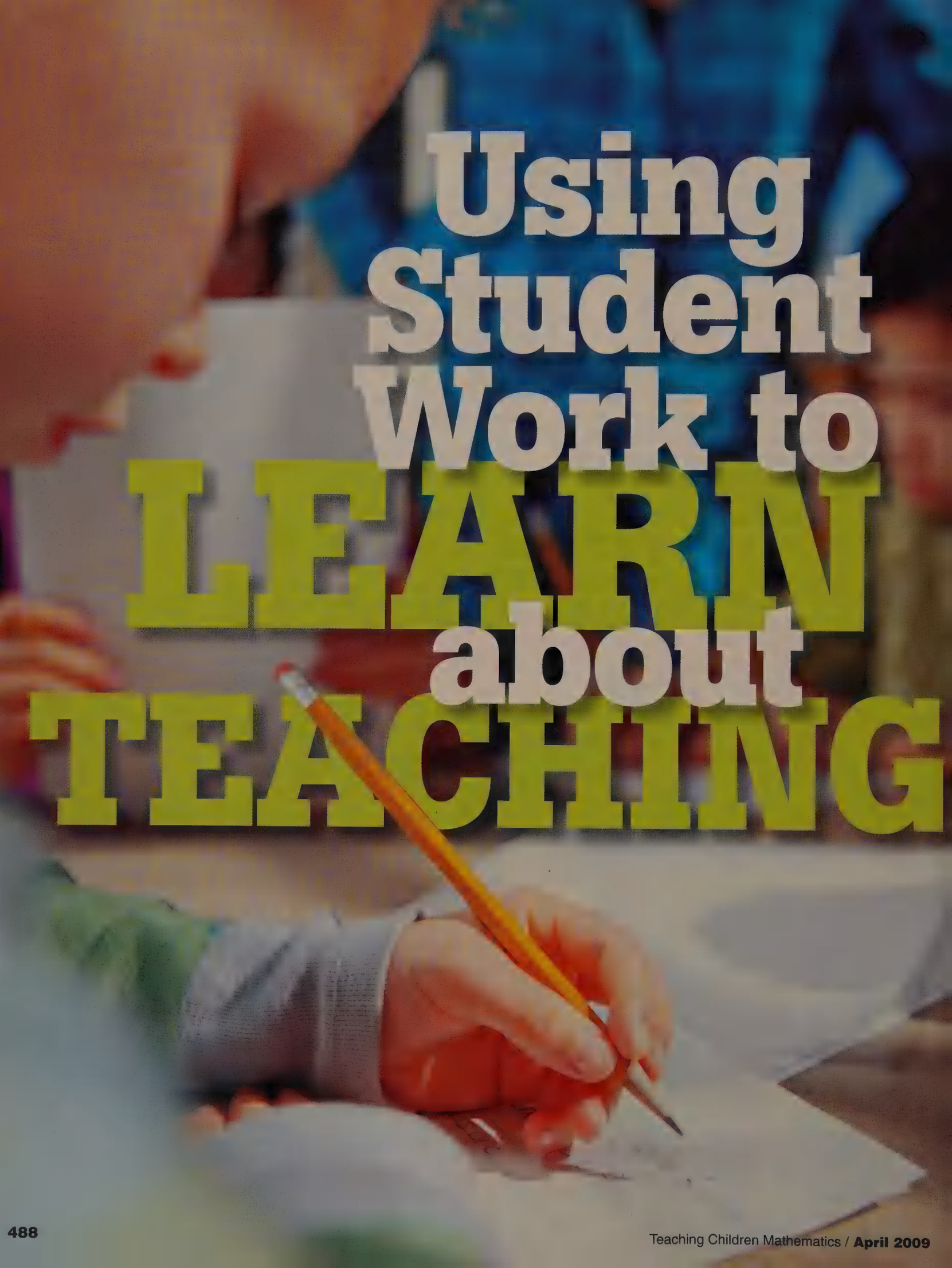


Using Student Work to LEARN about TEACHING

Many teachers struggle with strategy development. Examining student work provides helpful learning opportunities.

Analyzing student work can help teachers improve the teaching and learning of mathematics. Kazemi and Franke state:

> Discussions of student work allow teachers to raise their own questions about practice and to deliberate about what it is that they want and need students to learn. Such professional inquiry can allow teachers to form generalizations and conclusions from the particular instances of students' reasoning that would guide future interactions in their classroom. (2003, p. 6)

Fortunately, the four authors have experienced professional development that supports teachers' professional growth and provides tools to begin the challenge of analyzing student work. However, as they began to assume leadership roles in their schools and district, they recognized that other teachers had not had opportunities to learn how to examine student work to enhance teaching and learning. As a result, they heard comments such as, "I have this good work from my students, but I don't know what to do with it."

Many roadblocks prevent teachers from routinely and effectively using student work: limited time, inexperience with standards-based curricula, insufficient mathematical knowledge, and limited teaching experience. As a result of their own observations, the authors initiated a process for teachers in their schools to come together to look at student work. Beginning to design these opportunities, they considered questions posed by colleagues about using strategies that appeared in the student work:

- Should I expect students to use the same strategies to solve addition problems all year long?
- What strategies do students use to solve addition problems at the next grade level?
- How do I move students toward the use of more efficient strategies?

From their conversations came the realization that many teachers struggle with strategy development within and across grade levels. The authors decided that focusing on strategy development when examining student work would provide helpful learning opportunities for teachers. They created two professional development experiences based on teachers' needs within their schools. Marilee Cameron and Vickie Rorvig had already begun this task and had worked with a team of teachers. After hearing of their successful experience, Jenine Loesing modified the task and used it with the entire faculty in her school. This article examines the professional development activities that the team facilitated in their respective schools related to examining students' addition strategies in grades K–5 and the benefits resulting from the process.

Case 1

Cameron and Rorvig teach at Derby Ridge Elementary, a Title I school with a population of about seven hundred K–5 students. Its staff consists of thirty-four regular classroom teachers, eight special education teachers, and two full-time and one part-time Title I teachers. To support this large staff in implementing the Investigations in Number, Data, and Space curriculum series, the school initiated a professional development plan that involved a vertical mathematics team. One teacher from each grade level and a special education teacher volunteered to participate on the team and meet once a month. The

By Marilee Cameron, Jenine Loesing, Vickie Rorvig, and Kathryn B. Chval

Marilee Cameron is a kindergarten teacher at Derby Ridge Elementary School in Columbia, Missouri. Jenine Loesing is a mathematics coach and former first-grade teacher at Paxton Keeley Elementary School in Columbia. Vickie Rorvig is a mathematics coach and former third-grade teacher at Derby Ridge Elementary School in Columbia. Kathryn B. Chval is an assistant professor at the University of Missouri. All the authors are interested in improving the teaching and learning of mathematics in elementary classrooms.

purpose of the vertical team was to discuss issues related to developing mathematical thinking in the school and to help teachers make informed instructional decisions. Occasionally, an administrator and the district mathematics coordinator attended meetings to observe and to provide guidance.

During a vertical team meeting, a colleague, Anna, expressed her difficulty in understanding some of the addition strategies her students use to solve problems. She asked, "Is anyone else really confused by all the different strategies the children are using? Some of my students use strategies that fill a whole sheet of paper, while others seem to perform most of their thinking mentally. Am I the only one experiencing this same frustration? Is it normal to have such a diversity of strategies?"

As Anna shared, others began to express similar concerns. Matt added, "I understand and agree with Anna. At the same time, I'm struggling with another issue. I'm not sure if the strategies being used by my students are efficient for third graders. How am I supposed to know?" He continued, "If we don't know what an efficient strategy is for each grade level, how can we move [students] past inefficiency?"

Figure 1

These sample student problems illustrate vertical alignment across grades K–5.

K—Ann and Jack baked some cookies. Ann made 7 cookies, and Jack made 4 cookies. How many cookies did they make?

Grade 1—Ann and Jack baked some cookies. Ann made 17 cookies, and Jack made 14 cookies. How many cookies did they make?

Bill had 15 pieces of candy. He gave his sister 7 pieces of candy. How many pieces of candy does Bill have now?

Grade 2—Sue and Shawn are collecting jars of peanut butter for Meals on Wheels. Sue got 27 jars from the second grade, and Shawn got 38 jars from the third grade. How many jars of peanut butter did they collect?

Rachel had 35 animal stamps in her collection. Her friend had 27 stamps in her collection. How many more stamps did Rachel have than her friend?

Grade 3—Michael began a new book and read 87 pages on Saturday. He read 137 pages on Sunday. How many pages did he read on the weekend?

Beth had 98 pages left to read in her new book. She read 53 pages on Saturday. How many more pages does she need to read to finish her book?

Grade 4—Andrea drove 86 miles on Friday and 225 miles on Satuday. What was the total number of miles that Andrea drove on Friday and Saturday?

Beth had 198 pages left to read in her new book. She read 53 pages on Saturday. How many more pages must she read to finish her book?

Grade 5—Stephen had collected 398 stamps. His uncle gave him 245 stamps to add to his collection. How many stamps does Stephen have now?

Michelle decided to read 450 pages in her new book by Saturday. By Thursday evening, she had read 137 pages. How many more pages must she read to reach her goal of 450 pages?

As the conversation proceeded, Molly interjected, "I thought the whole point of a reform curriculum was to allow children to produce a wide range of strategies that work best for them. Students are *encouraged* to explore different ways to solve problems! Should I worry about my second graders who are still counting to solve addition?"

Everyone agreed with Molly, but then Mike entered the conversation: "We *do* value the mathematical thinking our students share, but at the same time, our students need to be efficient. The question is, then, how can we move children toward efficient strategies if we don't look at the learning trajectories?"

This discussion made clear our team's need to devote several meetings to using student work samples to analyze addition strategies. The question was where to begin. We agreed that once we determined exactly which addition strategies our students were using in grades 1–5, we could ponder efficient strategies for each grade level and how to move children toward efficiency.

As this meeting ended, we looked at one another and realized that we had just collaborated successfully for the first time as a vertical team. Anna's initial question had opened the door for others to comfortably join in and share their thinking. Past meetings had been littered with complaints and disagreements, and very little, if anything, had been accomplished. This meeting was a turning point for our team. We now had a defined purpose. Our next step was to devise a plan to accomplish this purpose.

Analyzing Student Strategies

Teachers at Derby Ridge were familiar with and comfortable with analyzing student writing. The entire staff had participated in a writing Gallery Walk, a process in which student writing was displayed like art in a gallery so that all the teachers could study it, pose questions, and make comments about specific writing pieces. The vertical writing team had organized the Gallery Walk so that each faculty member would be more knowledgeable about student writing across the grades and would be involved in the process of selecting student writing samples to use as benchmark papers at each grade level. To accomplish this goal, two writing samples from each grade level were enlarged and hung on the wall for accessible viewing. Then the faculty, organized in vertical teams, walked around, read and discussed the writing samples, and came to consensus about which papers would become

benchmarks. All the teachers enthusiastically embraced this activity and felt that examining student writing samples essentially painted a picture of how children develop as writers from kindergarten to fifth grade.

Because the writing Gallery Walk was so successful, the mathematics vertical team thought we should use a similar approach to examine students' addition strategies. However, we decided to start the process with the vertical team members before attempting it with the entire faculty. We asked the K–5 district mathematics coordinator to design appropriate contextual problems for each grade level (see **fig. 1**). Team members then asked their students to solve the problems independently and show their thinking. Separately, each team member analyzed and sorted his or her student work by common strategies and brought a sample of each strategy to the next meeting.

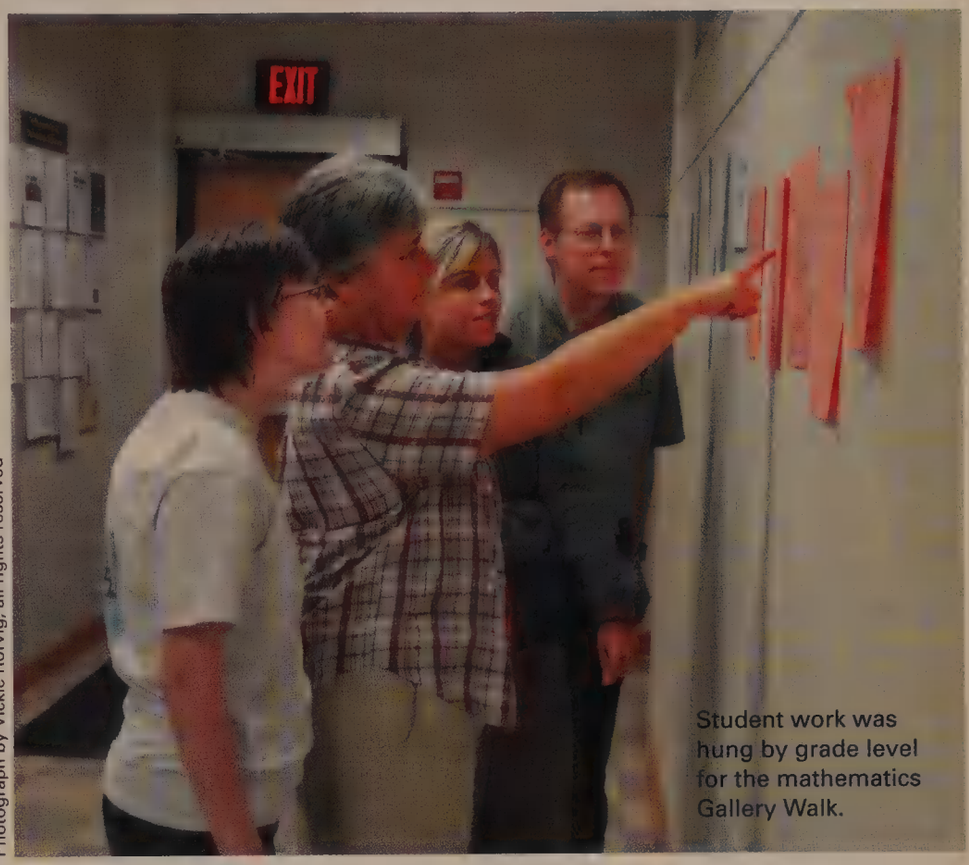
Everyone on the vertical team was amazed at the number of strategies that students used. In order for teachers to view each grade's work, we created our version of the writing Gallery Walk by hanging students' mathematics work in the hallway by grade level. Next, the vertical team members took turns explaining the strategies their students used to solve the problems. This gave teachers from other grade levels the opportunity to become familiar with strategies at every stage of development by asking questions and comparing strategies.

Conversations began to filter through the hallway as Jill, our first-grade teacher, proudly began to share her students' work. "As you can see, Billy is holding the first number in his head and then counting on the rest. See the think bubble that has the number eight? That means it is in his brain; then he counts on four more by saying, 'Eight' (that is in his head), 'nine, ten, eleven,' then 'twelve,' which is four more."

Molly asked, "Is that the strategy used by a lot of first graders?"

Jill explained, "Yes, this is a more efficient strategy because they are not counting the first group of eight, then the group of four, combining the two groups, and finally recounting all to get the sum of twelve. That's nice to see some first graders do this early in the year."

Matt shared one of his third grader's strategies. During his explanation, Anna, who teaches second grade, immediately recognized something: "Some of my students are using the same strategy of decomposing all the numbers and then adding them together as your third graders [do], Matt. But I see some students making their numbers easier to work with by making tens. How do we get children to use friendly numbers to solve problems?"



Photograph by Vickie Rorvig; all rights reserved

Student work was hung by grade level for the mathematics Gallery Walk.

Matt explained how the Investigations curriculum series encourages such thinking. Our fourth-grade teacher made connections with what third graders were doing and was pleased that children experienced such thinking before they came to fourth grade.

These were just a few of the rich conversations we heard as we walked through our student work gallery. A meeting scheduled for one hour lasted two! Everyone left feeling that they understood the strategies and why their students were using them. However, it was obvious that our work had just chipped away a small piece of the iceberg. The team agreed that more could be done with student work samples: Determining grade-level efficiency was next on our agenda.

Case 2

Loesing teaches at Paxton Keeley Elementary, a K–5 school with an enrollment of almost seven hundred students. The building houses twenty-eight regular classroom teachers, four full-time and one part-time special education teachers, and two ESL teachers who support eighty English Language

Learners. As a result of a summer mathematics institute sponsored by the district, several teachers from Paxton Keeley expressed interest in learning about strategy development across grade levels. Further conversations with these teachers led Loesing to consider ways to address this issue. Loesing remembered that Cameron and Rorvig had dealt with a similar issue in their school. After she talked with them to get more details, Loesing designed a professional development activity for the entire faculty at Paxton Keeley that focused on addition strategies across grade levels and excluded discussions of mathematical efficiency.

Loesing gave the same problems (see **fig. 1**) to all the classroom teachers at Paxton Keeley. Before the professional development activity, children were asked to solve the problems independently. (All work was anonymous— papers had no student or teacher names.) Teachers collected and brought the student work to a schoolwide meeting.

Initially, classroom teachers sat in grade level groups while specialists (i.e., art, music, and physical educators) were invited to join any group on the basis of the ages of the children they taught. To establish common language and clarify which student work samples exemplified specific strategies, teachers discussed the difference between a strategy and a representation. For example, using an open number line is a representation, and using counting on is a strategy.

Photograph by Vickie Rorvig; all rights reserved

Next, each group examined its grade-level samples and sorted them by strategy. To determine which strategies were predominantly used at each grade level, teachers used the sorted samples to make graphs by placing similar samples in columns on the floor. In grade-level groups, teachers then discussed their observations.

Then teachers formed vertical teams to view the student work across grade levels. Teams stopped to view each grade level of student work for a few minutes. The teacher from that particular grade level briefly commented on the different strategies. Once they completed this process, faculty members reconvened to briefly share their reactions.

Benefits

This professional development activity was successful for several reasons. First, each classroom teacher played a role in the nonthreatening process. Teachers contributed to the activity by bringing student work from their classrooms. They felt knowledgeable about the mathematical strategies used at their grade levels. Teachers could see that over time children obviously develop more sophisticated thinking about addition strategies and their use. For example, one teacher noted that few children were using counting strategies in fifth grade. Using the graphs, teachers were also able to identify each grade level's most prevalent strategies.

Most important, the event built a more coherent professional community among the faculty. Teachers had previously attended focused professional development activities for specific grade levels at the school or district level, and such events varied according to teacher needs and years of teaching experience. In contrast, the cases we describe not only brought the entire faculty together to think about mathematical alignment and coherence across grade levels but also provided context and a common experience for teachers to continue the conversation beyond one event.

On the basis of this successful experience, teachers initiated grade-level conversations around student work. Furthermore, teachers worked with administrators to change the nature and focus of faculty meetings. Rather than spending time "delivering information" to the faculty, teachers now use that time for professional development centered on student learning. Moreover, the following academic year, the district hired a full-time mathematics coach to support such efforts at both schools.

Summary

These professional development cases were designed to support teachers' understanding of student addition strategies within and across grade levels. Although the two schools used different approaches, both greatly influenced participating teachers. Using student work enhanced and focused communication among teachers within and across grade levels regarding important mathematical ideas needed to effectively teach mathematics to elementary school students.

Too often, teachers attend professional development sessions without providing input into their design. These teachers felt invested in the activity because they were involved in creating the experience as well as in leading parts of the discussion. Their concerns about student strategies were validated and addressed in positive ways. The experience helped teachers recognize and interpret student work and create a common language for future discussions.

Moreover, the experience impacted instruction. During classroom observations, the mathematics coaches noted that the teachers now appear more comfortable facilitating whole-class discussions that involve student thinking. The teachers have a better understanding of efficiency, resulting in more discussions about efficiency with their students. Furthermore, the teachers are better equipped to provide specific student feedback and support. Future plans include continuing to analyze student work in professional development settings for other purposes, such as improving teachers' mathematical content knowledge.

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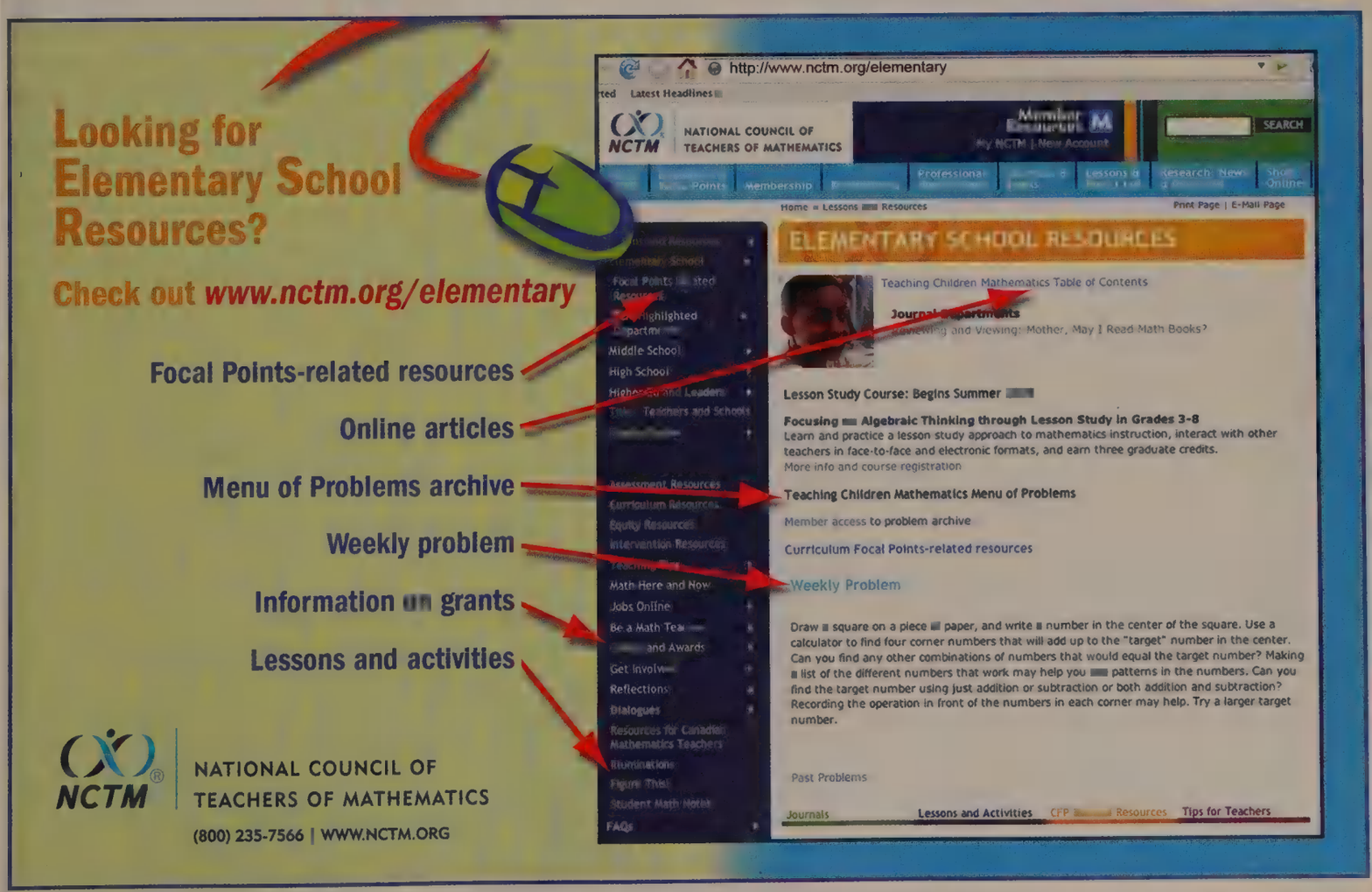
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Weekly Problem
Draw a square on a piece of paper, and write a number in the center of the square. Use a calculator to find four corner numbers that will add up to the "target" number in the center. Can you find any other combinations of numbers that would equal the target number? Making a list of the different numbers that work may help you see patterns in the numbers. Can you find the target number using just addition or subtraction or both addition and subtraction? Recording the operation in front of the numbers in each corner may help. Try a larger target number.
Past Problems
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Teaching Fractions

To support the development of preservice elementary teachers' mathematical knowledge for teaching fractions, the authors created and implemented a mathematics-for-teaching course module centered on student written work and verbal explanations.

by Laura Kondek McLeman and Heather A. Cavell

Laura Kondek McLeman, lkondekm@willamette.edu, works with teacher candidates at Willamette University in Salem, Oregon. She is interested in researching the development of preservice teachers' mathematical knowledge for teaching as well as how to help preservice teachers meet the needs of English Language Learners (ELLs). Heather A. Cavell, hcavell@email.arizona.edu, teaches preservice teachers at the University of Arizona in Tucson. She is interested in researching ELLs' needs in the mathematics classroom and developing preservice teachers' knowledge to meet the needs of growing diversity in mathematics classrooms.

Teacher educators are constantly searching for ways to better prepare preservice teachers to enter the education field. One way that has gained prominence in the last twenty years is through examining and discussing authentic student work samples. Research suggests that doing so helps preservice teachers to (1) gain insight into their own beliefs about what it means to know and do mathematics (Timmerman 2004) and (2) acquire some elements of the knowledge they need for teaching (Tirosh 2000).

This type of know-how is what Hill, Schilling, and Ball (2004) describe as *mathematical knowledge for teaching* (MKT), which they argue is different from the mathematical knowledge that mathematicians need. MKT encompasses, among other things, understanding the concepts that underlie procedures, performing error analysis to determine where a student's mistake lies, and choosing effective representations to model concepts.

Encouraged by the many benefits of using student work in other teacher education courses, we were interested in incorporating it into a mathematics-for-teaching course offered at our university. Specifically, we aimed to support the development of preservice elementary teachers' mathematical knowledge for teaching fractions (MKTF).

To accomplish our goal, we created a three-day module from four fifth graders' written work and verbal explanations, which we implemented in one section of our spring 2006 course. Twenty-three

3/4 - 1/4 = 1/2
6/7
1/8
4/5
3/5
5/8
11/15
9/4

female preservice teachers participated. All were juniors or seniors.

This article briefly discusses the module's development and implementation, some observations and reflections from the pilot implementation, and some of the questions that arose. And finally, we appeal to our colleagues to facilitate and report on similar activities in their own courses.

Figure 1

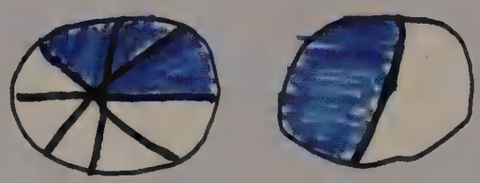
A unit was developed around student Y's and student D's written work and student Y's videotaped explanation. The teacher candidates looked at the student work first.

Section 1. Compare student Y's and student D's written work, specifically looking at problem 1a.

Student Y's work

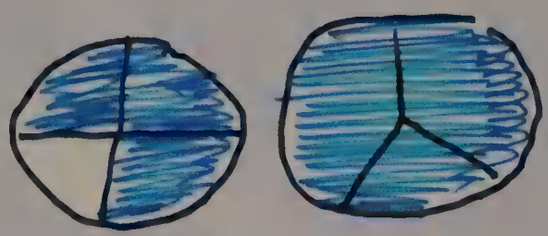
1. Circle the larger fraction in each pair. Write = if you think they are the same size. Next to each pair, show or write about how you decided.

a. $\frac{3}{8}$ $\frac{1}{2}$



b. $\frac{2}{3}$ $\frac{5}{6}$

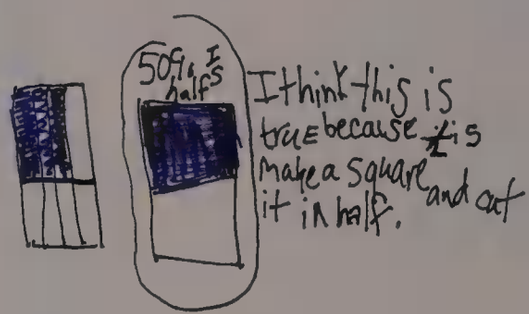
c. $\frac{3}{4}$ $\frac{4}{3}$



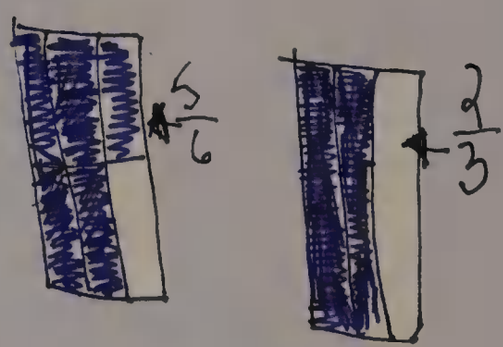
Student D's work

1. Circle the larger fraction in each pair. Write = if you think they are the same size. Next to each pair, show or write about how you decided.

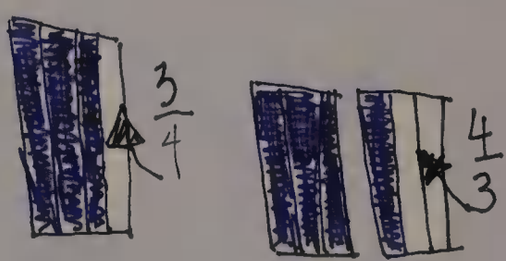
a. $\frac{3}{8}$ $\frac{1}{2}$



b. $\frac{2}{3}$ $\frac{5}{6}$



c. $\frac{3}{4}$ $\frac{4}{3}$



Module Development and Implementation

In December 2005, facilitators for a multischool, grades 3–6, teacher study group within the Center for the Mathematics Education of Latinos/as (CEMELA) invited four fifth graders to share their thinking about how to compare fractions. The students, two Hispanic females and two African-American males, were asked to participate because of their varying levels of fraction knowledge.

The students were to complete the "Comparing Fractions" activity sheet from the fourth-grade Investigations in Number, Data, and Space unit. Students were individually videotaped sharing their thinking about one problem they had completed. After the students left, the study group teachers engaged in a rich discussion of the youngsters' thinking about fractions, mainly focusing on the fragile understanding that they appeared to have.

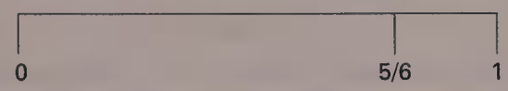
While reviewing the video clips, CEMELA members decided that the elementary students' written work and explanations should be developed into a university mathematics course module for elementary education majors. The course, which examines K–8 mathematics from an advanced perspective, has a history of exploring student explanations, specifically around the concepts of multiplication and division. Because this part of the course is generally met with a great deal of enthusiasm, we were interested in seeing how preservice teachers would engage with student thinking when the content is fractions, an area that many preservice teachers generally find difficult (Ball 1990).

During spring 2006, the authors and the course instructor developed four units, one for each child (whom we call students Y, D, A, and T). Each unit comprised two sections: section 1 focused on the student's written work, and section 2 focused on the student's verbal explanations of their written work. **Figure 1** shows an example of a unit.

Before implementing the activities that specifically focus on the student work, the course instructor gave teacher candidates a prior-knowledge assessment and then held small-group and whole-class discussions about their responses. The assessment consisted of six questions related to fraction topics such as representation, comparison, the whole, and equivalency. After the discussions, preservice teachers completed section 1 of each unit (an analysis of each child's written work) followed by section 2 (an investigation into the student's verbal explanations of their written work). The format—having the teachers look at written work independent of verbal explanations—was chosen deliberately on the hypothesis that the teacher candidates would be more engaged in listening to students' explanations once they had spent time analyzing students' written work.

Figure 2

One preservice teacher represented the fraction 5/6 in this way.



While observing the preservice teachers' classroom discussions and their completed work, we noted how they were developing mathematical knowledge for teaching fractions. Two items were of particular interest: (1) the preservice teachers' understanding about the representation of fractions, both of their own and that of their future students, and (2) the preservice teachers' mathematical understanding of fractions through their discussions of students' work.

Fraction representations

One area of focus throughout the module was the representation of fractions. Hill, Schilling, and Ball (2004) argue that choosing effective representations to model concepts is part of possessing MKT. To assess the preservice teachers' MKT with regard to their representation of fractions, we asked them to represent 5/6 in two different ways on the prior-knowledge assessment. In the discussion that followed, one preservice teacher shared two different area models: (1) a hexagon divided into six equal pieces with five pieces shaded and (2) a rectangle divided into six equal pieces with five pieces shaded. Another preservice teacher, PT 1, offered her representation of five-sixths (see **fig. 2**). After PT 1 drew her representation, a third class member, PT 2, commented that the number line drawn by PT 1 should be divided into equal-sized segments in order to accurately represent five-sixths:

PT 2: But is that another representation [of five-sixths], or is it just another way to write it? Because I don't think that [the number line representation of the fraction] shows me what five-sixths is. It just puts it between zero and one.

PT 1: Well, it's like this [*pointing to the two area models on the board*] is more concrete. You know,

you can touch it. This is just like a number line. It's showing that five-sixths is less than one, greater than zero but closer to one.
PT 2: Yeah, but you still didn't partition them off so I could see …
PT 1: Oh, well … I could do one-sixth, two-sixths, three-sixths, four-sixths [*drawing tick marks on the number line*]. I could do it that way.
PT 2: That way, it's easier for me to see. It's not that; it's just sort …
PT 1: But then it's just like that [*pointing to the two area models*].
PT 3: But at least we can see the progression. It'd be good for students to see …

PT 2's argument seems to be that the number line should show parts of the whole with partitions into sixths. However, PT 1 seems to have a different idea than PT 2 of the meaning of 5/6. PT 1 viewed the fraction as a number within a larger set of numbers. The problem for PT 2 was how to comprehend what PT 1 was drawing in order for her own representation to make sense. What PT 2 needed to see was the number line showing parts of a whole, evidenced by her repeated use of phrases such as "so I could see" and "it doesn't show me." Such difficulty in coordinating two different interpretations is not uncommon and is something that elementary-age children may struggle with.

In fact, at the end of the exchange, a third preservice teacher joined the conversation and switched the focus from the preservice teachers' own understanding of five-sixths to what representation might aid elementary students. The focus on elementary students continued after this discussion. After observing preservice teachers' in-class interactions and looking through their completed work, we noted that many of them commented that it would be important to expose children to different representations in order to help them make sense of fractional concepts. For example, one preservice teacher commented that she would use "realistic things so [a] child could relate," and another cautioned against using only area models: "I believe that [the] area model, while visually appealing, could confuse the child."

Another example of how the preservice teachers worked to make sense of their own mathematical understanding about representations occurred when three of them discussed student T's work. Student T had concluded that two-thirds and five-sixths are equivalent by focusing on the fact that each fraction has one leftover piece (see **fig. 3**).

PT 1: His whole equal sign thing … I don't know.
PT 2: I don't know what it means.
PT 3: Maybe he was thinking of comparing the equivalent fraction with [a] denominator like six? It still would be wrong, so I don't know.
PT 1: Yeah.
PT 2: The way he has written it, he needs a better model. The divisions aren't the same.
PT 1 and PT 3: Yeah.
PT 3: He is having trouble with which leftover piece is bigger. Again, though, the rectangles are the same size, and they both show that one box is left, so …
PT 2: You know, I see that he thinks he needs to add that piece, but he doesn't; and [he] doesn't shade it in.
PT 1: You know what? His number of pieces left are equal size, 'cause there is one piece left in both.

In their discussion, the three preservice teachers reflected on their own understanding of fractions to make sense of how someone could conclude that two-thirds and five-sixths were equivalent. In the beginning, they struggled to understand what the student was doing, which is evident in their repeated use of phrases such as "I don't know." However, toward the end of the exchange, the preservice teachers connected their own knowledge of fractions with the representations drawn, concluding that because one piece was left in each drawing, it was possible that student T believed they were equivalent fractions. We posit that this reflection and use of mathematical knowledge was possible because of the module activities.

These examples showcase that during the student thinking activities, the preservice teachers were able to reflect on and use their knowledge of fractional representations as well as to consider what representations might be most beneficial to students. By doing these things, the preservice teachers were developing MKTF.

Another intriguing element of the preservice teachers' engagement with the activities was how they made sense of their own mathematical understanding by analyzing the language that the children used.

Language

Another important component of MKT is the choice of words that teachers use to express mathematical concepts, as this language provides insight into their mathematical understanding. Throughout the module activities, preservice teachers expressed

their mathematical understanding of fractions through their discussion of what the children were doing. While analyzing students' written work, preservice teachers' conversations and written responses contained statements such as these:

"They realize [that the numerator] is part of a whole and think in terms of area."

"Student Y didn't understand how to [compare] improper fractions."

Investigating the students' verbal explanations, a preservice teacher wrote, "She sees three from three-eighths and only one from one-half, so she concluded three-eighths was bigger. She clearly understands how to compare fractions or sees the shading as three-eighths and the one-half as maybe four-eighths."

Another commented, "He understands the relationship between the numerator and the denominator and that he must show each fraction with the same unit."

Such language highlights the active involvement it takes both physically and mentally for participants to make connections from their ordinary language to mathematical language (Pirie 1998). Word choices such as *whole*, *bigger*, *area*, and *relationship* have specific contextual meaning that allows participants to make sense of classroom activities and make links between student language and mathematics' symbolic language. Using these terms as a means to evaluate what the children are thinking shows that the preservice teachers have a specific mathematical understanding of fractions and that the language the child uses is the tool by which the teachers connect to their own knowledge base. For example, in a previous statement, a preservice teacher discussed with her classmates that a child understood fractions by comparing numerators. The child's explanation of how she compared fractions apparently triggered recognition in the preservice teacher's own mathematical register. Specifically, the preservice teacher was

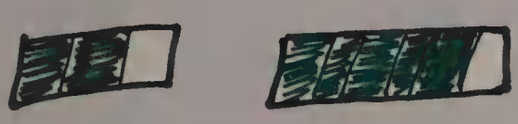
Figure 3

Three preservice teachers worked to make sense of their own mathematical understanding about representations by discussing student T's written work and explanation of problem 1b.

Comparing Fractions

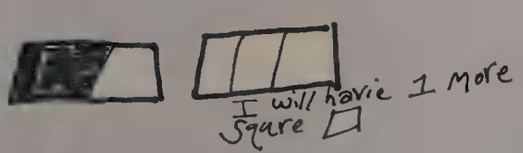
1. Circle the larger fraction in each pair. Write = if you think they are the same size. Next to each pair, show or write about how you decided.

a. $\left(\frac{3}{8}\right)$ $\frac{1}{2}$



b. $\frac{2}{3}$ = $\frac{5}{6}$

c. $\left(\frac{3}{4}\right)$ $\frac{4}{3}$



Key
F = Facilitator; *T* = Student T; *OCI* = Off-Camera Individual

T: I think they're equal because two take away three equals.... You have one left; and five take away six—I mean six take away five—you still have five left on each one. So I thought they were both equal.

F: So you have one piece left on each one of your drawings, don't you? [F *gestures toward* T's *drawing.*] And so you think they're equal because you have one left on each one? Do you think those pieces that are left are the same size?

T: Mmm-hmmm.

F: Yeah? You think they're the same size?

[T *looks at his worksheet and thinks for about 25 seconds. Teachers whisper to each other.*]

OCI: What did you say about "b" again? Can you explain it again? You're looking at how many pieces are left?

T: I thought it was equal because there's both one left on each one [*gesturing toward the two unshaded portions of his drawing*], and I thought it was equal.

OCI: What do you mean you thought they were equal?

T: On this one [*gesturing toward the right side of his drawing*], two. You have three, and you take away two out of that group, and you have one left. Then you have six, [*you*] take away five; you have one left, and they're equal 'cause you both have one left for each one.

able to use the child's language to reflect on a familiar way of comparing fractions (i.e., getting a common denominator and comparing numerators). The MKTF module provided space that allowed preservice teachers to verbalize the students' understanding of fractions, as well as their own, by discussing the language that the children used; by doing so, the teachers acquired components of MKTF.

As an aside, note the confident language of the teachers' statements: *clearly understands* and *realize*. Such language points to an important aspect of using authentic samples of student thinking with preservice teachers, namely, to be careful not to use small, out-of-context segments to declare knowledge of what someone else is thinking. Preservice teachers generally have very little teaching experience, so they might be prone to think in a concrete fashion, expressing sentiments such as, "If he had just done _____, then he would have understood."

Teacher candidates' responses during some of the module activities underscore the necessity for teacher educators to challenge such a concrete mentality by presenting multiple scholarly experiences regarding elementary students' thinking. When challenged in this way, preservice teachers start to realize that understanding is complex and is not easily diagnosed.

Module Reflection

The preservice teachers were extremely engaged both in analyzing the student written work and in examining their verbal explanations. However, the mathematical concepts that the preservice teachers were learning were not prominent enough in the activities. In retrospect, the module activities were too evaluative in terms of correcting student mathematical mistakes through teaching interventions and not exploratory enough to allow the preservice teachers to examine, at a deeper level, the content inherent in the student work. Therefore, we ought to modify the activities from a teaching-centered focus to a more foundational mathematical focus. This would allow a clearer connection between fraction concepts and analyses of the student thinking.

The preservice teachers struggled with the related issue of how to discuss mathematical content. This struggle did not surprise us, because addressing mathematical content by examining student work was a new experience for all participants and focusing on pedagogy is probably more interesting to preservice teachers, given their

Acquiring mathematical knowledge *for teaching*—which differs from mathematical knowledge—better prepares preservice teachers for working with students.

chosen careers. We do believe that using group work and video technology provided preservice teachers opportunities for rich interactions around course content—opportunities limited by a traditional lecture format. However, in future implementations of the module, we will set aside time for the course instructor to discuss the module goals in the beginning and to have these goals revisited throughout the activities' implementation. Coupled with a clearer focus on content, this should make for more content-rich conversations.

Future Research

Several questions naturally arose during the implementation of the activities: What was it about the student written work and explanations that promoted the conversations that occurred? Was it the framing of the activities by the course instructor, or was it the dynamics of this particular group of preservice teachers? Observing a classroom of predominantly Anglo-American females in their early twenties analyzing and making generalizations about students' understanding of fractions was intriguing. How would the discussions have been enriched if the classroom dynamics were more heterogeneous? Finally, what specific mathematical content did the preservice teachers learn from participating in the activities, and was the mathematical understanding about fractions that was achieved in this setting any different from the understanding of the teacher candidates who did not engage in the activities?

Call to Action

Incorporating field-based materials is important for developing preservice teachers' knowledge base of MKT. Developing and implementing our module activities showed us the value of instructional materials that incorporate student written work and verbal explanations. Preservice teachers who engaged in the module activities developed their MKTF both by reflecting on what it means to represent a fraction and by discussing their thoughts about what they saw in the student work and heard in the student explanations.

Using student work and explanations also allowed the preservice teachers a chance to "interact" with children in a course that has no field-based practicum. Seeing and hearing students' mathematical thinking inspired the preservice teachers to understand the mathematical concepts more deeply.

This article provides a brief glimpse into an important way to help preservice teachers acquire the mathematical knowledge they need for teaching by using activities centered on students' thinking. However, using materials such as the MKTF module in a mathematics-for-teaching course is uncommon, and a prescribed structure for implementation does not yet exist. Therefore, we call on our colleagues in the field to develop and use these types of materials in order to help other teacher educators better organize and facilitate similar activities in their own courses. Having children explain their thinking deepens preservice teachers' own mathematical understanding and helps them develop flexibility and proficiency in understanding students' cognitive processes (Tirosh 2000).

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A. Deanie Sullivan and Amy Roth McDuffie

Connecting Multiplication to Contexts and Language

This department features children's explorations in mathematics and presents teachers with open-ended investigations to enhance mathematics instruction. The tasks invoke problem solving and reasoning, require communication skills, connect various mathematical concepts and principles, and have been classroom tested.

The investigations present scripted sequences and directions for the purpose of communicating what happened in a particular classroom. *Principles and Standards for School Mathematics* (NCTM 2000) encourages teachers and students to explore multiple approaches and representations when engaging in mathematical activities.

The Investigation

The activities in the Connecting Multiplication investigation were designed around the adopted curriculum in a particular school district to provide more explicit opportunities for students to find real-world multiplication applications that are all around them and to understand that multiplication is a way to mathematically describe equal-sized groups. Collective nouns are a specialized part of speech designed to describe groups. Learning collective nouns and using visual images to support their work will likely engage and motivate students to create and illustrate their own multiplication word problems and to develop deeper understanding of the meaning of multiplication by seeing, organizing, and interpreting their world mathematically (Fosnot and Dolk 2001).

Learning goals, rationale, and pedagogical context

Curriculum Focal Points (NCTM 2006) recommends multiplication as a primary focus of third-grade mathematics, and this is the practice in many states. Students need to develop understanding of varied multiplication applications (e.g., sets of equal-sized groups and the relationship between repeated addition and multiplication) through the use of representations and models (NCTM 2006). Developing these mathematical models and connecting the models to the real world is crucial to students' understanding of key mathematical concepts such as multiplication (Fosnot and Dolk 2001). However, making connections to real-life situations often challenges students, especially because the word *times*, which is most commonly used to describe multiplication, holds very little meaning for students and disguises multiplication's true meaning.

After analyzing our students' results on the Washington Assessment of Student Learning, we found that the two strands on which our students scored lowest were number sense and making connections. Recognizing a need for more attention to these areas motivated our work when we developed this investigation. In the past when we had asked students to write their own story problems involving multiplication, the problems were dull, somewhat artificial, and often filled with the word *groups*. Students frequently failed to demonstrate understanding of the distinction between the size of the group and the number in each group, or they confused multiplication and addition. For example, to show five times three, a student might mistakenly write, "There were five dogs, and three more came. How many dogs in all?"

When we encouraged students to use descriptive, interesting, and effective words for a specific

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Edited by **LouAnn H. Lovin**, *lovin@jmu.edu, who teaches mathematics methods and content courses to prospective elementary teachers and certified classroom teachers at James Madison University in Harrisonburg, Virginia.* *"Investigations" highlights classroom-tested multilesson units that develop conceptual understanding of mathematics topics. Classroom teachers may reproduce this material for use with their own students without requesting permission from the National Council of Teachers of Mathematics. Readers are encouraged to submit manuscripts appropriate for this department by accessing* **tcm.msubmit.net**. *Double-spaced typed manuscripts should not exceed 3000 words and two reproducible pages of activities.*

audience and purpose, we also saw a connection to our writing instruction. In many states, selecting and using specific descriptive words and specialized vocabulary to clarify and improve writing effectiveness is a key learning goal in third-grade language arts. By using collective nouns in place of the word *groups* in student-created story problems, students actively connect multiplication to the world around them; recognize the relationship between groups and the number in each group; apply interesting, specific vocabulary; and fully engage in the learning process.

The lessons that we describe were taught to twenty-four third graders in Washington State. Two students were receiving additional special education mathematical support, and one student had a full-time aide for a physical disability. The class had no identified gifted students or English Language Learners. The school is in a suburban district; a little more than 50 percent of the students receive free or reduced lunch.

Objectives

Students use context to—

- develop their understanding of the role of equal-sized groups in multiplication and the relationship between repeated addition and multiplication;
- write, illustrate, and solve problems that involve multiplication in a real-life context using story elements and collective nouns; and
- engage in mathematical modeling of the real world and understand that multiplication is all around them in their daily lives.

Materials

Lesson 1

- A set of images with examples of groups for introductory activity (see **fig. 1**)

Lessons 2 and 3

- Manipulatives (Unifix® cubes, foam shapes, or chips)

Lesson 4

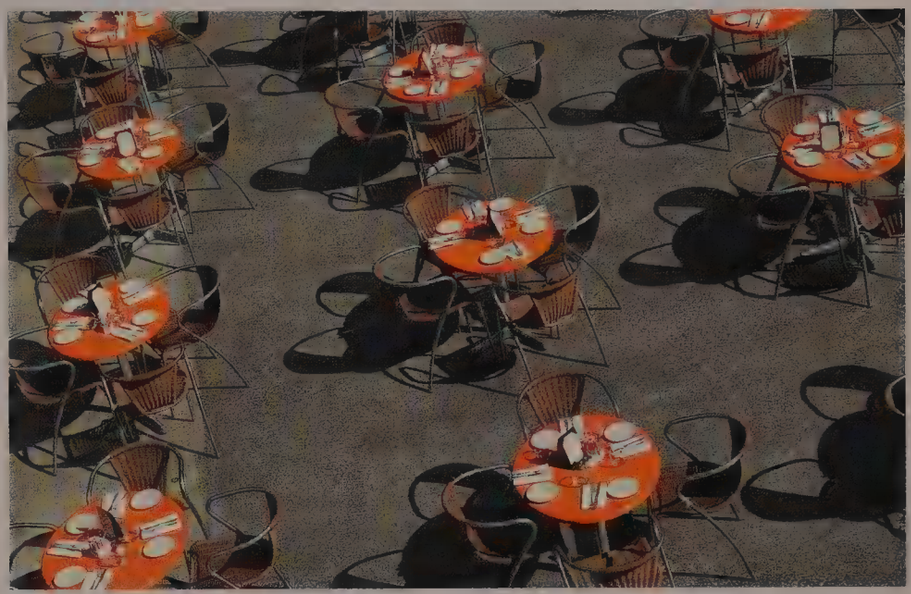
- Manipulatives (Unifix cubes, foam shapes, or chips)
- Index cards

Lesson 5

- A teacher-made big book, *The Most Important Thing about Collective Nouns* (see **fig. 2**)

Figure 1

Showing images during the investigation's introductory activity establishes the context that multiplication is a way to mathematically describe *groups of groups*.



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Lesson 6

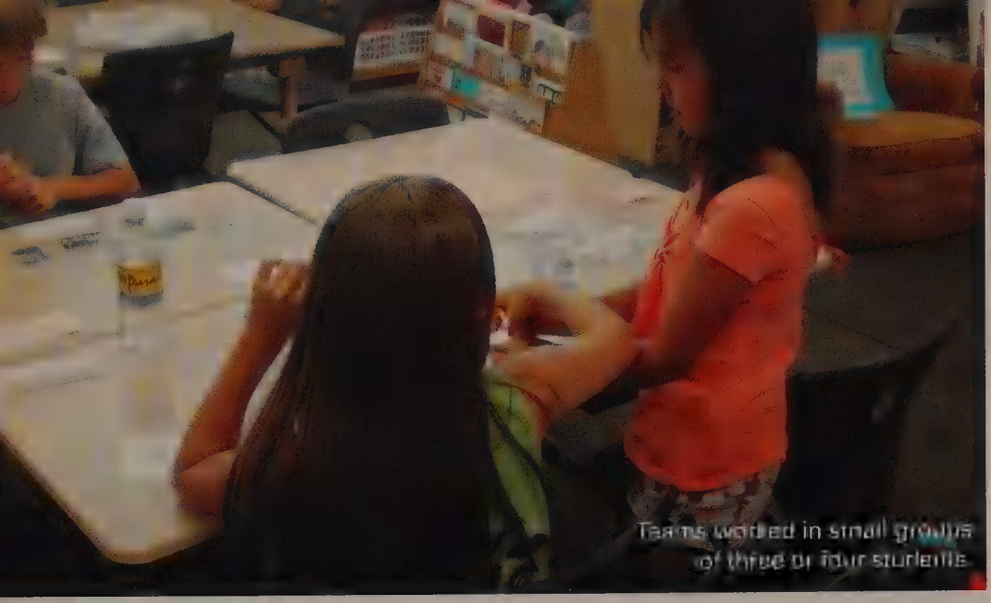
- A set of collective noun images laminated on individual index cards (see **fig. 3**)
- Index cards
- **Activity sheets 1** and **2** (see **p. 512**)

Previous knowledge

Students do not need prior exposure to multiplication to engage in this lesson; however, they should be familiar with using addition to find the total, or sum, of a quantity. Students should also be familiar with story elements (character, setting, events, problem, and solution) and be able to use these elements in their own writing.

Lesson 1: Establishing Context

The goal of the introductory activity is for students to recognize that *groups of groups* occur in many places, instances, and examples in everyday life. To inform our planning and implementation of



Teams worked in small groups of three or four students.

Photograph by A. Deanie Sullivan; all rights reserved

lesson 1, we used Guided Language Acquisition Design strategies (Brechtel 2001), which provide an organizational structure and model that promote language and vocabulary skills and increase students' background knowledge.

To introduce this unit's key focus—that multiplication is a way to mathematically describe *groups of groups*—we used Google Images (images.google.com) to find pictures of real-world items that are in groups and that would engage and motivate students (see **fig. 1**). We collected, printed, and laminated color photographs of groups of lion cubs, tropical fish, cars in a parking lot, ice-cream cones, a choir, children on swings, birds in a nest, flowers in pots, and pastries in a case.

So that each student could not simply pick a favorite image, we gave the teams one fewer image than the number of students on the team. Students worked in small teams of three or four to choose one image, discuss it, and report an observation and a question to the whole class.

Table 1

Things That Come in Groups
(Unit Overview in Investigations in Number, Data, and Space Series)

Investigation	Mathematics Learning Goals
1. Things That Come in Groups	• Identifying groups of items in the world • Writing with multiplication notation to describe *groups of groups* • Illustrating multiplication situations
2. Skip Counting and 100s Charts	• Connecting skip counting to multiples of the same number and to multiplication • Recognizing patterns in multiples of 2 through 12 on the 100s chart
3. Arrays and Skip Counting	• Recognizing that multiplication can be used to find the area of a rectangle • Using arrays to skip count and to multiply and divide • Identifying factor pairs

We recorded their observations and questions on a chart and posted each team's selected image alongside the chart. Teams shared observations such as "two spiders sitting on a leaf" and "lots of cars parked on a field" and questions such as "What kind of spiders are they?" and "Why are there cars parked on a field?"

Teams tried to determine from all the images and other teams' observations and questions how the images were related and what key word or phrase was common to all the images. Our goal was for students to recognize that all the images showed *groups of groups*. If your students have difficulty reaching this conclusion about the images, you can point out similarities that relate to groups, for example, groups of legs swinging on the swing, tires on cars, or rows of donuts in the pastry case.

Once students had found the key word or phrase, we wrote *groups of groups* on the chart and left the chart posted for the duration of the multiplication unit for students to refer to as needed. This activity anchored the entire unit in the real world and emphasized that multiplication is a mathematical way to describe group items.

Lessons 2–4: Describing Groups, Implementing an Investigation

For the next several lessons, we followed our district-adopted materials, *Things That Come in Groups* (Russell et al. 1998), from the Investigations series. To provide context and a sense of the sequence of activities in these materials, this article briefly describes the curriculum's relevant goals and investigations (see **table 1**). The first investigation emphasizes finding things that come in groups and using multiplication notation to describe the number of groups and how many are in each group.

We referred to the images from our introductory lesson 1 activity to create charts for items that come in twos, threes, fours, and so on. For example, we used a spider image as the heading for our "Things That Come in 8s" chart (spider legs), and we used an image from an auto show for our "Things That Come in 4s" chart (car tires).

The next two investigations in the curriculum focus on skip counting, looking for patterns in multiples, and using arrays to skip count. Again we referred to the introductory activity images for visual examples of multiple patterns and arrays. This was another opportunity for students to make real-world connections with multiplication. When we were confident that most students were comfortable with the skills, we wrote a story problem on chart paper using the

word *groups* to describe three times four (e.g., Three groups of cookies were on the table. Each group had four cookies in it. How many cookies were there in all?) We drew a picture to illustrate the problem, and we showed the repeated addition and multiplication equations that describe the problem. We emphasized that the symbol × in this problem indicates *groups of*, and whenever a student said the word *times*, we reiterated that the word also means *groups of*.

We then modeled the problem on chart paper and acted it out with manipulatives (e.g., Unifix cubes or chips) in front of the entire class. We gave each team a set of manipulatives and a set of multiplication expressions (e.g., 3×4, 2×6, 7×5). Students spent the rest of the class period in their small groups, acting out the multiplication expressions with manipulatives and then drawing pictures on an index card to show the repeated addition and the multiplication expressions. This activity provided a good introduction to writing, illustrating, and solving multiplication problems. However, as we described earlier, many students continued to confuse multiplication and addition. We found that most students needed more experiences to develop strong understanding of multiplication. After recognizing this need, we further refined lessons 5 and 6 to include additional opportunities for students to build understanding of the role of groups in multiplication and a deeper awareness of multiplication in the world around them.

Lesson 5: Collective Nouns

The fifth lesson of our investigation included a printed list of collective nouns from Enchanted Learning (www.enchantedlearning.com/grammar/partsofspeech/nouns/collective/). Using *A Cache of Jewels and Other Collective Nouns* (Heller 1987) as a reference, we chose about fifty collective nouns that students would find engaging. From Google Images, we downloaded pictures of each collective noun and created a big book patterned after Margaret Wise Brown's *The Important Book* (1949). We titled ours *The Important Book about Collective Nouns* (see **fig. 2**).

Many of the collective nouns were animal groups, so we organized those images by animal habitats, such as ocean, forest, and desert. We also included nonanimal collective nouns such as *columns*, *rows*, *towers*, *stacks*, *teams*, *convoys*, and *flotillas* (see **fig. 3**). We repeated the statement that "the most important thing about collective nouns is that they describe groups" to emphasize our key focus in this multiplication unit as a way to mathematically describe *groups of groups*.

Figure 2

For lesson 2 we downloaded pictures of collective nouns to create our own big book patterned after Margaret Wise Brown's *The Important Book*.

The Important Book About Collective Nouns
Mrs. Sullivan Grade 3
Big birds...
And little birds...
Collective nouns can describe people...
And how people move around...
herd of cattle
pride of lions
clan of hyenas
Animals who live in grasslands...
Collective nouns can describe things that swim and slither...
And crawl and fly.
But the most important thing about collective nouns is that they can describe groups.

Figure 3

We printed another set of our collective noun images on card stock, laminated them, and then cut out each image so that we had a collection of individual image cards.

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Students enthusiastically revised mathematical word problems to include all the story elements they were learning in language arts.

We read our big book several times to the class and connected the importance of good word choices—to make written communication specific and descriptive—to the importance of specific, descriptive mathematical language, especially in problem solving. We then read to the class several examples of student-created multiplication story problems that used the word *groups*. Students were eager to suggest that the word problems could be rewritten using collective nouns instead of *groups*. For example, where students had written, "There were four groups of whales and six whales in each group. How many whales in all?" one student suggested replacing the word *groups* with *pods*. We changed the word problem to read, "There were four pods of whales and six whales in each pod. How many whales in all?"

The whole class used our big book to create multiplication word problems that included a collective noun such as the whale problem described above. Many of the collective nouns are interesting, highly unusual words (e.g., *tribe* of monkeys, *leap* of leopards, *sloth* of bears). Some students—and even teachers—had never heard these terms used in this way. The most difficult part of this task was for the class to decide which collective nouns to use for examples because most students had favorites. We decided on five class-created examples, wrote them on chart paper, and posted them in the classroom for students to refer to in the next lesson.

Lesson 6: Student-Created Word Problems

We printed one set of the collective noun images on card stock and laminated them. Each image was approximately three inches by three inches (several images fit on one 8 1/2 × 11-inch page). We cut out each image so that we had a collection of individual image cards (see **fig. 3**). With a permanent marker, we wrote the appropriate collective noun on the back of each card. Student teams were given a stack of cards, a collection of multiplication expressions, and blank index cards on which to write multiplication word problems. The teams first chose a multiplication expression and then a collective noun image to go with it.

Students referred to the posted word problem examples from the previous lesson for a model. Their word problems had to include a collective noun instead of the word *groups*, an illustration of the problem, and two equations: the repeated addition and the multiplication expressions (see **fig. 4** and **activity sheet 1 on p. 512**). We observed that most

students wrote their multiplication equation last (although it appeared on the top of their cards), suggesting that they needed to work through the more concrete step of repeated addition before they were ready to make the connection to multiplication.

We encouraged students to trade collective noun images and multiplication word problems with other teams. Viewing and handling the color photographs while learning interesting terms for collective nouns (e.g., *horde* of goblins, *rafter* of turkeys, or *flotilla* of ships) engaged all the students. They enthusiastically traded and shared with their classmates, waving cards in the air and shouting requests: "Who wants the army of ants?" "Does anyone have the stack of pancakes?" "I'll trade a herd of antelope for a convoy of trucks!"

Beyond the Lesson

This investigation focused primarily on using collective nouns to describe groups in multiplication and to connect mathematical problems to real-life contexts. However, besides specific descriptive words, many correlations and connections to writing and language arts naturally emerged during the lessons. Indeed, the multiplication word problems had all the story elements that we had been working on during reading and writing classes: character, setting, event or action, problem, and solution. After mapping these elements, class members revised their multiplication word problems to include all the elements (see **activity sheet 2 on p. 512**). For instance, one student wrote, "Four colonies of bats

Figure 4

Even if they wrote the multiplication equation at the top of their card, most students needed to work through the more concrete step of repeated addition before making the connection to multiplication.

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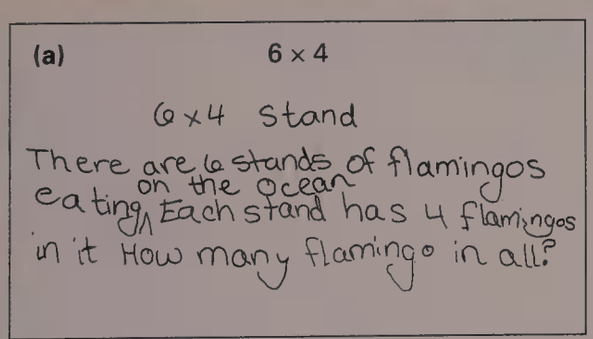
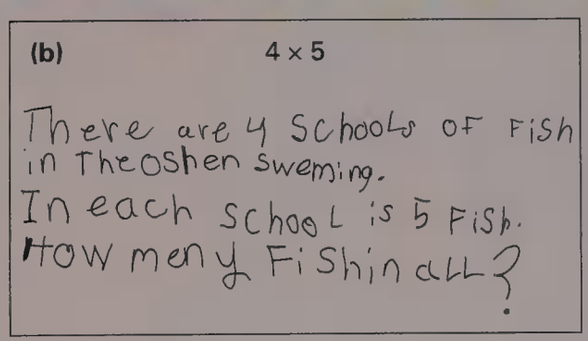
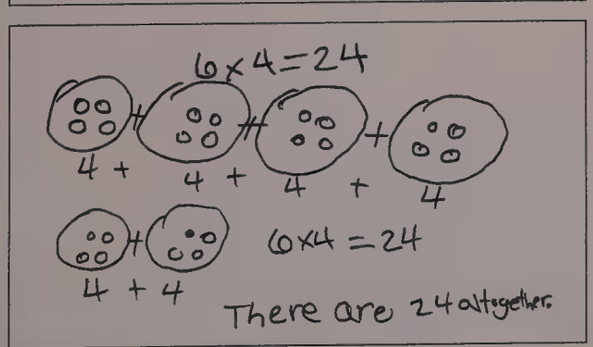
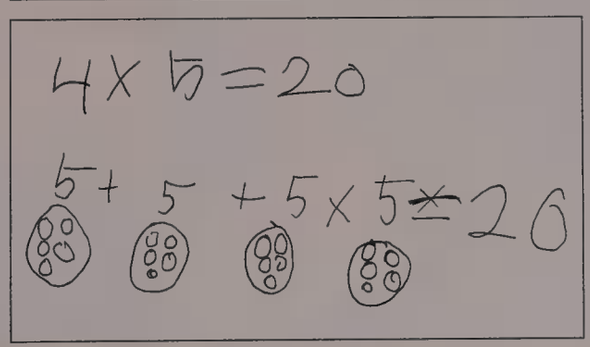
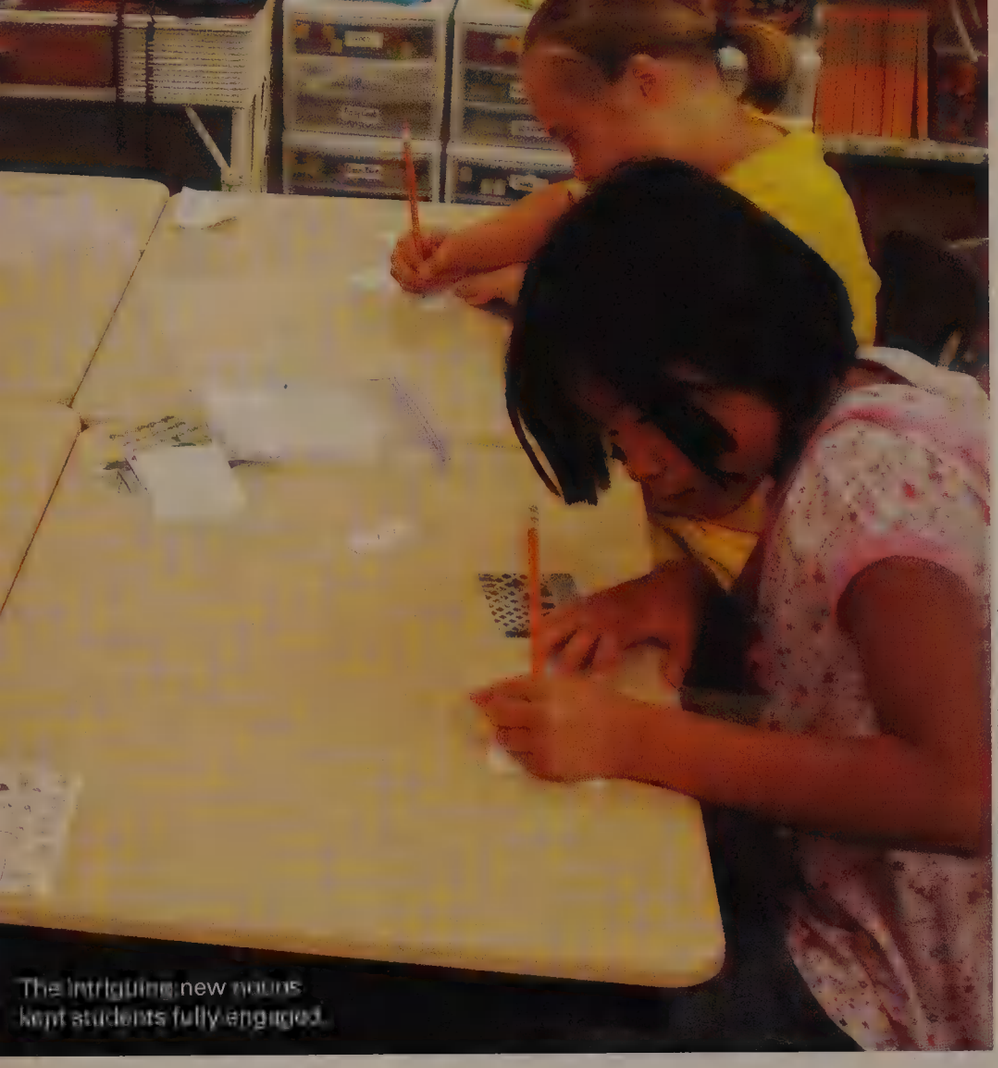










The intriguing new nouns kept students fully engaged.

Photograph by A. Deanie Sullivan; all rights reserved

(*character*) were flying (*event* or *action*) in a cave (*setting*). How many bats were in the cave (*problem*)? 3 + 3 + 3 + 3 = 12, or 4 × 3 = 12 (*solution*)."

Students were eager to make sure that all their word problems included the story elements, and they were quick to notice if a classmate's word problem did not include all the story elements or if the repeated addition or multiplication was incorrect. Through this process, they refined their use of language and their connections between problem statements and multiplication. We discussed how students could play the roles of editors and peer assessors by revising, editing, and correcting errors or omissions.

Identifying the event or action and the setting seemed to challenge most students. We reminded them that the action could be *sitting* on the table or *lying* on the ground and that the event or action does not require movement. For example, a student initially wrote, "Three batches of cookies had twelve cookies in each batch. How many cookies in all?" After peer conferencing, she revised her problem to read, "Three batches of cookies (*character*) were sitting (*event* or *action*) on a plate in the kitchen (*setting*). Each batch had twelve cookies.

Figure 5

Without good understanding of the role of groups in multiplication, students often confuse multiplication and addition.

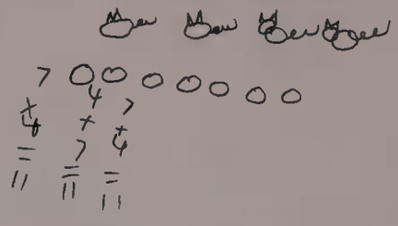
(a)

3. Write a story problem to describe the multiplication expression below.

7 × 4

There are 7 pigs 4 more pigs come along How many are There in all

Draw a picture that represents the problem; then solve the problem. Write both the repeated addition number sentence (equation) and the multiplication number sentence (equation) that describes the problem.



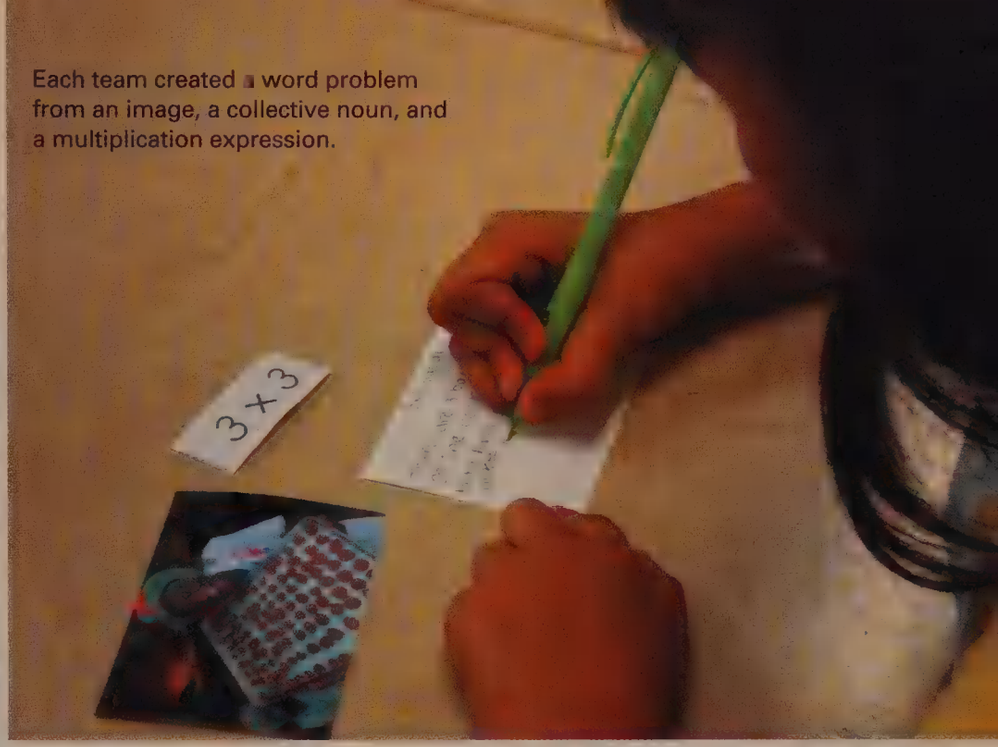
(b)

3. Write a story problem to describe the multiplication expression below.

7 × 4

There was 7 kids and 4 terchers how meany altogeth.

Draw a picture that represents the problem; then solve the problem. Write both the repeated addition number sentence (equation) and the multiplication number sentence (equation) that describes the problem.



Each team created a word problem from an image, a collective noun, and a multiplication expression.

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How many cookies were on the plate (*problem*)? 12 + 12 + 12 = 36, or 3 × 12 = 36 (*solution*)."

We also planned for future lessons to continue using the collective noun context to introduce division. For instance, a parcel of penguins could illustrate 15 ÷ 3: Fifteen penguins are resting on an iceberg. Each parcel has five penguins in it. How many parcels are on the iceberg?

Reflections

We found that this series of lessons promoted learning and engaging on a higher level than lessons we had tried in the past. All the third graders in the school were given the same unit postassessment from the district-adopted curriculum materials. Students across the four classes had similar mathematical skills and achievement, but students in this class consistently outperformed other classes on this unit. The other students held the common misconception of confusing the relationship between addition and multiplication (similar to our observations before developing these lessons). On questions such as "Three spiders on a web each have eight legs. How many legs are there altogether?" some students in other classes added (3 + 8 = 11), showing very little understanding of multiplication as describing groups. Other students confused the role of the number of groups with the size of each group by drawing a picture of eight spiders with three legs each.

Many students in the other classes also struggled with questions such as "Write a story problem to

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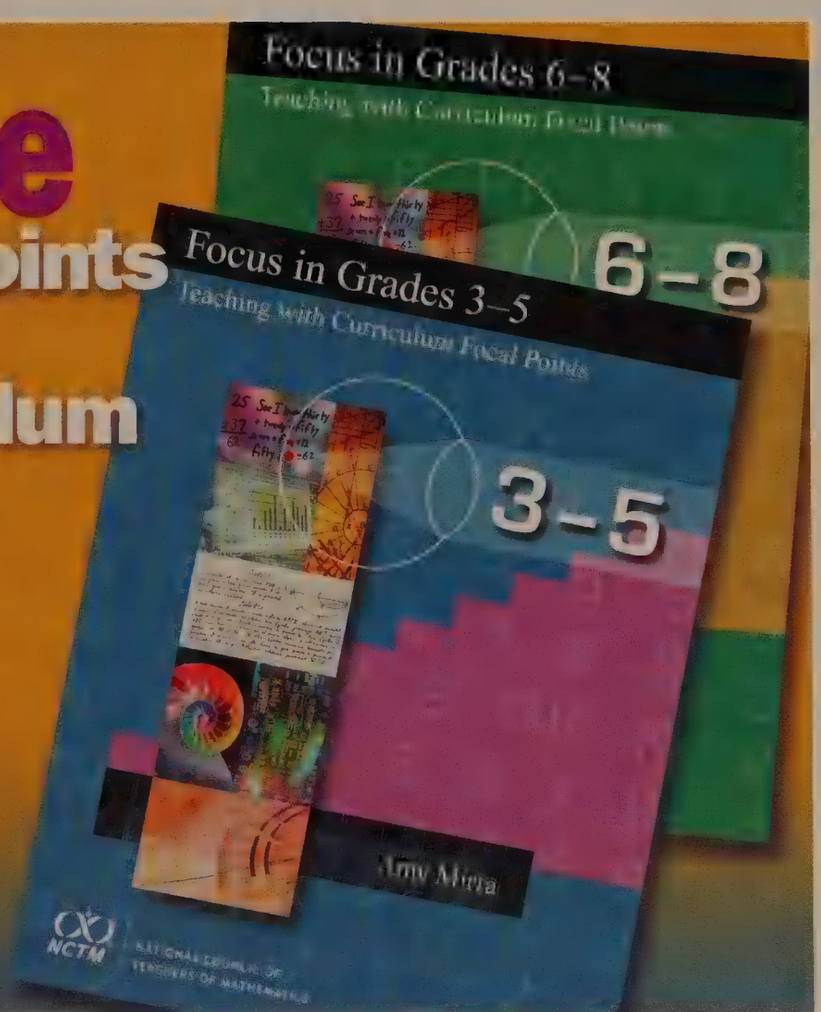

describe the multiplication expression 7 × 4." These students wrote problems involving additive situations instead of using multiplication (e.g., 7 × 4 = 11 or 7 + 4 = 28 [see **fig. 5**]). However, students who had participated in our lessons demonstrated deeper understanding by correctly describing a given multiplication word problem, both visually with an illustration and mathematically with repeated addition and multiplication equations. Our students were also able to write their own word problems using story elements and collective nouns (see **fig. 6**).

During the unit, students had earned black-and-white copies of the collective noun images as award cards for positive behavior choices. Many students still had their award cards on the last day of school—an indicator of how much the students valued their experiences with these lessons.

These investigation activities parallel many *Things That Come in Groups* (Russell et al. 1998) sessions, such as students creating a classroom chart and drawing pictures of items that come in groups as well as writing and solving riddles involving groups. But because we used color photographs of real-life situations and engaging, rich vocabulary and language, our students made powerful connections between multiplication, language, and their world.

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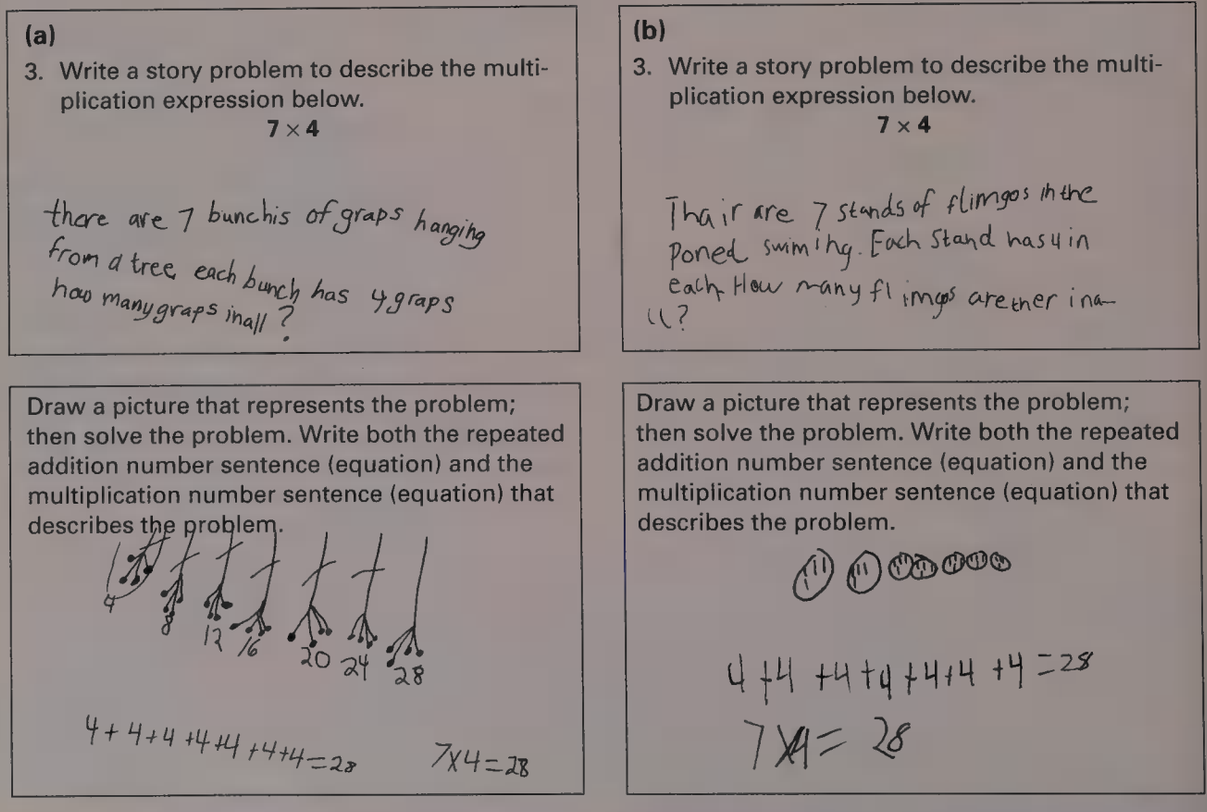
Activity sheets are on page 512.

Figure 6

These student-created word problems show deepening understanding of the role of groups in multiplication and the connection between repeated addition and multiplication.


(a)
3. Write a story problem to describe the multiplication expression below.
7 × 4
there are 7 bunchis of graps hanging from a tree each bunch has 4 graps hao many graps in all ?
Draw a picture that represents the problem; then solve the problem. Write both the repeated addition number sentence (equation) and the multiplication number sentence (equation) that describes the problem.
4 8 12 16 20 24 28
4+4+4+4+4+4+4=28
7X4=28
(b)
3. Write a story problem to describe the multiplication expression below.
7 × 4
Thair are 7 stands of flimgos in the poned swiming. Each stand has 4 in each How many flimgos are ther in all?
Draw a picture that represents the problem; then solve the problem. Write both the repeated addition number sentence (equation) and the multiplication number sentence (equation) that describes the problem.
4+4+4+4+4+4+4=28
7X4=28

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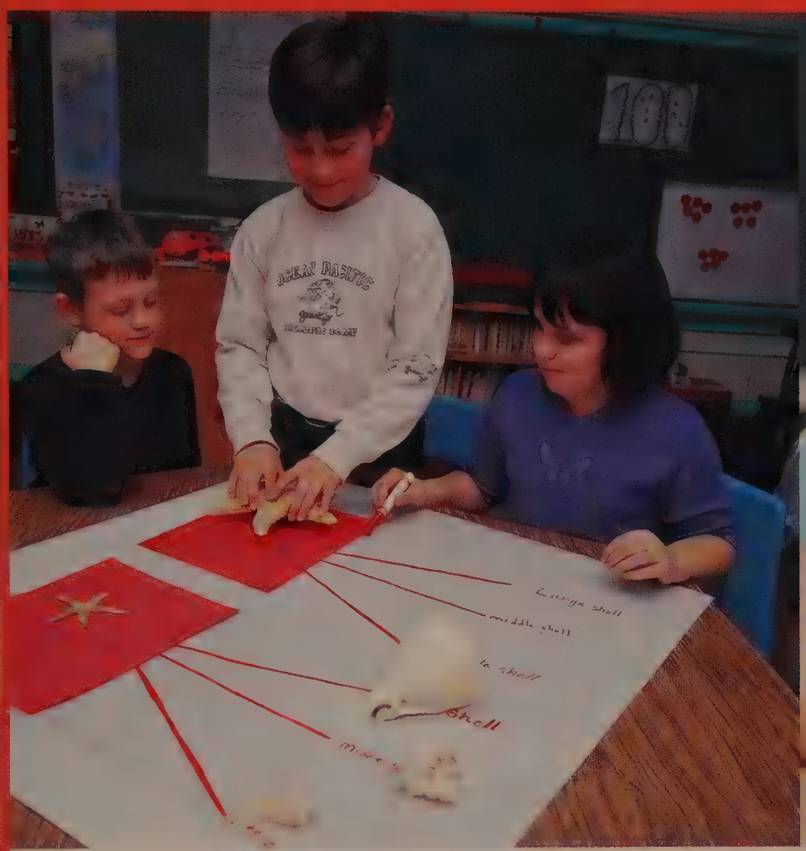

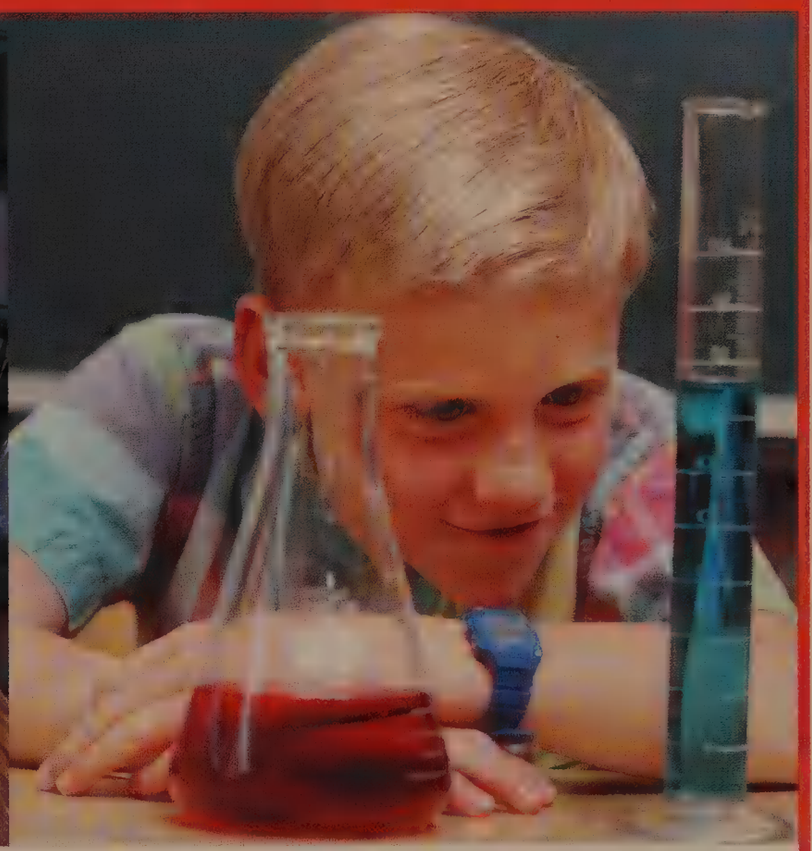

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Activity Sheet 1. Connecting Multiplication to Contexts and Language

COLLECTIVE NOUN WORD PROBLEMS

Name_____

You will need the following items:

- A set of collective noun images
- A set of multiplication expressions
- Index cards

Directions

1. Choose a multiplication expression.
2. Choose a collective noun image.
3. Write a word problem on the index card using the multiplication expression and the collective noun image that you chose (see the example).
4. Your word problem must include the following:

Front of card

- A collective noun instead of the word *groups*
- Illustration of the problem

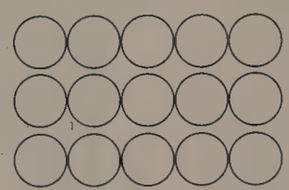
Back of card

- A repeated addition equation and a multiplication equation

Example

Multiplication expression: 3 x 5
Collective noun image: Stand of Flamingos
Word problem: Three stands of flamingos each had 5 flamingos in it. How many flamingos were there in all?

Illustration:



Repeated addition: 5 + 5 + 5 = 15
Multiplication: 3 × 5 = 15

From the April 2009 issue of TEACHING CHILDREN Mathematics

✂ .....

Activity Sheet 2. Connecting Multiplication to Contexts and Language

COLLECTIVE NOUN WORD PROBLEMS WITH LITERARY ELEMENTS

Name_____

You will need the following items:

- A set of collective noun images
- A set of multiplication expressions
- Index cards
- A list of literary elements (character, setting, event or action, problem, solution)

Directions

1. Choose a multiplication expression.
2. Choose a collective noun image.
3. Write a word problem on the index card using the multiplication expression and the collective noun image that you chose. Include all five literary elements (see the example).
4. Your word problem must include the following:

Front of card

- A collective noun instead of the word *groups*
- All five literary elements (character, setting, event or action, problem, and solution)
- An illustration of the problem

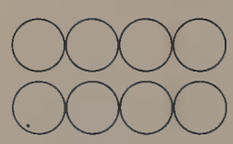
Back of card

- A repeated addition equation and a multiplication equation

Example

Multiplication expression: 2 x 4
Collective noun image: Pride of Lions
Word problem: Two prides of lions were napping in the grass. Each pride had 4 lions in it. How many lions were in the grass?

Illustration:



Repeated addition: 4 + 4 = 8
Multiplication: 2 × 4 = 8

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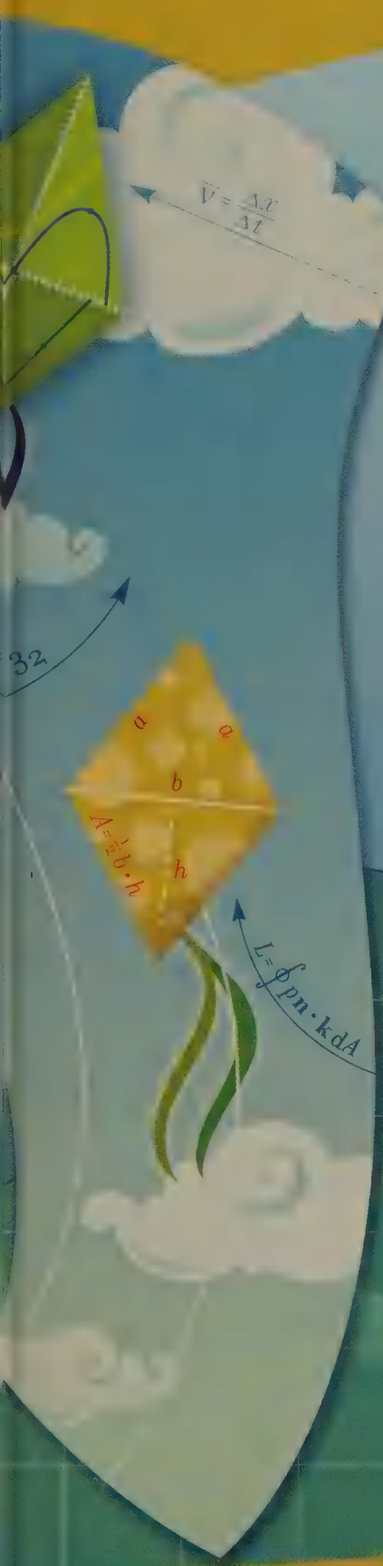

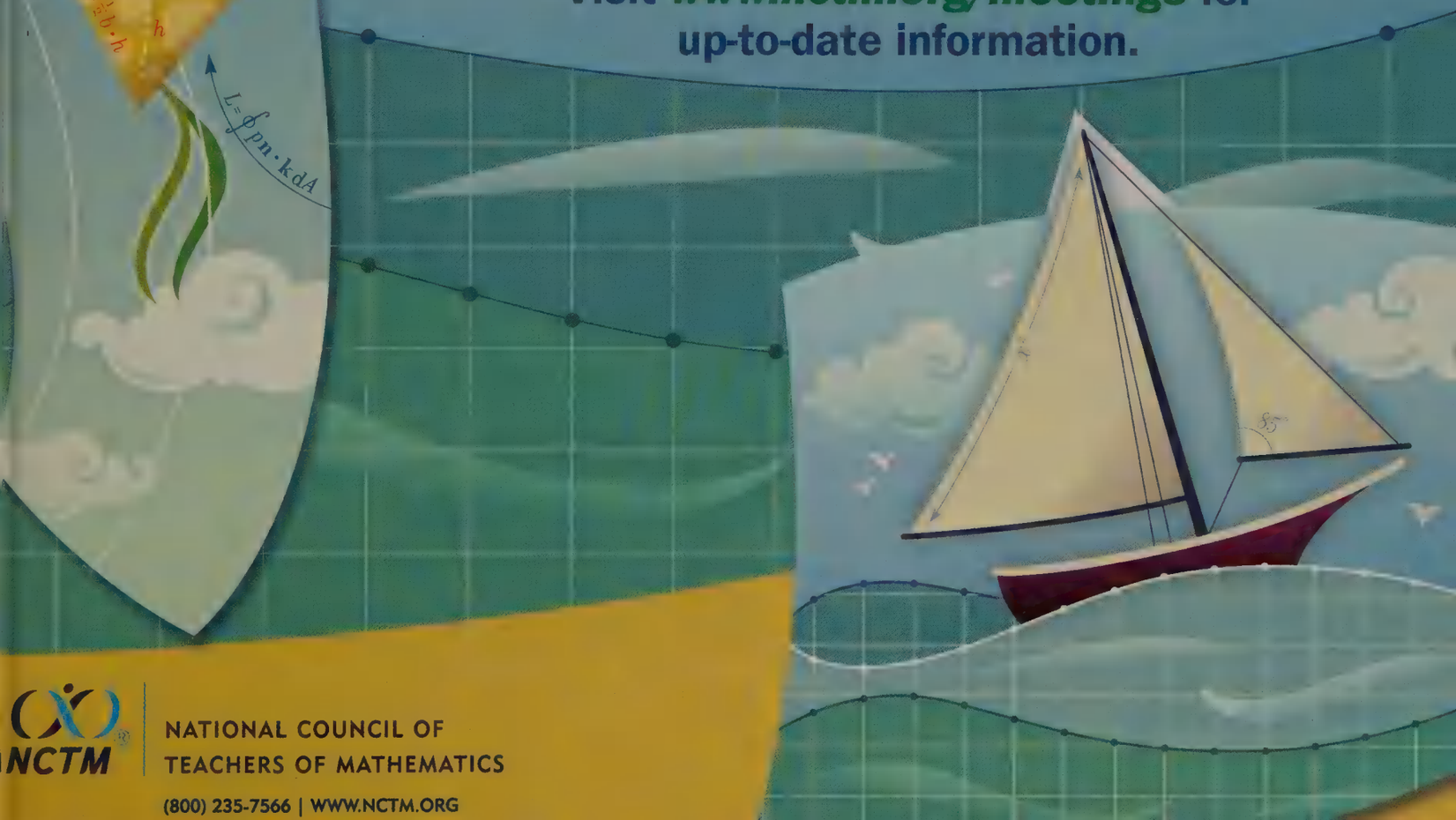

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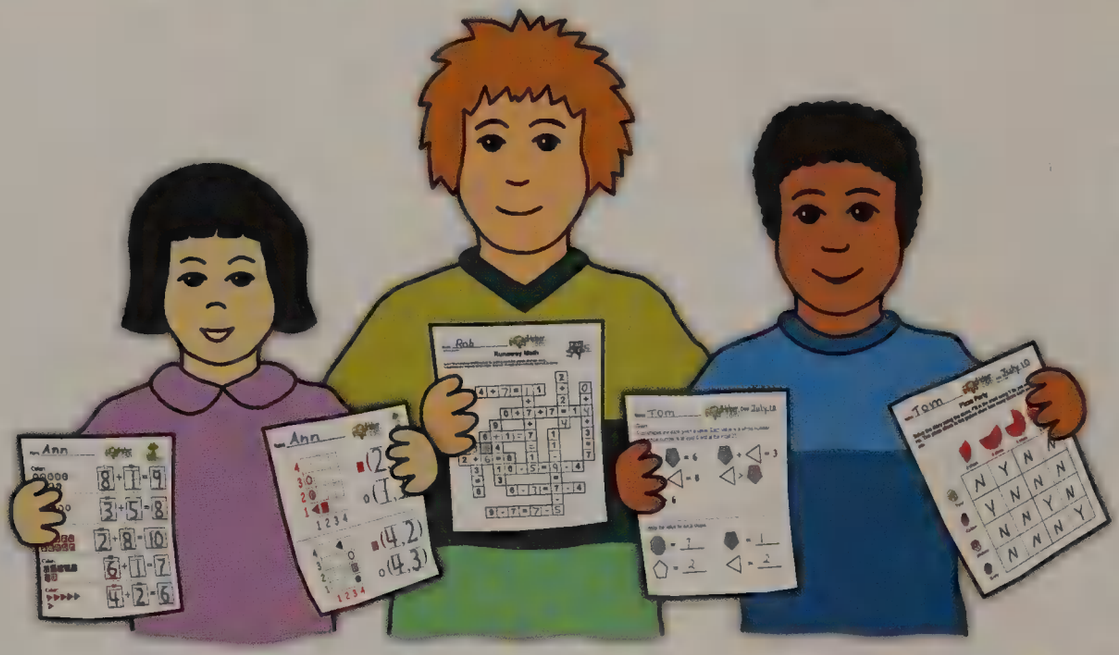

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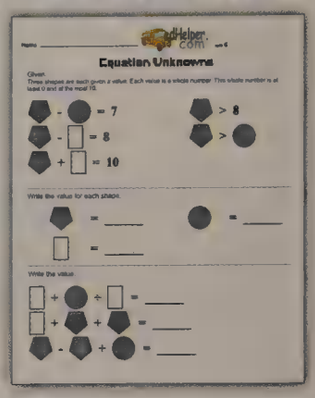

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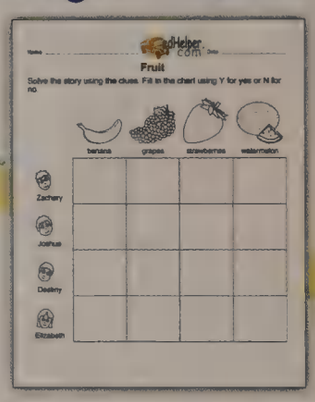

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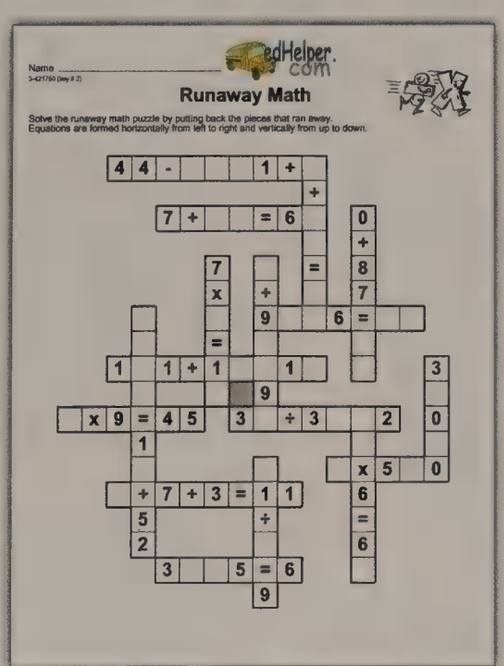
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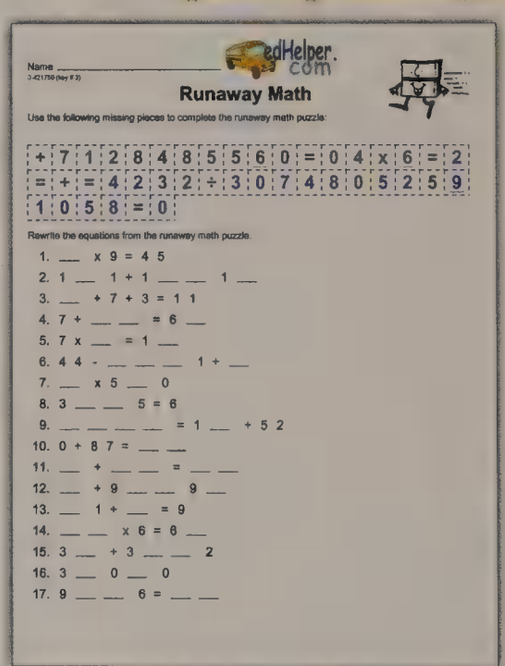
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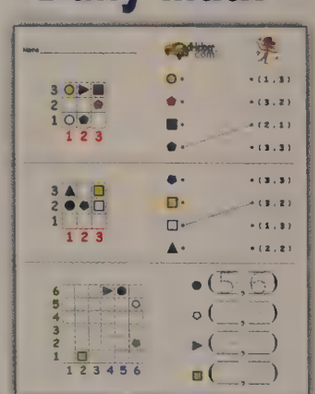

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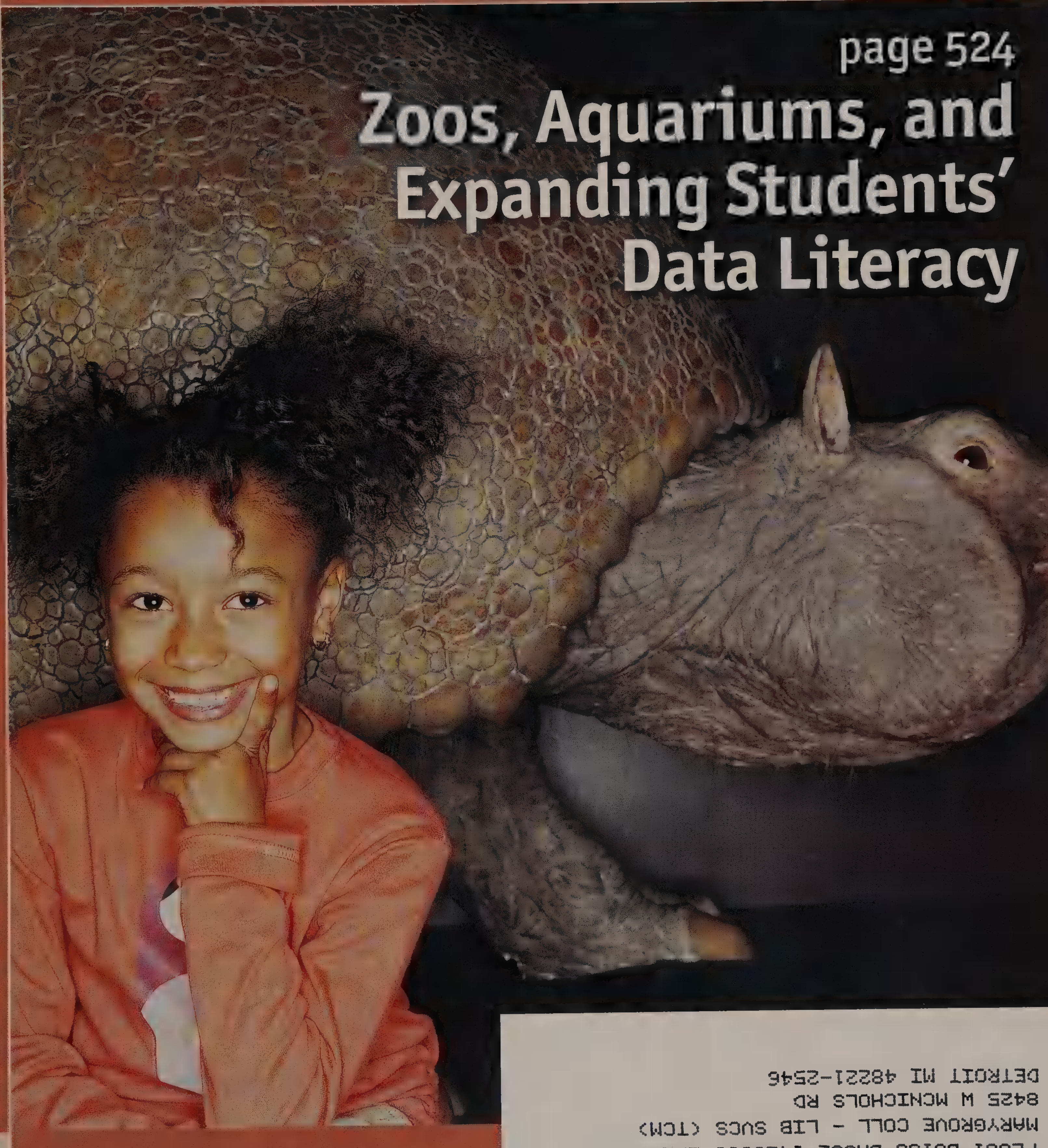

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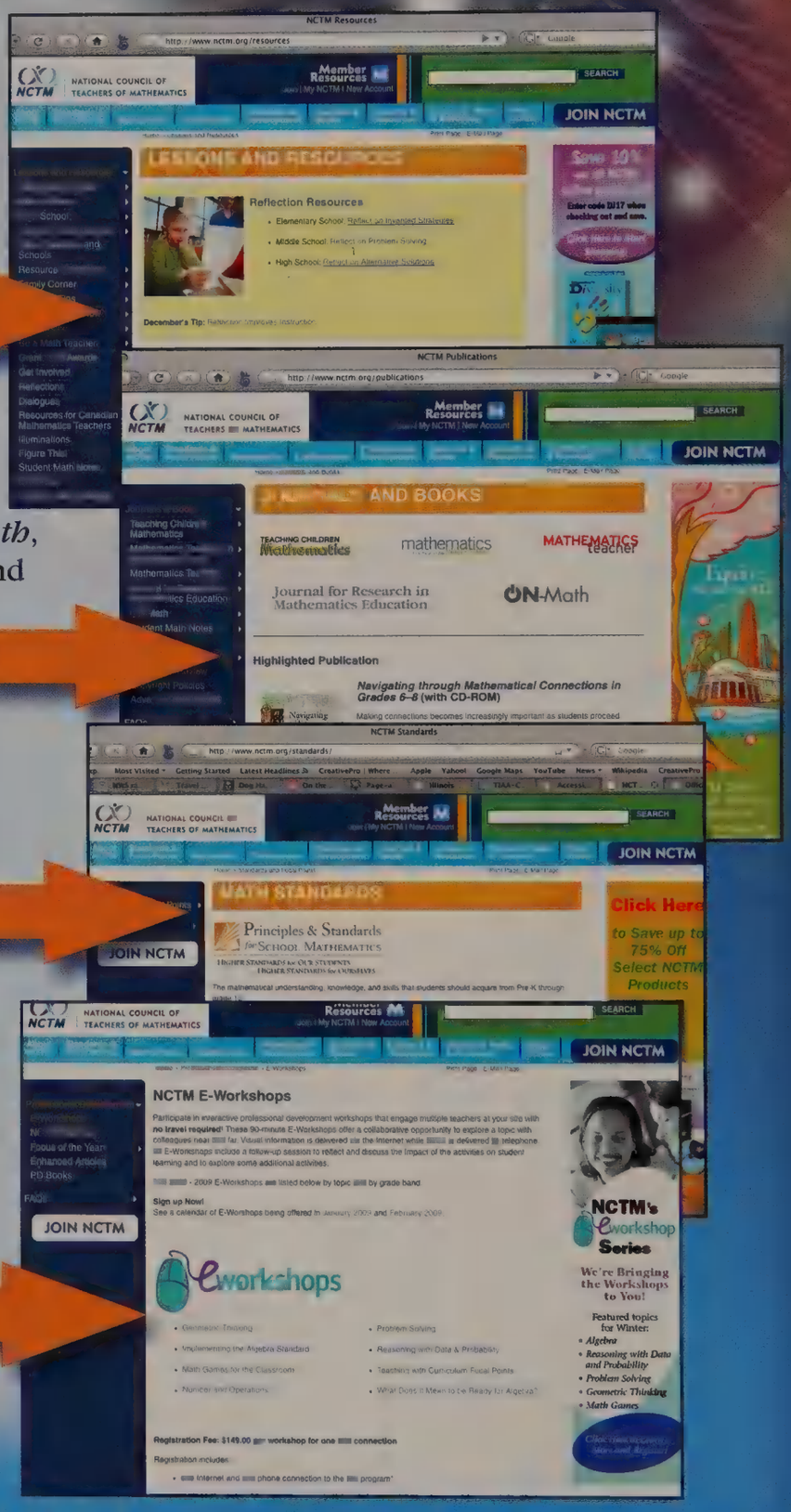

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Mathematics

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May 2009

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ON THE COVER

Zoo and aquarium educators are increasingly providing educationally rigorous programs that connect their animal collections with curriculum standards in mathematics as well as in science. Jan Mokros and Tracey Wright's article, "Zoos, Aquariums, and Expanding Students' Data Literacy," outlines step-by-step data activities for the classroom and field trips and shows how partnering with such institutions is a powerful way for teachers to provide students with opportunities to observe, collect, and analyze data.

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The Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Letters may be submitted to Teaching Children Mathematics *at* **tcm@nctm.org**. *Please include "Readers Exchange" in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit are subject to abridgment. Letters also are edited for style and content.*

Let's Shine a Light on Mathematics Learning

Math phobia plagues our citizenry and seeps into the classroom. Educators seek to improve mathematics dispositions, but a recurring theme pits us against ourselves in that effort. Consider these comments by two elementary education majors and a middle school mathematics teacher:

- The most amazing part about the lesson is that the students forget they are actually doing fractions or learning mathematics.
- It is a [math] game where the students are having fun and learning at the same time without even realizing it.
- Our main focus has been on problem solving, and with that I can trick my sixth-grade students into using basic mathematical computational skills to solve these problems, and they don't feel they are doing math.

For years I have heard these comments presented proudly as if it were a goal to hide mathematics learning in the shadows as an unpalatable monster. If we treat mathematics as a medicine that can only be tolerated with a "spoonful of sugar," we surely perpetuate bad ideas about mathematics. We imply that real mathematics is the "other stuff," perhaps inaccessible, distasteful, and useless. We do not clue students in to the fact that these other engaging activities are indeed mathematics. I suggest that we illuminate mathematics by letting students know when they are participating in this important discipline. Further, we should relay the ideal that real learning exacts cognitive demand that is to be valued and not sidestepped.

Lynda R. Wiest
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Sudoku Variation

I enjoyed reading "Using Sudoku Bulletin Boards to Teach Mathematical Reasoning" in [the November 2008 issue of] *TCM*. I thought you might be interested in an activity I use on a 9 × 9 Sudoku grid. In addition to your approach of using numbers and/or colors, you may want to challenge students to place the following shapes on the grid:

circle	square	pentagon
oval	rectangle	hexagon
triangle	trapezoid	octagon

Further, with regard to all you shared in your article related to cooperative learning and reasoning, shapes can be used to develop and strengthen mathematical language associated with identifying and classifying shapes and their properties.

Again, I enjoyed reading about your work with your students.

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EDITOR'S NOTE

Saving Dollars and Trees

➔ Look for journal indexes on www.nctm.org.

NCTM is always looking at ways to stretch your membership dollars and simultaneously promote a green work environment. Today's economy dictates that we find new ways to ensure that our limited resources are disseminated judiciously. One way to accomplish all tasks and add some value is to put each journal's index on the NCTM Web site. You will still find it in the same format, so you can search it alphabetically, by topic, author, or title. Because it is posted in Word, you will find it easier to get to from the NCTM search engine.

Each journal is still packed with the same great features and departments as previous issues.

We welcome you to log on to **www.nctm.org** and click on the respective journals and the May 2009 table of contents to view the indexes.

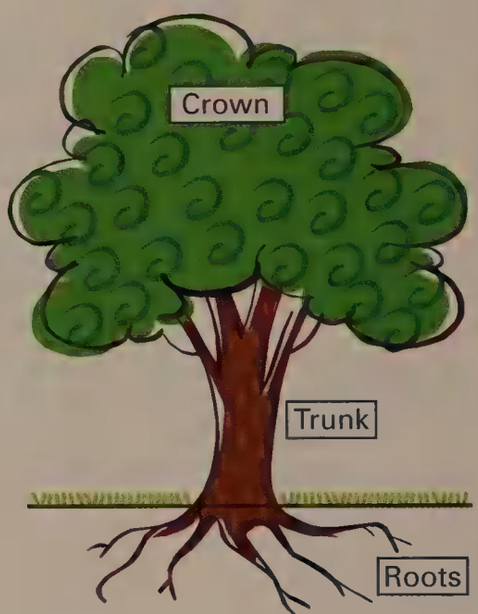
A Visit to Your School

Problem

Throughout the past nine months, our problem-solving excursions led us to exciting worldwide destinations. For our final adventure, our destination is *your* school. Whether it is in the city, country, or mountains or along a shore, your school is that special place where you and your friends go to learn. While you solve this month's problem using information that you gather at your school, we are going solve the problem using data from Pierce Lake Elementary School in Chelsea, Michigan, where one of our editors teaches.

As we walked from the school to the playground, we noticed several trees growing on the campus. Standing under one of them, we looked up toward the sky and saw many branches reaching out away from the trunk. We focused our attention on the tree's leaves.

Locate a tree near your school. Examine its branches and leaves. Focus on the size, shape, and number of leaves. Estimate the number and describe how you arrive at your estimate.



Now imagine arranging all the leaves from your tree into a puzzle with no space between the leaves. On the ground under the tree, use string or yarn to outline the circumference of the tree's crown. The *crown* is the part of the tree where the branches and leaves form (see the sketch above). Would the leaves fill the circular area marked by the string or yarn? If so, would they fill a single layer or many layers? Describe how you determined your answer.

Extensions

- Trace a leaf on grid or graph paper. Is it possible to use your tracing to determine how many leaves will be inside the circle around your tree?
- Teachers at the upper grades may want to extend this introduction by discussing terms such as *chlorophyll* and *photosynthesis*.

Edited by **Sarah Bunten**, *sbunten@gmail.chelsea.k12.mi.us, a third-grade teacher at Pierce Lake Elementary School in Chelsea, Michigan;* **Brian Schad**, *schad@aaps.k12.mi.us, a fifth-grade teacher at Lawton Elementary School in Ann Arbor; and* **Joseph Georgeson**, *jgeorgeson@usmk12.org, middle school mathematics department chair and teacher of eighth graders at the University School of Milwaukee, Wisconsin.* *Each month this section of the "Problem Solvers" department features a new problem for students. Readers are encouraged to submit problems to the editors to be considered for future "Problem Solvers" columns. Receipt of problems will not be acknowledged; however, problems selected for publication will be credited to the author.*

In the Classroom

Set the context for this problem by discussing the different parts of a tree. Emphasize the role that leaves play within our environment.

Point out a specific tree near the school and describe the estimation problem, showing students the photographs from Pierce Lake Elementary School (see **p. 518**). Emphasize the importance of working slowly and developing a strategy to solve the problem. Taking time to discuss various strategies before beginning the problem will enhance the problem-solving possibilities. For example, making the problem simpler by examining only a small portion of the leaves at a time would enhance a child's ability to successfully estimate the total.

Photograph by Sarah Bunten; all rights reserved

Where's the Math?

The problem this month involves measuring circles, estimating, and making some connections to nature. Students will consider how many leaves a tree produces in a summer. Anyone who has had to rake those leaves in the fall will know that the answer is "more than you think." Counting leaves one by one would take lots of time. Let mathematics come to the rescue.

Viewed from above, trees appear circular. So, the first strategy is to use the area of that circle to estimate the number of leaves. Assuming that they fall straight down and do not blow around, the leaves would fill up that circle in layers. Foresters have determined that, on average, trees yield about 4.5 layers of leaves in the circle under the crown. The illustration in **figure 1** might help students visualize the mathematics.

How many leaves would form one layer within the circle defined by the crown? This approach incorporates the idea of a tessellation, or covering. A leaf's shape is irregular, so it will not cover this circle completely without gaps or overlapping, but the method provides a way to estimate the area. For example, how many leaves would fill a square foot? Once we estimate that number, simple multiplication will tell us how many would fill the circle.

We can find the area of a circle by using a formula. Before you give the formula to your students, have them explore by cutting different-sized paper circles and estimating their areas. They must know a circle's radius and how it is related to its diameter. They could cut a circle as a pizza is cut into slices—through the center—and arrange those slices into a parallelogram. They could then use the area formula of a parallelogram to help determine the circle's area.

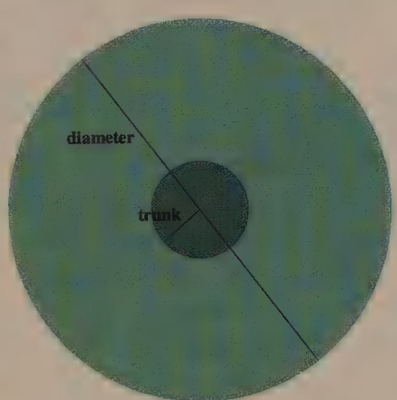
Another approach is to have students use grid paper to count the areas of two circles, one with a radius of four inches and another with a radius of eight inches. Ask them how the area changes. Such questions lead to an understanding of what the area of a circle represents and how to find it.

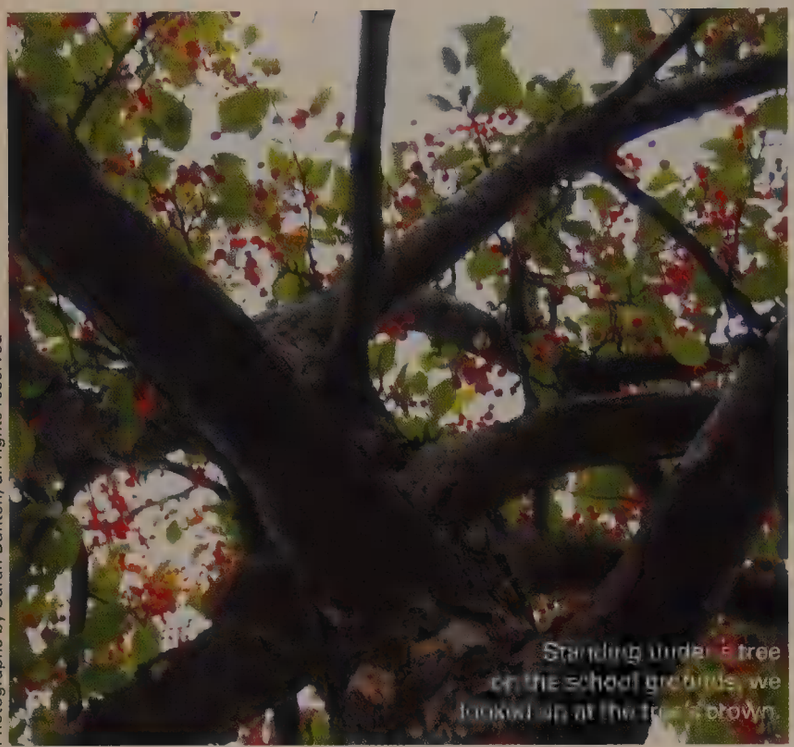
Formulas are important for solving and understanding this problem. A circle's diameter is twice its radius. Another way to state the same truth is to say that the radius measures halfway across the circle. Investigating the circumference of a circle is also interesting, and its relationship to the area is important to our analysis. However, although formulas are good to know, developing a profound understanding of area, volume, and the relationships among a circle's parts are key to studying this estimation problem.

For some of us, observing mathematics is a lifelong enterprise. Mathematics exists all around us—from our own backyards to the many exciting places in the world that we have explored this year. Nature exhibits some imperfect geometry that we may not see in textbooks, but it is real mathematics and part of our everyday lives. Nature provides a meaningful context for the knowledge and understanding that we expect students to gain by studying mathematics. So, the next time you are in the woods, start tessellating, counting, and estimating the leaves around you.

Figure 1

We can see a tree's crown from above, too. The green part represents the top of the tree. The dark part in the middle represents the trunk.





Standing under a tree on the school grounds, we looked up at the trees' brown

This leaf represents the average shape and size of the leaves on the trees at our school.

Photographs by Sarah Bunten; all rights reserved

Begin the activity by having students look up into a tree and make a quick estimate of the number of branches and how many leaves each branch holds. As they complete this problem, have students record their estimates, data, calculations, and other ideas. Monitor students as they work and provide them with the appropriate guidance, tools, and encouragement. Note the methods that they use to estimate and listen closely as they discuss their findings.

After students have completed the activity, discuss their strategies. Which ones did they find most helpful? Conclude the discussion by having students share their findings. Ask if anyone was surprised by a final estimate. How do these estimates compare with the ones students made at the beginning of the activity? How do they compare with others' estimates?

Extension

The problem may be extended for upper-grade students with the ideas presented in the Where's the Math? section.

Share Your Students' Work

Please try this problem in your classroom. We are interested in how your students responded to the problem, what problem-solving strategies they used, and how they explained or justified their reasoning. Include information about how you posed the problem, samples of student work, and photographs. E-mail your thoughts, reflections, scanned student work, and photographs by **July 1, 2009**, to **Sarah Bunten**, sbunten@gmail.chelsea.k12.mi.us, or you may send them to her at Pierce Lake Elementary School, 275 N. Freer, Chelsea, MI 48118. Include your name, grade level, and the school's name with your submission. Solutions to the **Bonus Problem** in the sidebar should be submitted to **Brian Schad**, schad@aaps.k12.mi.us, or mailed to him at Lawton School, 2250 S. Seventh St, Ann Arbor, MI 48103. ▲

(*Solutions to a previous problem begin on the next page.*)

Traveling the World Bonus Problem

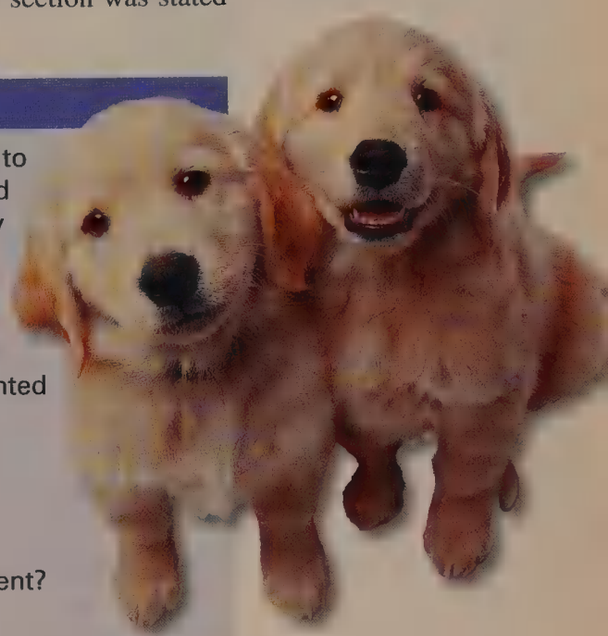
This past year the "Problem Solvers" department editors have led readers on "tours" to different geographic locations to present mathematics-related problems. In August we began by visiting Washington, D.C., and then proceeded to the Grand Hotel on Mackinac Island in Michigan; the Aquadom in Berlin, Germany; the city of Rome to see the Presidential Flag of Italy; the Liske Barn quilt in Sac County, Iowa; China's Silk Trail; the public library in Fredonia, Wisconsin; and the center of the continental United States in Lebanon, Kansas. We finished our adventures by looking at trees near our respective schools.

If we were to physically travel to all the locations in the order that they were presented in "Problem Solvers," how long a journey would it be? If you could visit the different locations in whatever order you choose, what would be the shortest possible distance that you cover? List the order. You must begin and end your journeys at the town where your school is located.

Solutions to the Walk for the Paws Problem

The Walk for the Paws problem calls for students to apply logical reasoning and algebraic thinking in a real-life scenario. The problem appearing in the May 2008 "Problem Solvers" section was stated as follows:

Problem



The Orange Beach Animal Shelter holds a Walk for the Paws event every year to raise money for dog care and adoptions. People sign up to walk three miles and may bring their dogs with them. The cost is $15 per person and $5 per dog. Every walker gets a "Walk for the Paws" shirt (for people) or a collar (for dogs).

This year's event is a big success. At the finish line, Evan and Crystal are helping count participants. Evan is told to count each person and each dog. Crystal is told to count the number of legs that cross the finish line. When everyone has finished, Evan has counted a total of 1000 people and dogs, and Crystal has counted 2500 legs. (Assume that each person had two legs and each dog had four legs.)

On the basis of information above, answer the following questions:

1. How many people participated in the Walk for the Paws?
2. How many dogs participated in the Walk for the Paws?
3. How much money did the Orange Beach Animal Shelter raise from this event?
4. How could next year's Walk for the Paws raise $15,000?

Variations

Younger students may be given more manageable numbers to work with by using 100 people and dogs and 250 legs. Also suggest that they explore a similar but easier situation with pictures to get a sense of how the number of people and dogs is related to the number of legs. Omit question 4 for younger students.

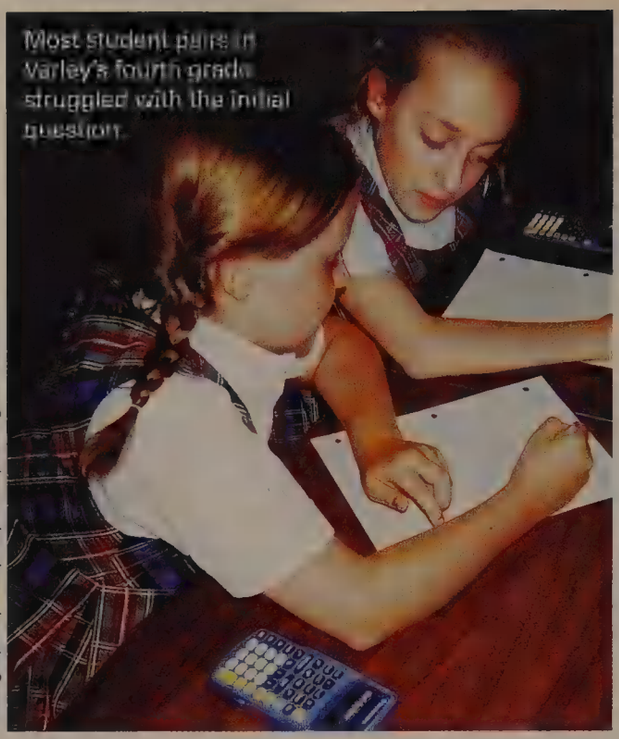
Teachers, is the following scenario familiar?

You just presented a problem-solving task that required a great deal of time and explanation. When you distributed the activity sheet, most of your students were unsure of what initial steps to take. Some of them quickly approached you with written answers but no documentation to validate their solution.

Such was the case with the Walk for the Paws problem for two dedicated teachers who used it in their classrooms and shared their teaching strategies. Reading their reflections about how they responded to such situations can help us all be more effective teachers.

Mary Kay Varley, a fourth-grade teacher at Fort Worth Country Day School in Texas, presented the Walk for the Paws problem as an opportunity for her students to problem solve in pairs and work with their calculators. Before posing the problem, Varley searched the Internet for information about actual animal shelter fund-raising walks and found one in Washington, D.C., to show to her class. She also informed students that 73 million households in the United States own dogs and about 40 percent of all households have at least one dog. Varley then created a tally chart to compare the class data to the national statistics. The class data gave students a chance to discuss how to turn their raw data into fractions,

Edited by **Mark Ellis**, *mellis@fullerton.edu, an assistant professor in the College of Education at California State University–Fullerton, where he prepares teachers of middle school mathematics and works with local classroom teachers in creating learning environments that support sense making in mathematics; and* **Cathery Yeh**, *catyeh@aol.com, who teaches fifth grade at Arnold Elementary School in Cypress, California. Each month this section of the "Problem Solvers" department discusses the classroom results of using problems presented in previous issues of* Teaching Children Mathematics.



Most student pairs in Varley's fourth grade struggled with the initial question.

Photograph by Mary Kay Varley; all rights reserved

decimals, and percentages (see **table 1**). They did long division to find the decimal for 6/7 in the *yes* column, rounded that to the nearest hundredth, and then multiplied by 100 to find 86%. They simply found the difference between 100% and 86% to get the answer that 14% of the students do not have a dog.

Following this introductory activity, Varley presented students with the Walk for the Paws problem. Her class experienced some initial difficulties with this problem. She recounts how she addressed the situation:

> Campbell and William were already on the fourth question when I realized that most of the student pairs were still struggling with the initial [questions]. I was surprised at how hard some of the students were making it for themselves. We had worked a long time on multiplying and dividing by 10s and 100s, but several groups wanted to start with very "uncomfortable" combinations that added up to 1000. Madeline convinced her partner, Katie, to start with the total number of legs and divide by 4. Madeline likes dividing more than any operation, but her answer didn't resolve their conflict. I asked if all the participants were dogs, and she knew that [they] couldn't be, but she and her partner hadn't realized that they had simply taken all the legs and found out how many dogs there might have been. After some discussion, they realized that there would have to be many fewer than 625 dogs but were stymied as to what to do next.

Varley brought the class back together and asked students to find some friendly combinations of numbers that add up to 1000. She was highly gratified to note the students' thought progression as they monitored the data:

> Their first combinations were 600 and 400. We saw that gave us too many legs. So, the second guess was 500 and 500, which turned out to be even more legs.... I loved the logical way the kids noticed how the data was changing. So, the third guess of 800 and 200 seemed to be a perfect guess. They were disappointed to see that we were 100 legs short. That's when William N. begged me to allow them to share his answer. He explained that if we need 100 more legs, maybe it made sense to take that 100 and make half of them be people and half be dogs. After checking to see if 750 people and 250 dogs equaled 2500 legs, the class seemed relieved to be able to move on to the third problem. They had no difficulty doing the third question of how much money was raised, since they had the use of their calculators. Even with the different amounts charged for dogs and people, the class found this question to be a simple matter of multiplication and addition. The fourth question brought the largest amount of discussion, confusion, and creative thinking, ... [but] once they realized that only $2500 more would be needed to get to $15,000, they were off and running as to the different ways to raise it.

An experienced teacher knows when and how to intervene. Varley's whole-class discussion of the problem, during which she modeled a guess-check-revise approach, led students to recognize a logical pathway to the solution.

Jessica Carter, a student teacher from Western Washington University, shared the Walk for

Table 1

Raw Data as Fractions, Decimals, and Percentages

Do You Have a Dog in Your House?

Over 73 million households in the USA own dogs. About 40 percent of all households have at least one dog. Does your household have a dog?

Yes, I have a dog.	No, I do not have ■ dog.
18 students	3 students
18/21 = 6/7	3/21 = 1/7
86%	14%

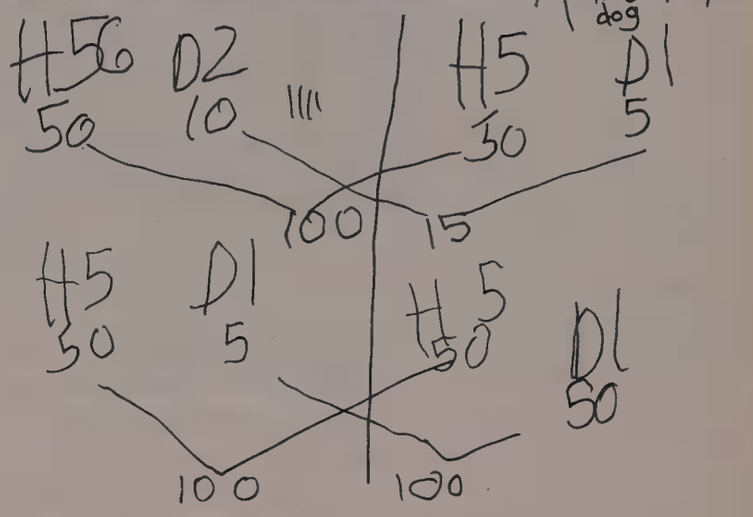
the Paws problem with supervising teacher Julia Ritchie's second graders at Geneva Elementary in Bellingham, Washington. Carter modified the numbers in the problem so that the second graders would be able to use manipulatives for support. Specifically, Carter changed the cost per person from \$15 to \$10; the combined number of people and dogs from 1000 to 25; the combined number of legs from 2500 to 60; and the dollar amount for question 4 from \$15,000 to \$300. Carter introduced the problem by reading it aloud with the students and asking them to underline key information on their copy of the problem. Students then had the options of using manipulatives and solving the problem independently or with a neighbor. Carter describes the experience:

> Many [students] were very quick to decide that they didn't know what to do. I found that many were confused about the total of 25 people and dogs. Some thought that this meant 25 people and didn't realize that it was people and dogs. I emphasized this point.... I also emphasized that the number of legs was important and would help them prove their answer.
>
> After a while, students started coming to me with written answers but no work to show how they got that answer. It looked like they had just guessed. For questions 1 and 2, many wrote some combination that added up to 25; for example, 10 people and 15 dogs. They figured that since it added up to 25, it had to be right. What they weren't paying attention to was that there had to be 60 legs. I redirected these students to go back and make sure they could prove their answer. I also reminded them that there had to be 60 legs total.
>
> Most students started off by using their pencil and paper. Eventually, some used whiteboards, links, Cuisenaire® rods, and snap cubes. One student used a guess-and-check strategy with tally marks on her paper to work through the problem. She started with 12 people and 12 dogs, knowing that this was not equal to 25 but was close. She put 2 tally marks to represent the legs for each person and 4 to represent the legs of each dog. She then switched her tally marks around, making some of the dogs into people until she reached the point where there were 60 legs and 25 dogs and people. To find the answer to question 2, she labeled each group of tally marks (representing either a person or a dog) with the amount of money it cost to enter the race. Then she added these numbers. For question 4, she used the tally system again. She added enough extra dogs so that the total amount of money would equal \$300.

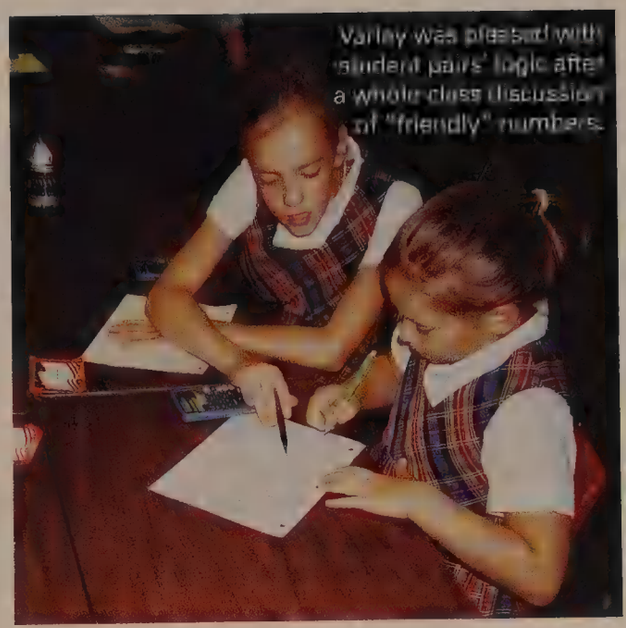
Figure 1

Carter had a group of second graders explain their answer to her until she was certain that she understood their thinking.

1. How many people participated in the Walk for the Paws? 20
2. How many dogs participated in the Walk for the Paws? 5
3. How much money did the Orange Beach Animal Shelter raise from this event? 225
4. How could next year's Walk for the Paws raise \$15,000? 7 more dog People



Carter had one group of students explain their answer to her several times. Although their work is difficult to read (see **fig. 1**), Carter finally understood their thinking and method:



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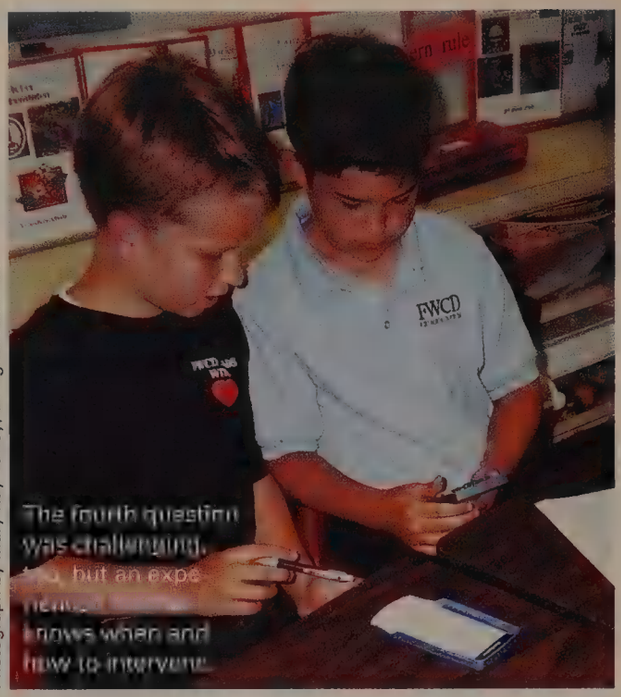

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They used a whiteboard to do their work and then transferred it to the paper. They made a chart with 4 sections and started with a guess. They decided to divide the 25 people and dogs into 4 families [with] 5 people in each section of the chart, so there were 20 people total. Then they started adding dogs. To do this, they wrote a [letter] *D* to symbolize *dog* in each section. They then added tally marks after the *D* to show their work. The first section [has] 2 tally marks, and … each of the other 3 sections [has 1 tally mark, for a total of] 5 dogs. The students then decided to look at the number of legs this produced. They knew that each family of 5 people had 10 legs, and they added to get 40 legs. They used tally marks to show the number of legs the dogs had and added these to get 20. This [totaled] 60 legs.

Although not all of Carter's second graders arrived at the correct answer, this problem provided them with an opportunity for growth in multiple areas. This was the first time many of them had spent so much time on just one problem; they learned that mathematics includes reasoning processes. Students revisited the problem through multiple strategies: They used manipulatives; drew visuals; worked individually and with partners; and explained their thinking on whiteboards, on paper, and to the whole class. Having students analyze a problem in so many ways showed them that "doing math" means more than getting the right answer. They learned that the problem-solving process may be just as rich as the solution.

The experience was also an opportunity for students to analyze a problem closely and to pay attention to all the important details. Many of their initial answers fulfilled the requirement of having a total of 25 people and dogs; however, they did not take into account the second criteria of having a total of 60 legs. Exploring problems that require an answer that meets two or more criteria exposes students to higher-order thinking, which helps build and strengthen their problem-solving skills.

All good problem-solving activities provide opportunities for students to learn about mathematics and for teachers to learn about their students' thinking. Carter saw this activity as a lens into her students' understanding of mathematics:

> I can see that [these second graders] know a lot. Their work shows me that they know how to add up to 25, 60, and 225 correctly. The way the first student shows her work is a nice setup for multiplication. It looks like she already has a basic foundation for it. In … students' examples, I notice the use of a symbol, *D*, to represent dogs. This is a great spontaneous use of algebra that came from a real-life example and, better yet, from the students themselves!

This "Problem Solvers" challenge dealt with logical reasoning and algebraic thinking in a context that intrigues children—an activity with people and pets. Because many of the students experienced initial difficulty with the problem, the responses that Varley and Carter share can provide insight into strategies for scaffolding the problem for students.

Look at the problem-solving strategies; Carter's second-grade class and Varley's fourth-grade class both used the guess-check-revise method. Although this can be a time-consuming strategy if it stops at the guessing stage, reasonable and well-informed guessing requires students to use logic and mathematical reasoning.

This problem also presents an opportunity to encourage students to look for multiple solution methods. Ask your students to solve the problem using manipulatives, diagrams, tables, and (for older students) algebraic symbols. Exposing them to multiple approaches for solving one problem leads to in-depth discussions and helps expand their understanding of mathematics.

Special thanks to those teachers and students who contributed to this article:

Jessica Carter, Julia Ritchie, and the second-grade students at Geneva Elementary in Bellingham, Washington; and

Mary Kay Varley and her fourth-grade students at Fort Worth Country Day School in Texas. ▲

CALL FOR MANUSCRIPTS FOR THE OCTOBER 2010 FOCUS ISSUE

Teaching Mathematics in a Flat World

Author Thomas Friedman's "flat world" phrase points to our new century's globally connected nature, exponential technical advances, and instantaneous communication across the planet. Former U.S. Secretary of Education Richard Riley keenly observed, "We are preparing students for jobs that do not exist, using technologies that have not been invented, in order to solve problems we cannot even imagine at this time."

Teaching Tomorrow's Citizens Today

As we hurtle through this rapidly advancing flat world, what does it mean to "know," "do," and "teach" math? Attempting to prepare students for a dynamically changing environment, what mathematical knowledge and essential understanding do elementary teachers need today to help them better educate the workforce and leaders of tomorrow?

The *Teaching Children Mathematics* Editorial Panel is seeking articles to address themes related to the mathematical knowledge of these teachers. We strongly encourage submissions that reflect the value of teacher knowledge directly within the classroom and its influence on student learning and achievement. Moreover, we welcome articles related to preparing preservice teachers as well as practicing teachers. The following topics are to guide—not limit—authors in addressing one or more such issues.

Teachers' Essential Mathematical Knowledge

- Identify the skills, concepts, and dispositions elementary teachers must know to promote both the procedural fluency and the conceptual understanding of their students.
- Provide evidence regarding content areas or specific topics that enable teachers to become more confident and competent in teaching mathematics.
- Describe the *essential* mathematical understandings necessary for elementary teachers and how these may differ from typical mathematics requirements.
- Explain how these essential concepts allow teachers to empower *all* their students with the mathematical skills and innovative problem solving needed in today's flat world.
- Demonstrate the connection between teachers' essential knowledge and the elementary curriculum in state or national standards or within NCTM's *Curriculum Focal Points*.

How to Best Learn

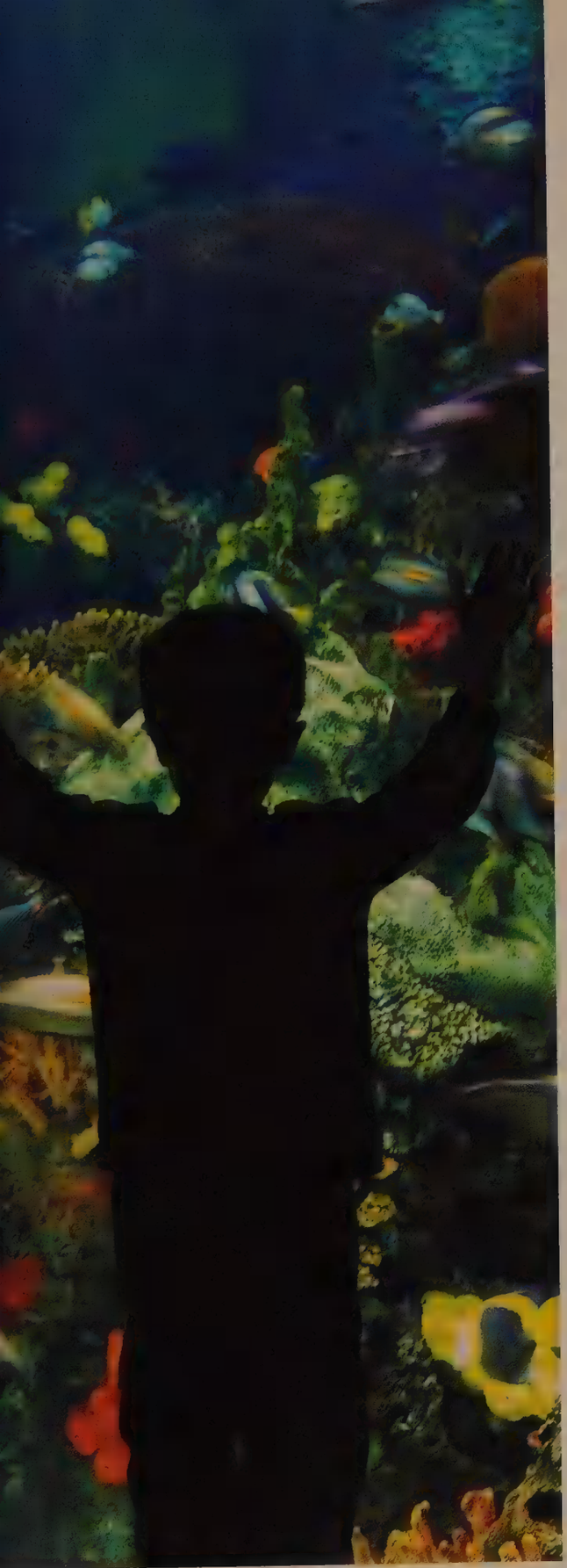
- Provide examples of innovative courses or professional development programs that have been successful in increasing teacher content knowledge and understanding.
- Describe effective, collaborative efforts—among school districts, institutes of higher education, and other agencies—for preparing elementary teachers to teach mathematics.
- Explain obstacles to increasing teachers' mathematical knowledge and how they have been effectively addressed.
- Clarify the importance of "mathematical habits of mind" and pedagogy in relation to learning and teaching mathematics in a flat world.
- Describe technology tools and other resources that support teachers in their work and in their acquisition of essential knowledge.

Impacting the Classroom

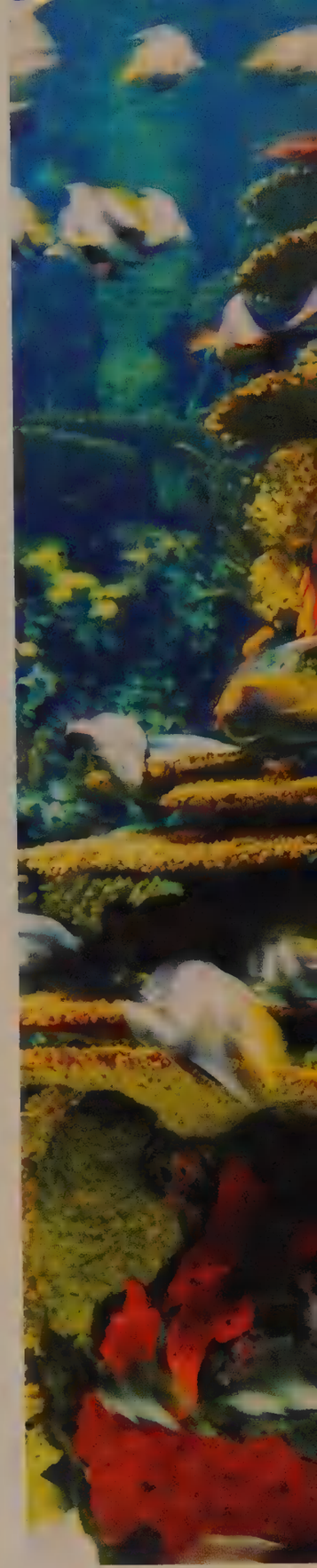
- Explicate the effects of an increase in a teacher's mathematical knowledge on the teacher's curriculum selection, teaching practices, and assessment choices.
- Discuss the connections between teachers' mathematical knowledge and student learning and achievement in mathematics.
- Describe specific classroom examples or experiences that demonstrate the importance of teacher knowledge in learning a particular mathematical concept or idea.
- Explore beliefs that teachers hold with regard to the value and type of mathematical knowledge needed in their classrooms.
- Explore the roles of math coaches and elementary math specialists in advancing the understanding of mathematics for both teachers and students.
- Examine the role of teacher collaboration in aiding teachers to reflect on and improve their mathematical knowledge and classroom practice.
- Discuss the level of mathematical understanding required for teachers to successfully address the needs of diverse learners and differentiated instruction.

Manuscripts should not exceed 2500 words; include figures and photographs at the end. Submit completed manuscripts to *Teaching Children Mathematics* by accessing **tcm.msubmit.net** by July 31, 2009. On the cover page, please state clearly that the manuscript is being submitted for the October 2010 *TCM* Focus issue. Author identification should appear only on the cover page. For manuscript preparation guidelines, visit my.nctm.org/eresources/submission_tcm.asp.

Zoos, Aquariums, and Expanding Students' Data Literacy



Nicholas Eveleigh/Getty Images



Field trips provide compelling venues for teaching students to work with observational data and make important connections between mathematics and science.

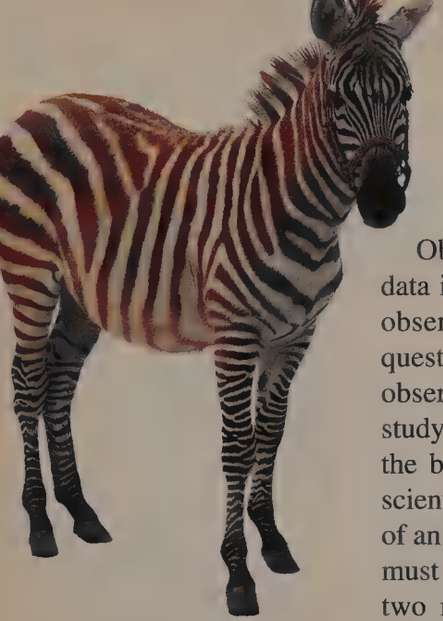
Collecting survey data by asking people questions is a staple ingredient of most elementary mathematics classrooms. Experiences with conducting surveys help students learn to deal with "messy" data and provide important models of the work of social scientists. Although elementary school students often have experiences with gathering, representing, and making sense of survey data, they seldom gather and analyze the kinds of observational data that are central to scientific research. Third through fifth graders ought to "collect data using observations, surveys, and experiments" (NCTM 2000, p. 176) in order to connect what they are learning in mathematics to other contexts (p. 64), especially contexts that are rich with real data. Another important piece of learning about data that is expected of students at this grade level is the ability to link data that they collect with conclusions that they formulate (p. 176).

Why Learn about Observational Data?

Because collecting observational data is at the heart of natural scientists' work, students should have more experiences doing so. A recently conducted project found that zoos and aquariums are compelling venues for teaching students to work with observational data.

By Jan Mokros and Tracey Wright

Jan Mokros, jmokros@mmsa.org, is the executive director of the Maine Mathematics and Science Alliance. Tracey Wright, Tracey_Wright@terc.edu, is a senior researcher and developer at TERC in Cambridge, Massachusetts.



Observational data are different from survey data in a few important ways. Most obviously, the observational data gatherer does not formulate questions for participants to answer but instead observes participants' behaviors. Scientists who study animals must make careful observations on the basis of agreed-upon protocols. For example, scientists in a zoo might collect systematic samples of an animal's behavior over a period of time. They must decide how to sample behaviors (once every two minutes or every time they see a behavior change), establish reliable ways of collecting data (ensuring that observers agree on what is being observed), and identify methods of collecting data that are meaningful and make sense. The work is often more unpredictable than conducting surveys, because the observed organism determines the investigation's direction.

Why During Math Class?

Why teach this type of data gathering in mathematics class, when it might legitimately be taught in a science class? Although some students experience these data collection methods in science class, such experiences are becoming increasingly rare. Unfortunately, science has been crowded out of the elementary curriculum: A recent study in California found that 80 percent of K–5 teachers taught science for one hour per week or less (Dorph et al. 2007). One outcome of this study was the recommendation that science be integrated into mathematics programs. In the case of data, this makes good sense, because collecting and analyzing observational data are as integral to mathematics as they are to science.

Teachers often face another significant barrier to helping children learn to collect and analyze observational data: Classrooms do not offer much to observe other than people's behavior. Plants are sometimes "observed" and measured but offer few, if any, events that students can classify during class time. Classroom pets, which were once a source of learning about animal behavior, are less common than they used to be. Significant benefits exist to making animal behavior more available to students as an exciting way to help them learn about the mathematics of data.

Figure 1

Work with a zoo educator to make the most of your field trip.

As you consider various species, identify what is scientifically interesting about the species:

- What do animal behaviorists look for when they observe this species?
- Do scientists observe particular behaviors to determine whether a species may be suffering from environmental problems, such as overcrowding or lack of food?

Questions from a pragmatic level are equally important:

- Do the animals under consideration exhibit a range of behaviors that viewers can observe during the time that the class will visit? Animals that are inactive during the day do not make good subjects.
- Is there enough behavior to make the activity interesting but not so much behavior that it is impossible to keep track of it?

Why Observe Animals?

A powerful way of providing children with more opportunities to observe, collect, and analyze scientific data involves partnering with zoos and aquariums in the service of expanding students' data literacy. Zoo and aquarium educators are increasingly providing educationally rigorous programs that connect their animal collections with curriculum standards in mathematics as well as science. Especially relevant to classroom teachers is the work that zoo and aquarium educators are doing with animal behavior studies that involve *ethograms*, descriptions of behaviors associated with a species. The Bronx Zoo, for example, involves middle and high school students in observing and making ethograms of primate behavior. Younger children would find these activities accessible as well. (See www.bronxzoo.com; follow links to education, education programs for all ages, school programs, primate behavior.)

Regular field trips to informal learning institutions can involve significant work with data. For example, field trips to science centers have engaged upper elementary students in work with measurement and data gathering (Sedzielarz and Robinson 2007). Emerging work conducted by the Math in Zoos and Aquariums (MiZA) project shows a strong parallel phenomenon, and even virtual field trips are beginning to support these efforts.

Before your field trip

A critical lesson from the MiZA project is that optimal field trip experiences focusing on animal

observation start in the classroom. Students should spend time studying the animal behavior that they will observe and practicing data-collection methods. Teachers and zoo educators suggest the following teacher tips to make data-gathering trips to the zoo or aquarium more fruitful for students:

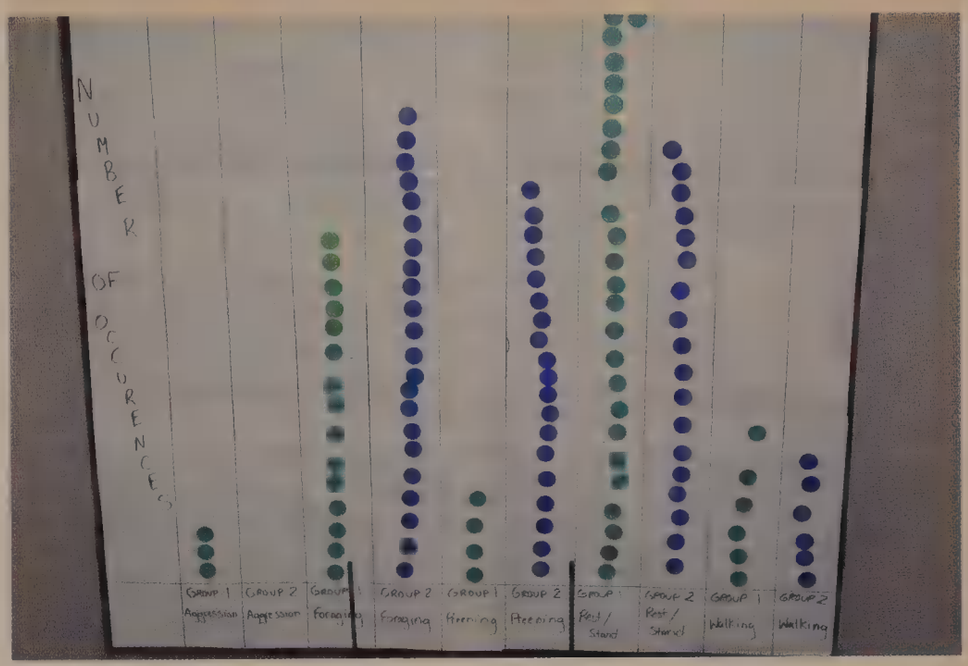
1. Identify animal collections to observe. Before making the field trip, visit the zoo on your own or contact the zoo's education department to identify an animal collection that is particularly suitable for observation and data collection. You may choose to observe a collection of large animals that are particularly engaging or a group of lesser-known but interesting small animals, such as flamingos or prairie dogs. Insect behaviors are fascinating to children. For example, diving beetles, which are a common collection in aquariums, have the ability to fly, dive, swim, and even engage in cannibalism. (See **fig. 1** for other suggestions.)

Once you have identified the animal collection to observe, work with the zoo or aquarium educator to make a list of five to ten common behaviors that your students are likely to see. Often these behaviors involve eating, grooming, forms of locomotion, and social interactions. Select behaviors that are not hard to interpret. A behavior such as "preening" is often easy to observe, whereas "playing" is harder to pinpoint and is more open to interpretation. Many zoos and aquariums have developed lists of behaviors that visitors can use to guide their observations. For example, the Phoenix Zoo has an easy-to-use catalog of flamingo behaviors. For information on this or the New England Aquarium's Penguin Chart, see www.phoenixzoo.org/animalmath or e-mail Gabrielle Hebert at the Phoenix Zoo: ghebert@thephxzoo.com. The key is to connect with your local zoo or aquarium educators to focus on behaviors of a species that they are highlighting.

2. Enact animal behaviors and practice observations through simulations. An interesting kinesthetic way to familiarize students with animal behaviors involves simulations. This activity was developed by Rebekah Stendahl at the New England Aquarium, where it is used with students in grades K–2 to help them learn about penguin behavior.

Figure 2

To record their data, students can use charts such as this one from a New England Aquarium activity for observing penguin behaviors.



Using the list of animal behaviors that you have developed, prepare a chart in which the behaviors are listed along one axis and a series of time intervals (about ten seconds) are listed along the other. For example, the New England Aquarium's activity involves observing penguin behaviors using the chart shown in **figure 2**.

Ask a student to demonstrate each behavior, or demonstrate the behaviors yourself. Encourage students to mimic the ways the animal would engage in the behavior. For instance, birds usually preen with their beaks, not their claws. Be prepared for student enactments to be energetic!

Collecting and analyzing observational data are as integral to mathematics as they are to science

Afterward, ask students to turn their attention to the role of scientists. Use this opportunity to demonstrate what scientists do and how they do it. Distribute the checklist of animal behaviors. Explain that scientists use different systems to collect and record observational data and that they need to be careful observers. One observation method is to sample what an animal does every few seconds or minutes, known as *time sampling*.

Next, ask half the children to "be the scientists" and half to "be the animals" that you are planning to observe. Each scientist chooses one animal to observe. On one side of the room are animals; on the other side are scientists equipped with clipboards, pencils, and observation charts. A timekeeper calls time every ten seconds. The scientists take mental snapshots and categorize what their animal is doing when time is called. Each scientist makes a check mark in the appropriate column and moves on to the next observation. Repeat this activity for at least eight time intervals.

Meanwhile, the animals act like animals, paying no attention to the scientists. At the end of the observation, scientists and animals switch roles and repeat the process.

This activity is likely to raise questions about how scientists collect behavior samples. Students often find that the most interesting behaviors happen at times other than during assessment periods, so discuss the various ways that scientists sample data. Sometimes they record every behavior that they see for a relatively short period of time (for example, at feeding time). Other times, they take data once every few minutes over a longer period of time. The way they sample data depends on the question that they are addressing.

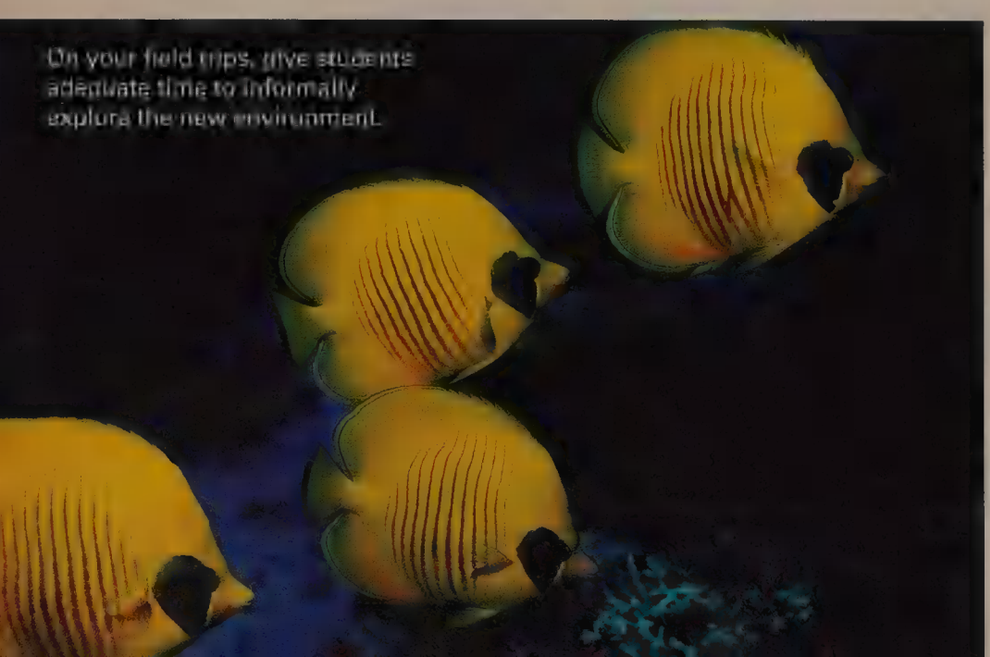


Figure 3

Students learn to take mental snapshots.

Whenever time is called, students form a mental image of what the animal is doing and find that behavior on the checklist:

- If they see a behavior that is not on the checklist, they either use the "other" category (if you have included one) or do not record anything during that time period.
- If they see several behaviors, they need record only the behavior they saw when "time" was called.
- If they keep seeing the same behavior (such as resting), they continue to record this behavior whenever "time" is called.

On a practical level, this activity gives children experiences recording data accurately, using a chart format that involves columns and rows. Younger children often have difficulty figuring out where they should make a check mark. As you work to clarify this skill, use the terms *column* and *row* so that students begin to develop an understanding of the differences between the two.

At the zoo

Giving students time to explore the new environment is important, much as you would encourage them to explore a new mathematics manipulative.

1. Examine the animal collection. First, give students time to simply look at the animal collection they will be working with. This introduction to the animals should involve opportunities for unstructured observations and "getting the lay of the land" in terms of where animals are located, how quickly they move, and how to identify and keep track of an individual animal.

2. Review procedures. Every student needs one animal to observe. An easy way to assign animals to students is to have them point to the animal that they wish to observe; make reassignments if two students choose the same animal. If there are not enough animals for each student to observe one, then assign teams or have students make consecutive observations. Remind them to use the visual snapshot method (see details in **fig. 3**).

3. Ready, set, observe! If the zoo is busy, ask the zoo educator if it is possible to keep other visitors clear of the exhibit area during the short time

that your students will be observing. Keeping the observation periods relatively short—a period of two minutes, divided into ten- or fifteen-second segments—works well for both animals and humans; intervals of ten or fifteen seconds give students time to check the appropriate box.

In most cases, the teacher or another adult serves as the timekeeper and calls out the time intervals in order ("Time 1, time 2, time 3") to help students keep track of the observation intervals. Have other adults, such as parent chaperones, available during this time to help students record in the appropriate places on their checklists. Chaperones often appreciate having a substantial role to play during the observations.

Before ending the observation period, take time to answer any questions students have about their charts. You may want to use the first observational round as practice and a second round for classroom use. Students are bound to make errors in completing the charts, in part because animals are not predictable. Animals disappear behind enclosures, sight lines are lost, or animals may have a flurry of activity that is difficult to code. Students often make mistakes when they are learning the difficult process of observation. They might end up recording two observations in the same column or losing track of the time intervals. Before leaving the aquarium or zoo, acknowledge that collecting data is difficult not only for students but also for experienced scientists.

Back in the classroom

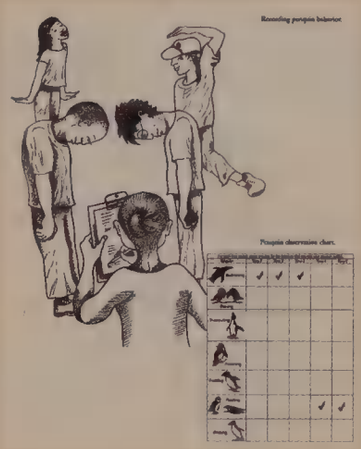
The data analysis phase is critical, as it helps students understand what can be learned from data and why scientists care about their data.

1. Work with observational data. A good way to begin is to review the observational work students have done, asking questions such as, "What was hard about observing the animals?" Remind students that these challenges are the same ones that scientists encounter. Part of a scientist's job is to determine whether his or her data collection processes have been "clean" enough to inspire confidence in the findings.

Next, ask students for their predictions about the collective behaviors of the animal groups they observed. Which behaviors do they think will be most and least frequent and why? Students often make predictions on the basis of the individual animal they observe and assume that all the animals behave as theirs does. Encourage students to think about more than just their own individual observations.

Figure 4

Two groups of upper elementary school students recorded flamingo behavior at the Phoenix Zoo at different times.



Determine a system to combine the data from all the observations. An easy system is a line plot chart listing each of the animal behaviors along a horizontal line on the board, leaving plenty of room for each student to post data in a "stack" over each behavior. Ask students to use small stick-on notes representing the number of times they saw each behavior (three stick-on squares for swimming, two circles for resting, etc.). As they finish this task, invite them to come to the board and post their stick-on notes in the proper categories. Encourage them to neatly stack their data so that observations are easy to count. As the chart is completed, ask for volunteers to count and record the number of data points in each category. The chart in **figure 4** shows flamingo behavior observed at the Phoenix Zoo; upper elementary school students compared two different groups at two different points in time.

2. Analyze observational data. First, ask students to reflect on and describe the data set. Then have them talk with a partner or within small groups about questions such as (1) which behaviors were the most or least common in their animal observations and (2) how their observations are like or unlike the data from the entire class. Have them spend a good portion of time describing what they found and then discussing the data set with their peers. They may be eager to speculate about why they found what they did, but they need to first familiarize themselves with the data. After they have done so, ask for (3)

Collecting observational data is at the heart of natural scientists' work

some hypotheses that might explain their results. Students are usually eager to share their ideas with the entire group. As you engage in class discussion, be sure to bring up the notion of time sampling. You might encourage students to think about what would happen if they observed for a longer period of time or at different times during the day. Of course, students will have many predictions about these differences; following up on them could easily become an occasion for another data-gathering field trip. We have worked with some teachers whose classes make multiple zoo visits to follow up on students' questions.

Conclusions

The Math in Zoos and Aquariums project found that studying animal behavior can deepen students' understanding of the entire cycle of asking questions; collecting, representing, and interpreting data; and using the findings to generate a next set of questions. Engaging in this cycle is a key component of zoo and aquarium scientists' work. Colleagues at the Phoenix Zoo, for example, have done a series of studies, each informed by a previous study, to better understand how environmental changes could enrich the lives of elephants (Tresz and Wright 2006). After observing how much time elephants spent foraging and engaging in aggressive and stereotypical behaviors, the scientists subsequently observed the effects of interventions, such as toys and new activities, on these behaviors.

Students' field and class work while studying data is analogous to scientists themselves spending only part of their time in the field and a good part of time back in their offices examining data.

As well as increasing students' understanding of scientists' work, learning in zoo and aquarium contexts can greatly enrich the connections that students make with observational data. Making the most of zoo and aquarium visits involves careful planning to connect previsit, visit, and postvisit work with data. Students could also learn many of the same data skills by working with other animals that are more familiar to them and accessible during school hours. For example, studying pigeon behavior is sometimes possible on or near school grounds (Corwin and Russell 1990, p. 69). However, it is important that this be a group activity and that there are enough easy-to-distinguish animals to accumulate a significant body of data. Such requirements are easier to fulfill at a zoo or aquarium than in most other places. Although the expense of field trips is a legitimate concern, note that 80 percent of American Zoo and Aquarium Association members surveyed in 2002 offered free educational programs, reaching over 3.7 million students (AZA 2003).

Increasingly, teachers are asked to justify the learning opportunities afforded by field trips. Focusing on animal behavior provides a way not only to link a zoo or aquarium field trip with mathematics and science but also to demonstrate the centrality of data analysis in the work of animal scientists.

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HOT Topics

Call for Manuscripts

Teaching Children Mathematics is interested in publishing articles on "hot topics" that *TCM* readers have noted as subjects of high interest on reader surveys. By highlighting the issues and challenges that face mathematics educators today, the Editorial Panel hopes to provide teachers and teacher educators with resources to assist them in their efforts to meet the mathematical needs of all students. We invite authors to share their classroom experiences and their ideas. Manuscripts that include photographs and samples of student work or dialogue are especially encouraged.

The following list of topics and related questions is intended as a guide for authors. Manuscripts that address related issues are always welcome.

Economics and elementary mathematics

- What lessons from the current world economic situation can be taught within the context of an elementary mathematics classroom?
- How can we help children analyze and make sense of economic information in the media by using mathematical content (e.g., rate, ratio, percent, equations, graphs)?
- How can economics topics (e.g., supply and demand; saving and borrowing; budgeting; inflation; interest) be used to enhance the learning and application of elementary mathematics?

Curriculum focal points

- In what ways can NCTM's *Curriculum Focal Points* inform the teaching and learning of mathematics?
- How can the Focal Points be addressed within the framework of state goals?
- How can the Focal Points be used in the mathematics classroom to shape student learning?

Differentiated instruction

- What are the challenges of differentiated instruction and how do teachers deal with them?
- How do we develop meaningful mathematical tasks that address students' diverse academic needs and ensure that all students have the opportunity for success in learning and understanding mathematics?
- In an inclusive mathematics classroom, what are the best ways to manage differentiated instruction?

Intervention strategies

- What evidence supports the use of instructional intervention strategies to ensure that all students succeed in today's high-stakes testing environment?
- What is the most effective way to manage intervention while still moving students forward?
- Which classroom structures effectively support intervention strategies?
- How can teachers successfully manage intervention strategies while meeting the diverse needs of all students?

Elementary school mathematics specialists

- What impact do elementary school mathematics specialists have on a school's professional community?
- What are the most effective strategies and best practices that specialists and coaches use for collaboration, co-teaching, and mentoring teachers?
- What are the roles of mathematics specialists?

Professional development strategies

- Which strategies successfully foster communication among teachers at the same grade level? Across grade levels?
- How can vertical teaming activities be developed and implemented in a multigrade school?
- How do we get started using book studies, lesson studies, and team-planning initiatives within our professional learning communities?
- How does team scoring affect student work and teaching practices?

If you have interesting ideas, research, or classroom-tested approaches concerning any of these topics, please share them by writing for the journal. Submit a manuscript to *TCM* by accessing **tcm.msubmit.net**. For more information, refer to "Instructions for Submission of Manuscripts" at nctm.org/publications/write.htm. Direct your questions to the journal editor at tcm@nctm.org.

Using the Mathematics We Do Every Day

Preservice teachers gain a new perspective by recognizing the mathematics in everyday situations.

Stop and think: During the past week, which of your daily, nonteaching tasks have required the use of some sort of mathematics? Have you mentally kept a running total of grocery prices to ensure that you do not exceed your budget? Have you measured wall space to hang a trio of pictures, to fit some furniture, or to install an appliance? Have you planned a meal and halved or doubled a recipe? These seemingly routine, nondescript experiences can be the foundation and infrastructure for a learning environment that not only supports but also fosters genuine mathematical understanding both within and outside the classroom.

The Assignment

An inaudible but visceral groan travels across the classroom as I explain the first of this quarter's written assignments to methods students. The weekly assignment requires students to submit a one-page summary of a situation in which they used mathematics over the past week in their daily, nonteaching lives. I offer both simple and elaborate examples: balancing a checkbook, shopping, counting calories, tipping, cooking, paying income tax, planning a vacation or party, detailing exercise routines and goals, and purchasing a home. Students must also describe how they solved their problems and offer at least one alternative solution that they discover on their own or solicit from a friend, relative, or co-worker.

Although initially unpopular, this one assignment has consistently proven to be pivotal for preservice teachers transitioning from their preconceptions about teaching elementary mathematics to a reform-based instructional approach and curriculum. The assignment supports growth in personal and professional mathematical understanding in at least three ways: It provides—

1. a window into students' growth as mathematical thinkers;

By Mindy S. Kalchman

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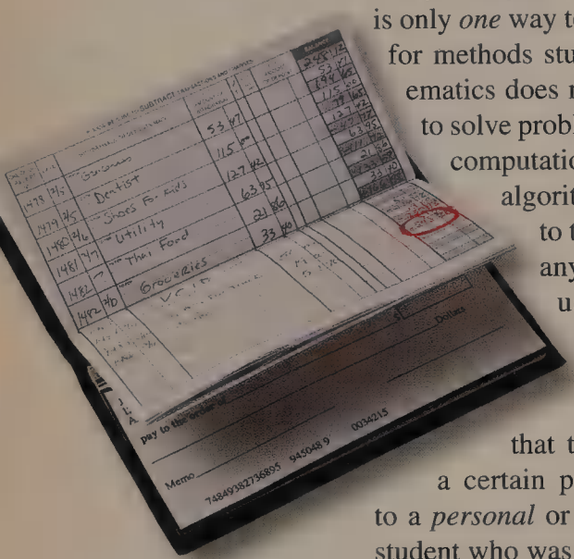
2. a curricular and instructional approach ensuring authentic mathematics while meeting NCTM and local standards for content, process, and professionalism; and
3. a model for classroom teaching that shows how students' everyday mathematical experiences are relevant to and inform school mathematics and how school mathematics are relevant to and inform students' everyday mathematical experiences.

A Window into Student Thinking

Two common complaints invariably characterize early submissions for the assignment: (1) "I didn't really do any mathematics this week" and (2) "There is only *one* way to solve my problem." It takes time for methods students to realize that doing mathematics does not necessarily mean sitting down to solve problems that involve paper-and-pencil computations using standard, school-taught algorithms. In fact, many students cling to their belief that they have not done any mathematics unless they have used a teacher-taught, teacher-approved procedure. In their early papers, students often ask my forgiveness as they confess that they cannot remember how to do a certain procedure and have thus resorted to a *personal* or *secret* method. For example, one student who was consistently hard on herself when she compared her nonstandard problem-solving approaches to some of her classmates' conventional procedures, wrote: "I'm sure there is some type of algebraic formula for this [problem], such as .20 × some number = some number – some number, … but … I simply used the good old-fashioned trial-and-error method." Imagine their skeptical glee as students begin to learn that *secret* methods are not only acceptable but also encouraged and valid.



Changing class profiles

During our introductory class, I ask students to tell us about their relationship with mathematics and why they believe that relationship exists. That is, do they like or dislike mathematics? Are they confident or intimidated by it? We always have some students who feel confident in their mathematical abilities and really enjoy the subject. These positive feelings usually stem from successful experiences with school mathematics. More students, however, feel intimated, anxious, and negative toward mathematics because teachers constantly admonished them for not remembering what formula they were supposed to use and when.

By midquarter, however, a new class profile often emerges: Students who believed themselves proficient and capable at the start begin to question what is at the root of that self-image. For example, one representative student wrote, "I am really beginning to wonder if I have any number sense at all. I know how to do the mathematics for my own purposes, but I am at a loss for explaining why I did it that way, and I really can't understand my boyfriend's method at all."

Another student explained, "When I heard how my classmate [calculated a gratuity] last week, I decided to do a poll of how my friends figure out [how much to] tip. I was so surprised to learn that not everyone used a calculator to figure out 20 percent of the bill by multiplying .2 times the total of the bill. Just about everyone I asked had a different method. Which method should I be using to teach my students?"

On the other hand, students who were initially intimidated or even resentful of mathematics also begin to change their attitudes: "I feel betrayed and cheated by my teachers. If I had been allowed to think about and solve problems in ways that make sense to me, I might have really enjoyed mathematics, and I might have been motivated to learn it."

Crowning moments always come when a student confronts a problem and writes something such as, "I would never have even tried to figure out the sale price without my calculator before doing this assignment. I didn't know it *could* be done in my head."

By the end of the quarter, the class profile has evolved even further. Usually by this point, the students whose confidence had somewhat waned at midquarter have been motivated to challenge themselves to broaden their thinking as they transition to more reform-based problem-solving approaches. Thus, they once again believe themselves to be capable mathematicians. However, their new definition of themselves as competent mathematicians now includes an appreciation and respect for alternative strategies that may seem confusing and illogical at first. Students who at midquarter had transitioned from feeling like incompetent and unsuccessful mathematicians to capable and confident learners generally maintain that feeling.

The most exciting part of my own analysis of this trend in the class profile has been the realization that students are not just rethinking mathematics

for the purpose of teaching it. Rather, most students genuinely change the ways that they approach and think about mathematics in their everyday lives. For example, one student confessed, "I am actually becoming quite the nerd when it comes to mathematics problems now. Things that I wouldn't have even considered caring about are now suddenly very important to me."

Another student wrote, "As I start to notice how much mathematics I am using, I have also been trying to assess different ways of doing the actual problem than I usually would."

Furthermore, this change in personal thinking and problem-solving habits significantly influences students' developing professional beliefs about teaching mathematics. Students *experience* the personal value of applying reform-style approaches to mathematical situations and thus are motivated to establish a classroom that reflects that value.

Developing individual progress

The shift in class profile is a rewarding result of a macroanalysis of collective learning. The success of any given class, however, must also be considered in the context of individual students' progress. Meet Alex and Morgan.

"Mathematics [makes] me feel inadequate," was Alex's opening remark. "I [have] an anxiety that was instilled at an early age and was reinforced over the years of my education." She followed up with details of how she "was always struggling to see what everyone else was able to see." Whenever she asked for explanations, teachers just repeated what was in the textbook. However, during our ten weeks together, Alex transitioned from being a passive participant unwilling to take risks in class (Ebby 2000) to being an emergent, active, reflective, and engaged mathematical thinker. After completing the course, Alex explained: "Manipulating numbers and problems began to clarify my sense of numbers and their relationships. Through the mathematics exercises, I was forced to reflect on and acknowledge the mathematics that I do naturally every day." Alex's weekly mathematics problems documented her development.

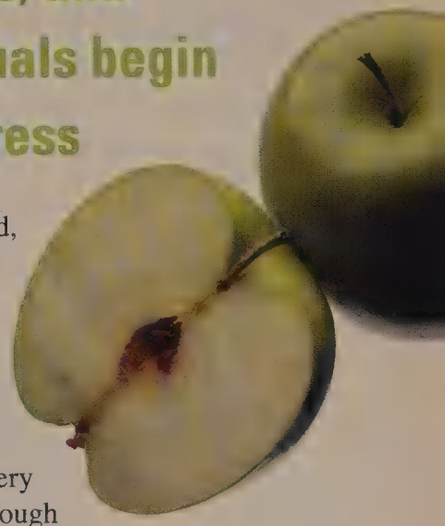
Alex's first problem was simple: How much more would it cost per week to take the train if there were a fare increase? Alex did not solve the problem or explain her reasoning. Instead, she described how she could connect this problem to other subjects by doing creative art projects and using the Internet to research public transportation. Likewise, for Alex's next problem, a simple banking situation with addition and subtraction, she did not offer a solution but described how she would discuss banking with children.

From weeks 3–5, however, Alex's weekly problems began to increase in complexity. In week 5, her mathematics problem was about how long it would take her to drive from Florida to Chicago, considering speed limits and time zones. Alex did not successfully solve the problem, but she began to explain her thinking with respect to why she did what she did, and she included the algorithms that she attempted to use.

By week 6, Alex had begun to explain her thinking and to offer multiple correct solutions to her weekly problems. For example, in week 6 Alex figured out how much a gum-chewing habit was costing her per week, per month, and per year. Alex solved the problem using two distinct solutions that reflected confidence, flexibility, and number sense—a tremendous accomplishment given her history.

Morgan's development reflected a different starting place from Alex's and thus, a different endpoint. Morgan began the course with confidence and relatively good number sense. She consistently showed creative, effective, and efficient ways to solve her daily mathematics challenges. However, Morgan was skeptical about using reform-based pedagogies to teach such an "established" subject. Moreover, she did not see the value of spending so much time listening to how her peers solved a problem that she had understood implicitly and had just solved quickly and easily.

When students learn that *secret* methods are valid, acceptable, and encouraged, the class profile changes, and individuals begin to progress



At the end of the course, Morgan reflected, "I feel much more prepared to teach mathematics in a constructivist way than any other subject." Morgan attributed much of this confident feeling to the weekly mathematics problems. "I think using the weekly mathematics discussions to examine how I looked at mathematics was very helpful for me." Morgan explained that although her "personal strategies did not change" from week

to week, her write-ups started "to reflect … additional perspectives my classmates really helped me [to gain] when they shared the different ways they solved problems.… I learned that people have different methods and it is not for me to correct how they solve something. Constructivism allows kids to figure things out their own way, and it became important for me to try my best to see how [students might be tackling] a problem."

A Curricular and Instructional Approach

As the quarter progresses, talk of reform, children's alternative strategies, Cognitively Guided Instruction (e.g., Carpenter, Fennema, and Franke [1996]), and other educational innovations invariably leads to the question, "To textbook or not to textbook?" Students are often drawn to vignettes from articles we read from the constructivist literature (e.g., Carpenter, Fennema, and Franke [1996]; Lampert [2001]; Schifter [1996]), and they wonder how to go about establishing and maintaining such classrooms without the aid of a textbook. Field experiences often complement this wonder in that students recognize the limitations of textbooks with respect to constructivist pedagogies yet feel overwhelmed by the idea of having to be the architects of an entire year's scope and sequence.

After a couple weeks of sharing students' weekly mathematics problems that I find intriguing and challenging, I begin to reveal my organizational strategy for the problems I collect. To begin with, I personally do each problem and classify it within the NCTM's content-area strands. A problem could involve more than one strand and thus be "multicategorized." Next, I evaluate the problem and decide on a grade range for it. Then, I file problems accordingly in a file box of strands and corresponding grade levels.

Eventually, I involve students in this filing process and have them suggest a strand for their own and their classmates' problems. Then, as a class, we look at the NCTM Standards to see if solving the problems in question would provide opportunity for students to meet a grade-based strand's requirements. This organizational strategy is intended to track the strands and corresponding Standards that have been met; the strategy is not meant to create a cadre of problems for future use. That is not to say that good problems cannot be reused. However, the idea behind the approach is that our daily lives contain ample mathematics to satisfy teachers' curricular mandates.

Figure 1

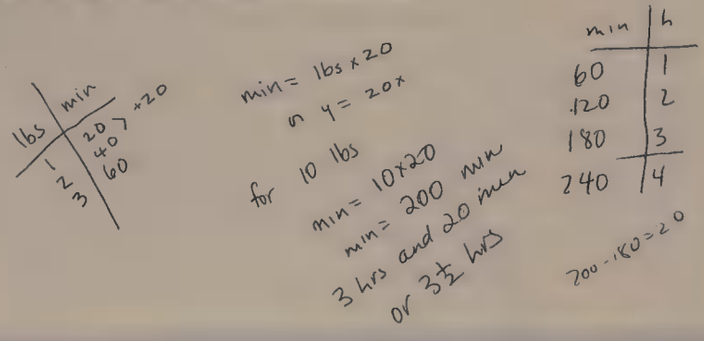
Before sharing her categorization and solution of an adult student's submitted problem, the instructor solicits feedback on the problem and interpretations of the content-area strand.

April 11, 2007

How math has been a part of my life this week…

My math dilemma for the week occurred when I tried to determine the cooking time for a large ham I was making for our Easter holiday family dinner. I have never made a ham before and had to read the cooking instructions that came with it. The instructions were simple: cook the ham 20 minutes for each pound it weighed. So, I set the oven temperature at 350 degrees and then had to figure out how long to set the timer for.

According to the price tag, my ham weighed 9.90 lbs. To make my calculations easier, I rounded up and just said my ham weighed 10 lbs. Now, how long should I cook my ham if it weighs 10 lbs. and I am looking it 20 minutes for each pound it weighs? Well, there are 60 minutes in an hour and if I divided 60 ÷ 20 = 3. That told me I could cook 3 lbs. per hour. So, in one hour I could cook a 3 lb. ham, in 2 hours I could cook a 6 lb. ham and in 3 hours I could cook a 9 lb. ham. Since my ham was 10 lbs., I figured it would take me 3 hours (for the first nine lbs.) plus 20 more minutes for the additional pound to cook my ham. It turned out to be the correct amount of time because the ham was ready when the timer went off after 3 hours and 20 minutes.



Meeting the Content Standards

In a recent class, a student submitted a seemingly simple problem about cooking a ham (see **fig. 1**). As I solved the problem, I realized I was using basic algebraic representations and methods. I referenced the NCTM Content Standards and decided I was involved in the Grades 3–5 Algebra Expectations by—

- describ[ing], extend[ing], and mak[ing] generalizations about … numeric patterns;
- investigat[ing] how a change in one variable relates to a change in a second variable;

Figure 2

Cooking examples for the weekly mathematics assignment often address many of the NCTM Content Standards.

One Saturday morning, I decided to make something different for breakfast. I had a recipe to make English Muffin Strata with Ham and Cheese. However, the recipe called for six muffins, and I was making this for only one other person, so I decided to use only three muffins, which means that I had to cut the quantity of ingredients in half. I could compute any whole number immediately, but I had to think about some of the other measurements. For example, the recipe called for 3/4 cup of chopped bacon, so I figured that I needed to multiply that by 1/2, and I got 3/8. A good way for younger children to do this is to draw a circle with four "pieces" and shade three of them. Draw another line in each of the four pieces and "remove" the shaded pieces to get three of eight "pieces." [*See* **fig. 3.**]

I was stumped again when it called for 2 1/2 cups of soy milk. I had to think back on how to work with mixed fractions. It did not take long before I remembered to turn it into a regular fraction by multiplying the whole number 2 by the 2 in the denominator and then adding the 1 in the numerator and putting that over 2, which gave me 5/2. Then I multiplied that by 1/2 and got 5/4, which is 1 1/2 cup when I put it back into a mixed fraction. This exercise can also be done using circles. Start with two whole circles and half a circle. Take away half of each to end up with 1 1/4. [*See* **fig. 4.**]

- represent[ing] the idea of a variable as an unknown quantity using a letter or symbol; and
- express[ing] mathematical relationships using equations. (NCTM 2000, p. 158)

I was also engaging in the Grades 6–8 Algebra Standards by—

- model[ing] and solv[ing] contextualized problems using various representations such as graphs, tables, and equations; and
- us[ing] symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships. (NCTM 2000, p. 222)

Before sharing my solution and categorization with the students, I solicited their feedback on the problem and their interpretations of the strand for this problem. They suggested Measurement (time); Numeration requiring computation with rational numbers; and Probability and Statistics.

Sometimes the strand of a problem is obvious, but the content illuminates the benefits of using real-life, context-driven situations to teach and learn a concept or skill. Cooking examples seem to drive this point home better than any other situation. Every quarter, I get students who admit to thinking that using a cooking situation would be a no-brainer for the weekly mathematics assignment. Then, as students begin the process, they realize that halving or doubling a recipe that calls for two-thirds cup of sugar or one-eighth teaspoon of pepper is not necessarily intuitive or mentally accessible. **Figure 2** is an example of a cooking problem that surprised a very confident, competent mathematics student.

Cooking examples address many Number and Operations Content Standards. For the example in

Figure 3

A simplified way to halve 3/4 cup of chopped bacon is to take 1/2 of 3/4 to equal 3/8.

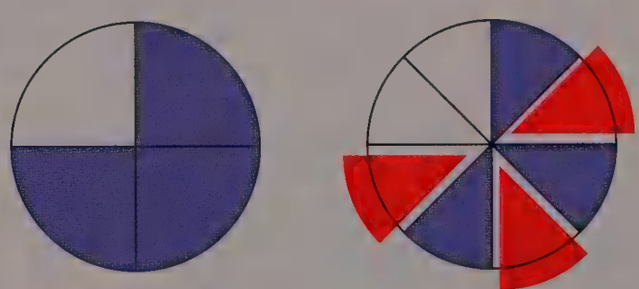


Figure 4

A simplified way to halve 2 1/2 cups of soy milk is to take 1/2 of 2 1/2 to equal 1/2 + 1/2 + 1/4, which yields 1 1/4.

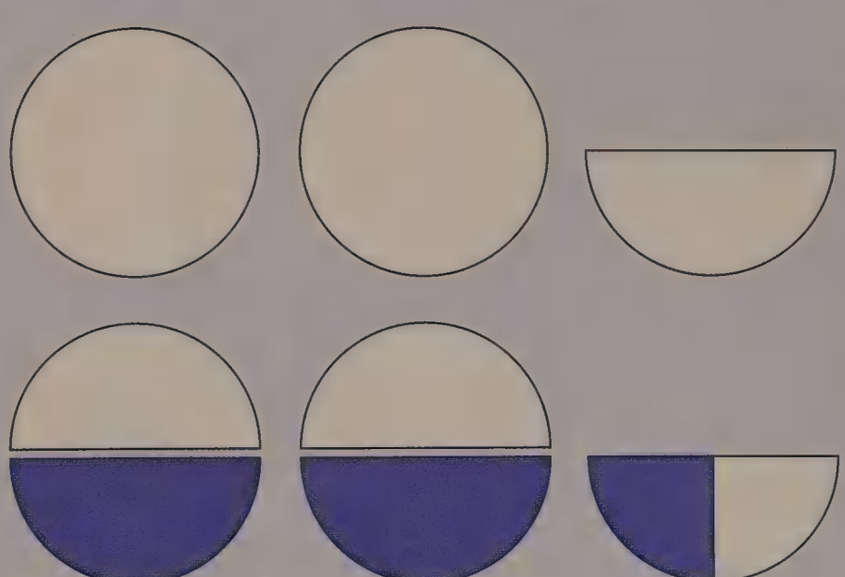


figure 2, we decided as a class on the following set of Standards that could be met by exploring this and similar types of problems:

Grades 3–5

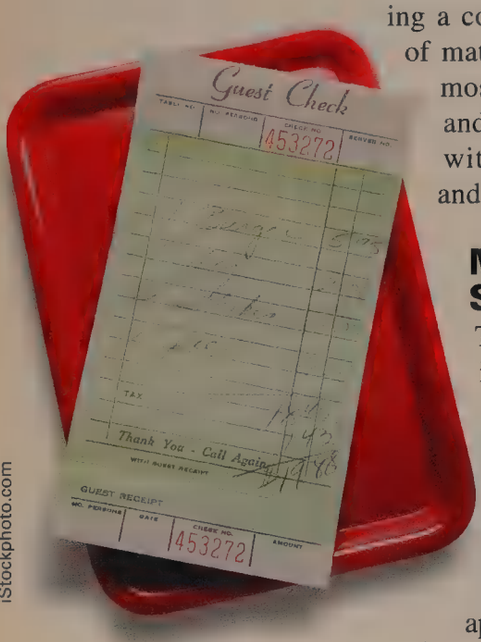
- Develop and use strategies to estimate computations involving fractions and decimals in situations relevant to students' experience.

Grades 6–8

- Work flexibly with fractions, decimals, and percentages to solve problems.
- Select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil—depending on the situation—and apply the selected methods.
- Develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use.
- Develop and use strategies to estimate the results of rational-number computations and judge the reasonableness of the results.

Meeting the Professional Standards

Professional Standards for Teaching Mathematics (NCTM 1991) concretely communicated to teachers their roles and responsibilities in the classroom. The weekly problem curricular approach provides ample opportunity for teachers to demonstrate their commitment to the teaching profession. In particular, the Standard of "providing a context that encourages the development of mathematical skill and proficiency" seems most relevant and gives students the support and encouragement they need to proceed with alternative models for curriculum and instruction.



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Meeting the Process Standards

The Process Standards are also easily attainable by using a curricular and instructional approach that focuses on the mathematics we do every day. For example, under Problem Solving, students are expected to solve problems that arise in mathematical and other contexts. In Connections, students recognize and apply mathematics in contexts outside mathematics. And for Communication, students communicate their mathematical thinking coherently and clearly to peers, teachers, and others.

To summarize, the sort of weekly problems students submit are everyday situations that are rich with the mathematics, processes, and professionalism required by district, state, and national standards. With some organization, planning, and enthusiasm, teachers can meet their professional mandates simply by engaging students in the mathematics that is around them. An important hurdle for many teacher candidates is realizing that one problem could, and would likely, provide a week's worth of lessons—or more. The single-problem context for meeting multiple standards within multiple strands provides consistency for students who are just beginning to discern the legitimacy of solution strategy choices based on context.

A Model for Classroom Teaching

Such real-life, mathematically rich situations as discussed above are not unique to adults. Elementary school students can also be asked to bring in mathematics problems that they confronted over the course of a week or weekend—a problem that might ultimately contribute to the scope and sequence of the class mathematics curriculum.

Moreover, preservice methods students recognized the degree to which using their own and classmates' problems motivated them to do mathematics and also listen to and learn about others' mathematical thinking. The early fear—"What if I don't understand what my students are saying or thinking?"—dissipates as students get excited, rather than terrified, by hearing and seeing the myriad ways different people approach the same problem. Students recognize and are inspired by the ways in which their own weekly analysis of a legitimate problem changed their competency, confidence, and attitude with respect to mathematics. Moreover, students come to see their roles as educators as giving their own students at least the opportunity to experience learning and doing mathematics in such an empowering way.

Conclusion

Using real-life problem-solving situations to animate mathematics in the classroom is not a new idea. Nor is it new to encourage methods students to develop and share multiple solution strategies

in order to illuminate and actualize the value and philosophy behind reform-based instruction and curriculum. What is innovative about this weekly-problem learning environment for supporting and fostering mathematical understanding is that it changes the ways that preservice teachers think about and practice mathematics *independent of* pedagogical goals. Many methods classes are commendably successful with respect to training new teachers to establish and maintain reform-based mathematical classrooms.

What we do not know about these successful new teachers is what sort of mathematics they practice at home. Is it the same mathematics that they expect from students in the classroom? Are teachers encouraging students to be reform-minded mathematical thinkers outside the classroom? The weekly-problem approach provides teachers with a good model for inspiring themselves and youngsters to be flexible and confident mathematics thinkers both within and outside the classroom walls. Finally, the learning environment and instructional and curricular approach suggested here provide an infrastructure and foundation for establishing and nurturing a reform-based mathematics classroom while meeting local and national content, professional, and process standards.

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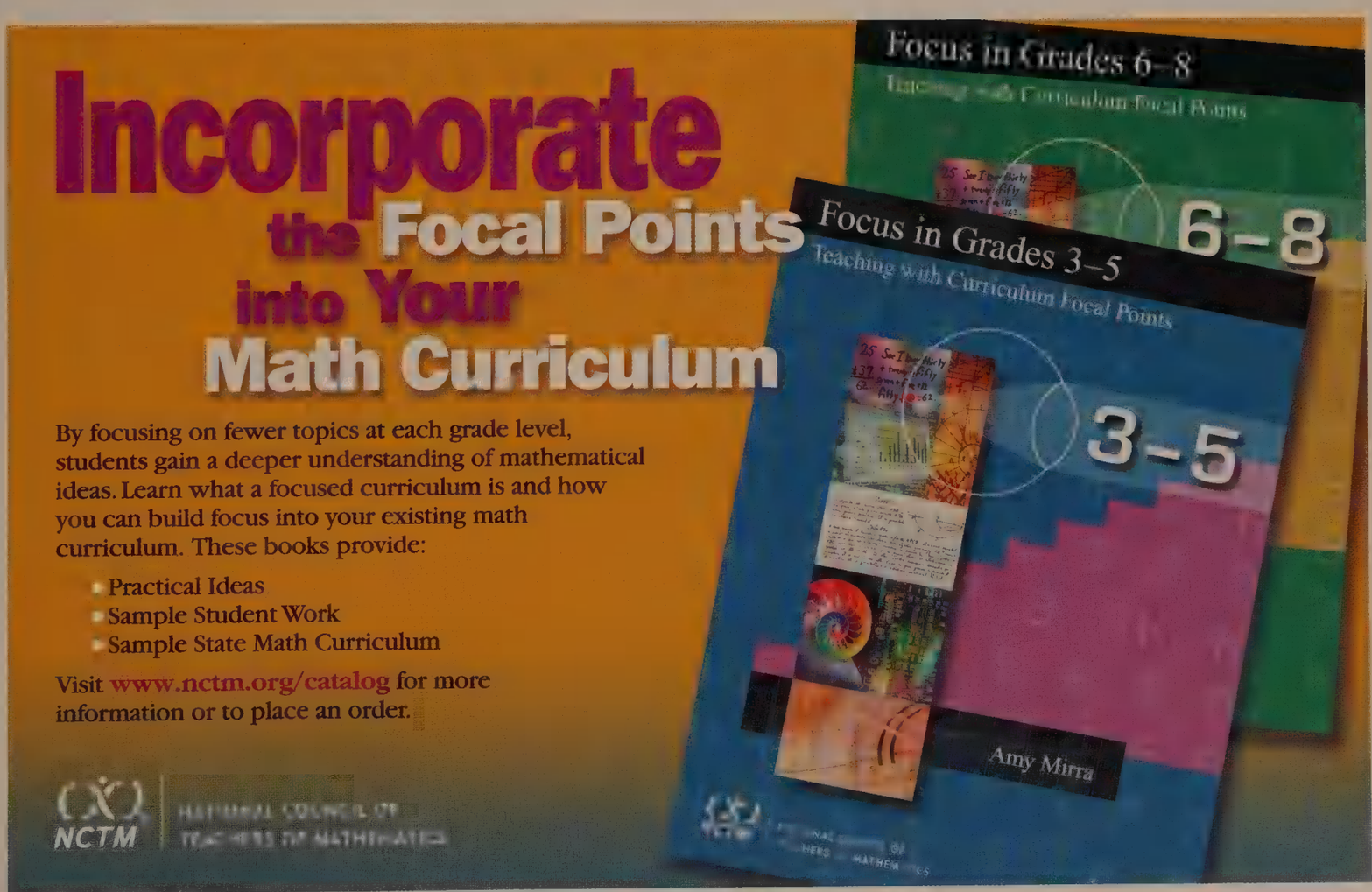
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Tantalizing Thirteen

"Math by the Month" activities are designed to engage students to think as mathematicians do. Students may work on the activities individually or in small groups, or the whole class may use these as problems of the week. Because no solutions are suggested, students will look to themselves for mathematical justification, thereby developing confidence to validate their work.

The following activities are designed to help you and your students discover connections to the number thirteen through mathematical investigations that involve number sense and operations, statistics, data collection, geometry, communication, and representation. ▲

WEEKLY ACTIVITIES

TANTALIZING THIRTEEN: K–2

MAY 2009

4 **Is 13 unlucky?** Some people think so. Try your luck with grouping 13 counters into groups of 2. On a sheet of paper, show how you grouped them. Next try making groups of 3, 4, 5, and 6. Be sure to record on your paper how you made your groups. What do you notice? What kind of luck did you have making your groups?

11 **A dozen is 12** items; a *baker's dozen* is 13. Hundreds of years ago, English bakers began the tradition of putting a little extra in the customer's basket because sometimes bakery items were lost, broken, eaten, burned, or otherwise ruined. A baker's dozen made sure customers still had 12 items when they got home. What kind of cookie would you want if you could have a baker's dozen? Predict your classmates' favorite cookies. Collect data to check your predictions and then create a graph to show your data.

18 **Polygons are shapes** with many sides. With a partner, create some polygons from a 5 × 5 geoboard (or geoboard paper) and geobands (rubber bands). Put 13 pegs inside the geoband, 13 pegs outside, and 13 pegs touching the geoband. How many different polygons can you create? Record your findings on geoboard paper. Compare your geometric shapes with your classmates'. Variation: Explore with a virtual geoboard at standards.nctm.org/document/eexamples/chap4/4.2.

25 **Brainstorm the number 13.** Create a graphic organizer, word web, or concept map with the number 13 in the center. How many different ways can you represent the number? For example, brainstorm ways to write 13, such as XIII or 1 ten and 3 ones. Show all the ways to make 13 cents using coins. Show how to use dominoes and 2 tens-frames to make 13. Find out how many students have—

- a 13 in their home address;
- a 13 in their telephone number;
- a birthday on the 13th day of a month;
- the 13th spot on the class list; and
- 13 letters in their first or last names.

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Edited by **Dana Islas**, *dana.islas@tusd1.org, who teaches kindergarten at Pueblo Gardens Elementary School in Tucson, Arizona. Readers are encouraged to submit problems to be considered for future "Math by the Month" columns to the department editor. Receipt of the problems will not be acknowledged; however, problems selected for publication will be credited to the author.*

TANTALIZING THIRTEEN: 3–4

MAY 2009

Military time uses the numbers 00–24 to express 24 hours in a day. Standard time uses the numbers 1–12. When a military clock's hour hand is on 13, the time is "13 hundred." When the hour hand is in this same position on a standard clock, the time is 1:00 p.m. Use a dinner-size paper plate to create an analog clock with the numbers 1–12 written on an outside ring and the numbers 13–24 on an inside ring. Make clock hands of construction paper and connect them to the plate with a brad. Practice telling military time. Then start at midnight and practice telling standard and military time, listing the times in a table. What is similar or different? Now play this game with a partner: Roll a die. Starting with midnight, predict what standard and military time you will land on if you move that many hours forward (or backward) on the clock. Then move the clock hands to show the time on the paper plate clock and record the 2 times on a sheet of paper. After several turns, talk with your partner about similarities, differences, and patterns you notice.

The first American flag had groups of 13 stars and 13 stripes. Look at the Great Seal of the United States to the left. Can you find any groups of 13? What similarities do you notice between the first U.S. flag and the Great Seal? Why do you think the number 13 is so important in these images? Create a seal to represent your school. Use numbers in your design and explain why you chose them.

Try ■ metric scavenger hunt. Look around your classroom, gym, and playground and estimate lengths of some objects and distances that are about 13 centimeters, 13 decimeters, and 13 meters. Find objects you can use to help you estimate the lengths. Record your estimates on a chart. Then use centimeter rulers, metersticks, and metric measuring tapes to measure and record the same objects and distances. Compare your chart with a classmate's. Which objects and distances were easier to estimate, and why do you think this is so?

Many famous sports figures wore jersey no. 13: The first Hispanic American elected to the Hall of Fame, Roberto Clemente, started his baseball career wearing no. 13. Quarterback Dan Marino of National Football League (NFL) fame wore no. 13, as did National Basketball Association (NBA) star Wilt Chamberlain. To collect data on sports jerseys that schoolmates wear, do an investigation in the cafeteria or one morning when students enter the building. Record how many sports jerseys you see and the numbers on them. Create a graph to show the most popular numbers.

WEEKLY ACTIVITIES

TANTALIZING THIRTEEN: 5–6

MAY 2009

Apollo 13—was the mission lucky or unlucky? Explain. What date did Apollo 13 lift off? What time did it lift off? Apollo 13 was planned as a lunar landing mission. How did teamwork and problem solving play a role in Apollo 13's safe return to earth? Use teamwork to solve the following problem: If each of 13 space research stations must carry out a space exploration with every other research station exactly once, how many space explorations would take place altogether? Show how you know.

What are Archimedean solids? What else can they be called? How many exist? List their names. Find a real-life example of each one. Create a chart of how many edges, faces, and vertices each Archimedean solid has. Subtract the number of edges plus the number of vertices from the number of faces for each one. Is there a pattern? Explain the result. This Web site can help you explore: www.scienceu.com/geometry/facts/solids/index.html.

A prime number has only two factors, itself and 1. Is 13 a prime? Using a hundreds chart, follow the rules of Eratosthenes, the Greek mathematician who devised ■ system for finding primes. First cross out 1. Next cross out all the multiples of 2 that are greater than 2. Cross out all the multiples of 3 that are greater than 3. Do the same for the multiples of 5 and 7. The remaining numbers are considered prime. How many prime numbers are in the first 100 counting numbers? Name them and color code each as odd or even. What do you notice? Which prime numbers come before 13? Which prime numbers follow 13? Why can you stop crossing out after the multiples of 7? If you want to find all the prime numbers up to 200, could you stop crossing out after the multiples of 7? Explain your thinking.

Triskaidekaphobia is fear of the number 13. Does your school have a classroom numbered 13? Who is the 13th student on your class list? List groups of 13 items in your home or school. *Paraskavedekatriaphobia* is fear of Friday the 13th. When is the next Friday the 13th on the calendar? How many does this year have? In the next 3 years, does every year have a Friday the 13th? Does every year have the same number of Fridays the 13th? Do you notice any patterns?

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MATH FAIR: Focus on Fractions

A math fair in India with hands-on experiences that simplify conceptual mathematics proves to be the perfect milieu to trigger children's interest in learning mathematics.

My primary focus in teaching mathematics for the past forty-five years—to create effective mathematics environments for students while enabling them to participate in mathematical experimentation—inspired the idea to host a math fair at our school in Bombay, India.

The basic concept of a math fair is threefold: (1) to create preplanned, organized, and intentionally developed mathematical learning environments; (2) to observe students' participation in and response to various activities; and (3) to use math fair activities as teaching aids and student enrichment in day-to-day classroom instruction.

The word *fair* itself reminds me of the very spirit of Indian culture and calls to mind the colorful event held in small Indian villages as people converge to worship their favorite deity. Adults and children enjoy participating in magic tricks, activities, booths, and rides such as roller coasters, Ferris wheels, merry-go-rounds, and so forth. Fairs have many interesting activities that engage children in play. The environment is vividly colorful, enthusiastic, full of energy, and pulsating with excitement and the hustle and bustle of life. We wanted to create a learning environment that would promote similar enthusiasm for mathematics.

By Neelima A. Mokashi

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Establishing Committees

In India the academic year commences in mid-June and ends the following April. Our math fair evolved in four distinct phases. A fair would require members' consent, so during the month of June we submitted a proposal for a math fair to our institution's managing committee. Our proposal included a basic description of the event, a proposed budget, and the fair's objectives. After the committee readily accepted this venture, I explained the idea to the four principals of the kindergarten and elementary sections of our school (one English-speaking and one Marathi-speaking, the regional language, at each level). All four principals also supported the idea.

During the first week in July, we established the organizing committee: adviser and chief coordinator (me), internal coordinators (the institution's administrator and superintendent), and coordinators of the four sections (the principals). Our working committee consisted of four supervisors for each section, four teacher representatives, and four parent representatives. Subcommittees for various tasks involved all teachers, including art, music, and physical education instructors. In mid-July, the organizing committee decided on the venue and the scope of the math fair, set the dates for January 2007, established a preliminary budget, projected a list of the resources that would be required, estimated the inaugural program's duration, and identified the chief guest. We wanted our chief guest to be a school alumnus who had a strong mathematics background. The total budget was Rs. 20,000 (rupees), which is about four hundred U.S. dollars. Each section received approximately one hundred dollars to work with.

Preparing and Planning

At the end of July, the various committees met and started the second phase. Our main objective was to trigger children's interest in learning mathematics with hands-on experiences that simplify conceptual mathematics. We had other objectives as well:

- Encourage reluctant students to become active participants in the established learning environment.
- Encourage mathematical understanding through play.
- Inspire teachers and parents to explore various methods of teaching key concepts.
- Develop simple, low-cost teaching aids that encourage class participation.

One teacher manages sixty to seventy students in classrooms in India; interacting with individual students is very difficult. The idea of a math fair with

Our threefold math fair goal was to create preplanned, organized, intentional mathematical learning environments; to observe students' response to the activities; and to use the activities as future teaching aids and student enrichment

thousands of children and parents simultaneously interacting at a single event appealed to many teachers, and they were willing to put in extensive time, effort, and preparation to make it happen. As much as possible, we wanted to highlight the entire mathematics curriculum from kindergarten to grade 4. With the help of parents and students, I distributed a list of related topics among all the teachers who were willing to put in the extra hours of work.

At the end of August, we again held a joint meeting of the organizing and working committees to refine two major components that needed extensive preparation: the entertainment programs and the exhibits.

Entertainment programs

We agreed that entertainment programs performed by students on the school auditorium stage would launch the inaugural event. Mathematical operations and signs were to provide the basis of the performances. The programs' duration was only half an hour in total. Hence, every section had only five to eight minutes on stage.

Exhibits

After extensive brainstorming sessions, we finalized our selection of specific activities and games as well as the resources needed to create them. Each section received printed instructions outlining needed preparation for specific topics in relation to the mathematics curriculum (to be covered in both English and Marathi).

Although preparations for the entertainment programs and exhibits were completed by the end of September, we still did not have a clear picture of the expenses. However, by the end of October, we had formulated a detailed budget. (Unfortunately, we overlooked the expense of snacks and lunches for the volunteers.) At this time, fairly late in the game, someone suggested the idea of sponsors.

The school year's first term ended in late October, followed by Diwali (the Festival of Light) and a three-week "vacation," during which the whole team steadily continued to prepare activities. During our one-week Christmas break in December, we made final touches, and the countdown began. Our next step was to collect all the prepared exhibits (activities and games). Storage proved to be a problem; we resorted to using some of the classrooms.

Math Fair in Action

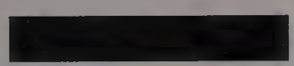
On January 11, 2007, our math fair began with great enthusiasm. Using the fair's inauguration date, we decorated the main entrance to the exhibition hall with a poster of a magic square, cells of which equaled the same total for each row, column, and diagonal. We also displayed an artistic image of Lord Ganesh, the Hindu god of knowledge, drawn solely using numbers. Right

Figure 1

Children were introduced to fraction symbols and then looked at posters and manipulatives that illustrated those symbols.

This is a picture of a cake. One-fourth has been removed. Three-fourths remains. The fraction is written as

$$\frac{3}{4} \quad \begin{matrix} \rightarrow \text{Numerator (N)} \\ \rightarrow \text{Denominator (D)} \end{matrix}$$

Figure	Total No. Shaded Rectangles	Total No. Rectangles in Figure	N (up) ↑	D (down) ↓	Fraction Shaded	In Words
	1	4	1	4	1/4	One-fourth
	2	4	2	4	2/4	Two-fourths
	3	4	3	4	3/4	Three-fourths
	4	4	4	4	4/4	Whole

away, the inauguration ceremony perfectly set the mood of the fair with the publication of manuscripts by the chief guests and with children's' performances that featured mathematical concepts:

- One action song was called "Train of Numbers."
- A fancy fashion show correlated mathematics with language. Students wore animal costumes to represent each letter of the word *mathematics*. The *M* was a monkey; the *A* was an antelope; the *T* was a tiger, and so forth.
- A skit told the story of *A Mouse with Ten Tails*.
- Physical activities included a flash card drill with mathematical signs.

During the following two-and-a-half days, more than eight thousand people visited the various booths lining the main exhibition hall. Hosted by teachers and student volunteers, most of the exhibits were hands-on games and puzzles, which instantly captured youngsters' attention. Each booth was devoted to one fundamental mathematical concept, such as prime numbers, multiplication, division, or fractions.

We certainly met our objectives from the Preparing and Planning phase and accomplished our main objective of triggering children's interest in learning mathematics. Students were involved in copying articles, drawing figures in manuscripts, and decorating the rooms. They acted as magicians, showing simple mathematical tricks to spectators. Some explained to visitors how the games and activities worked. Younger children participated in entertainment programs. It was an exciting learning environment for the whole crowd, as was evident from the visitors' active participation and their comments in the feedback book.

Focus on Fractions

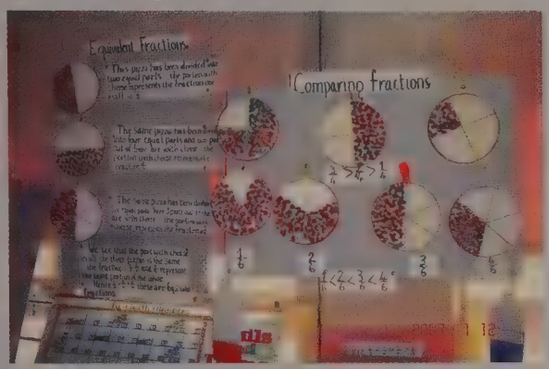
The math fair addressed such a wide variety of mathematical concepts at once that it is difficult to communicate the complete details of every activity in one article. I will relate some of our fraction activities and exhibits, although only about one-tenth of the event addressed the subject of fractions.

We distributed fraction activities and games into

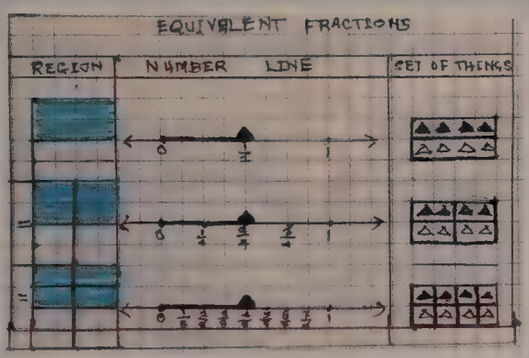
Figure 2

Posters illustrated the meaning of *fraction* and types of fractions.

(a) comparing fractions



(b) equivalent fractions



(c) like fractions

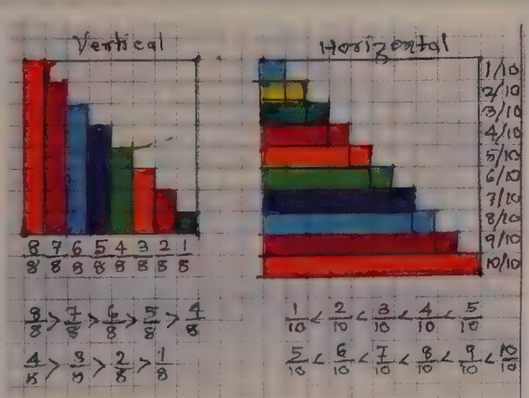
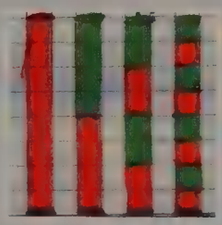


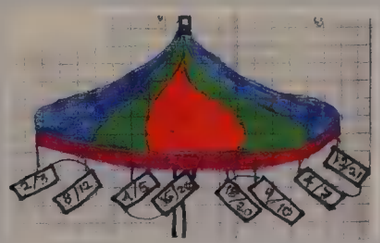
Figure 3

Hands-on manipulatives augmented these equivalent fractions posters.

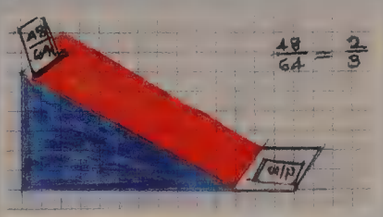
(a) cylinders (or Legos)



(b) merry-go-round



(c) magic slide



(d) distributing pizza

(e) poster

seven major categories according to the mathematics curriculum from kindergarten through grade 4: geometric patterns, distribution of a set of objects in equal parts, measurement problems, fraction symbols, fraction families, addition and subtraction of fractions, and comparison of fractional values. (See **table 1** for the objectives of some categories and their related activities.)

After children learned the symbols for fractions (see **fig. 1**), they looked at posters illustrating the meaning of the word *fraction* and various types of fractions, such as *unlike*, *proper* and *improper*, *mixed*, and *equivalent* (see **fig. 2**).

In addition to posters to illustrate families of equivalent fractions (see **fig. 3**), we used three-dimensional models such as cylinders; Legos® would work, too. In

Table 1

Selection of Fraction Activities and Games

	Objectives	Activity or Game
	Geometric Patterns	
1.	Recognize basic fractions (one-half, one-fourth, three-fourths).	Simple folding techniques (origami for third- and fourth-grade standards) Rangoli (a word used in India for joining dots to create a design using colored powder) Quilt and tile patterns (see **fig. 4**)
2.	Develop understanding of fractions as equal parts of unit wholes.	Cylinders (see **fig. 3a**) Circles: Divide models of pizza into two, three, and four equal parts (see **fig. 3d**). Rectangles: a chocolate bar Poster (see **fig. 3e**)
3.	Build skills in recognizing fractional numbers as equal parts of a whole.	Join the pieces to create a complete square.
	Distribution of a Set of Objects in Equal Parts	
1.	Develop understanding of fractions as parts of a whole set or group of items. Division plays an important role.	Use ice trays, egg cartons, and pebbles; divide twelve into two, three, and four equal parts to show one-half, one-third, and one-fourth. String 12 beads (see **fig. 5b**).
	Measurement Problems	
1.	Understand fractions in context of length, volume, or weight.	Show one-half, one-third, and one-fourth meter of a ribbon or cloth. Use different-shaped jars of 1 liter, 1/2 liter, 1/4 liter, and 1/8 liter. Use a balance with weights of 1 kg, 1/2 kg, 200 g, and so on.
	Fraction Symbols	
1.	Recognize, read, and write the fractions 1/2, 1/3, and 1/4 as symbols and in words.	Train of dominoes (see **fig. 6a**) Jigsaw puzzle (see **fig. 6b**)
2.	Develop understanding of fractions on a number line.	Use a number line.
3.	Develop combined understanding of a fraction of a whole region, a number line, and a set of things.	Use graphical representation for the whole region, a number line, and a set of items on one poster (see **fig. 2b**).
4.	Provide an opportunity to go beyond the textbook and classroom.	Are you a good "fitter"? On a grid of parallel lines, students fit triangles, rhombuses, trapeziums, and hexagons into a figure and answer questions such as, "What fraction of a hexagon is a triangle?"
5.	Provide insight into geometric figures and provide minds-on activities for fractional values.	Tangrams are another interesting tool for learning fractions.

keeping with the *fair* theme, we displayed a Ferris wheel model. Adjacent horses on a merry-go-round would also illustrate the concept. The magic slide is a wonderful teaching aid for students in kindergarten and older. Some of our more popular and interesting attractions were a magnetic board, the simple version of Cuisenaire® rods, a Dice game (also known as the I'm No. 1 game), and a Bowling game.

Follow-Up Phase

After the event was over, we continued to use the activities and games as teaching aids in everyday class work. Collecting objective and subjective data will be a demanding, ongoing process. We have yet to establish specific goals for statistical analysis. And yet, we must carry out a solid follow-up research project in the near future to enable us to capture objective data on the precise

impact the math fair had on students, parents, and teachers.

Conclusion

Planning for emergencies and unexpected hurdles during such a large event is necessary but nearly impossible. Of course, not everything ran smoothly:

- We dealt with shortages of time, money, and space to store the material in advance.
- Because we had waited so long, we had to search for sponsors in the month of December. We should have decided earlier, but no one had anticipated such a huge response to the fair.
- Very few teachers and parents in our school community have access to computers and Internet connections, and India has a scarcity of mathematics books, so our book exhibit was quite small.
- One of the principals had a health incident and was admitted to the hospital during the fair.
- As a retired principal of the institution, I was careful to avoid interfering in the school's day-to-day activities as we prepared for the fair. In an inauguration ceremony slide show, I demonstrated the fair's objectives and described the booths, but I depended on the high school authorities to carry out much of the program—although they were busy with many responsibilities during the month of January.

Our comment book

The fair targeted kindergartners through fourth graders, so participants were younger children. Their general feedback was, "I had fun." One student wrote, "If these fun games come to our classroom, I won't know it is a math class; it would be fun!"

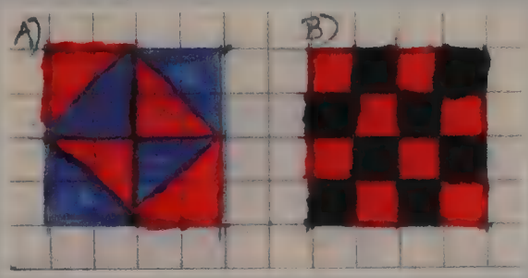
Specific comments from a parent's perspective included the following:

- We loved this; children were bubbling with energy.
- Is this really the subject of mathematics?
- If we had an opportunity to be part of something like this when we were children, maybe I would have liked math!
- This was a dreamlike fair; this should be … mobile … so other schools can also have a chance to experience this.

Figure 4

Quilt and tile patterns illustrated basic fractions.

(a) geometric patterns



(b) coloring patterns

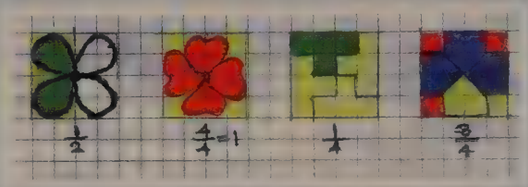
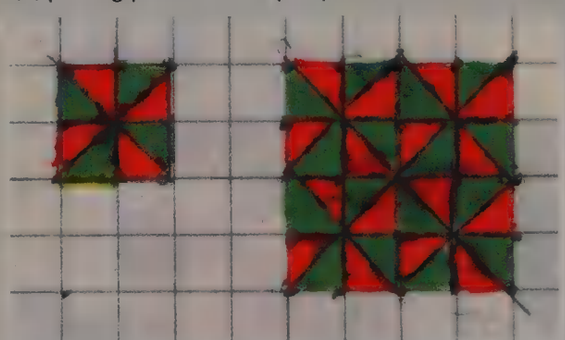


Figure 5

Other ways that students at the fair learned to recognize fractions were by

(a) putting patterns as equal parts into a whole and



(b) distributing beads equally.

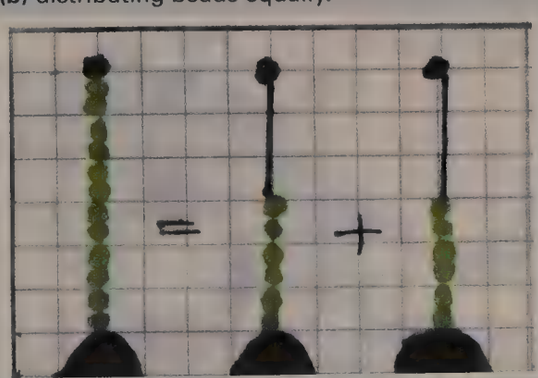


Figure 6

The objective of these displays was to teach children to recognize, read, and write fractions in words and in symbols.

(a) train of dominoes

(b) jigsaw puzzle

Among teachers' comments were the following:

- This should be part of a permanent hands-on exhibit at Homi Bhabha Science Center (a well-known institute promoting science and mathematics in Bombay, India).
- This should really go from the state to the national level.

One special visitor, the principal of an architectural firm in Bombay, wrote, "Can we borrow some of these ideas for our computer graphics section?"

The event also proved to be a great energizer and confidence booster for teachers. Numerous low-cost teaching aids became available after the fair and will help teachers continue to daily create interesting classroom learning environments.

A dream come true

Seeing one of my dreams of many years come true was personally satisfying. With the cooperation of our school, municipal educational officers, sponsors, media, parents, and—most important—our teachers and students, it has been my immense pleasure to

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Children learn to recognize fractions in different ways. At the fair, students assembled geometric parts into a whole.

watch young students explore mathematical concepts with teacher guidance in such a creative environment. I am eager to work on the action research to test the math fair's teaching aids in a math laboratory and to collect and analyze detailed objective data regarding the actual use of the aids in a day-to-day learning process that displays students' responses and progress in learning mathematics.

This is just the beginning. I believe that sharing such experiences of learning and creating innovative mathematics environments may trigger interest in many inquisitive minds in the international community.

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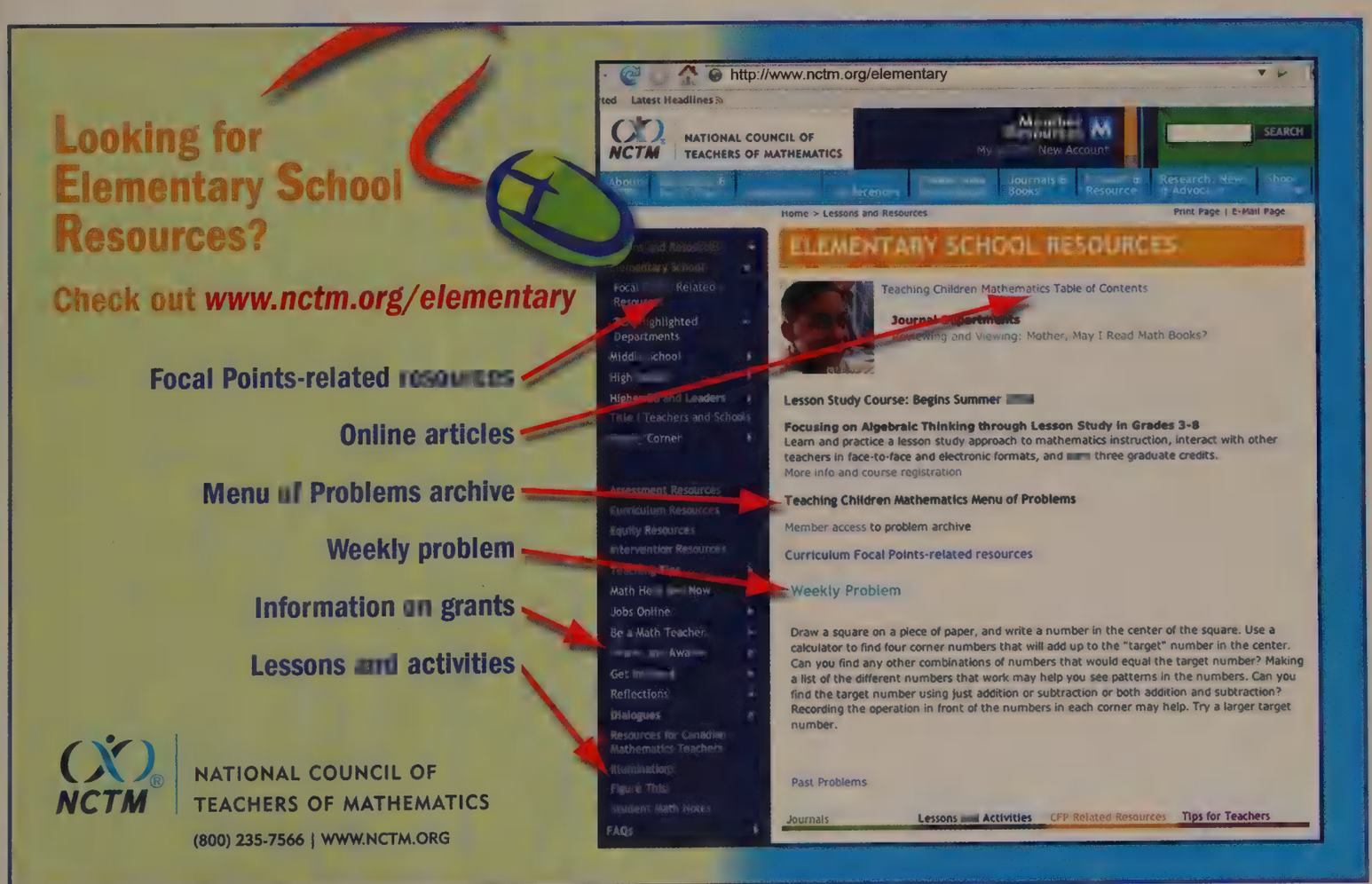
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Draw a square on a piece of paper, and write a number in the center of the square. Use a calculator to find four corner numbers that will add up to the "target" number in the center. Can you find any other combinations of numbers that would equal the target number? Making a list of the different numbers that work may help you see patterns in the numbers. Can you find the target number using just addition or subtraction or both addition and subtraction? Recording the operation in front of the numbers in each corner may help. Try a larger target number.
Past Problems
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Fraction Photo Frenzy: A New Exploration

Using technology to expand opportunities to share their thinking in a problem-solving environment, these preservice teachers discover exciting possibilities for sharing mathematical reasoning in elementary school classrooms.

Can technology be used to enhance the reflection and communication of mathematical ideas in the classroom? This article relates how a class of elementary preservice teachers used technology to share their thinking about a topic that is traditionally difficult to model and discuss—dividing fractions. One type of technology they used, a digital camera, was familiar to most students, but the other type, a SMART Board™, was relatively unfamiliar. Combining these technologies was new to everyone, including the instructor. Multiple factors impelled them to try this exploration with technology, particularly the desire to communicate about and unpack mathematical representations that are created when solving problems. The exploration offers some exciting possibilities for children to learn mathematically in their elementary classrooms.

By Daniel L. Canada

Daniel L. Canada, mathaction@gmail.com, currently works with preservice teachers at Eastern Washington University in Cheney, Washington. Among his interests are new and exciting ways of promoting a deeper conceptual understanding of rational numbers and proportional reasoning.

Setting the Stage

Because this exploration of technology-enhanced communication occurred well into a ten-week course for learning about elementary school mathematics teaching, the students and instructor had begun to develop a classroom culture and environment where all were willing to try new ideas. In the early days of the course, many—if not most—of the students held some reservations about working together in small groups instead of exclusively on their own. Nonetheless, "working in groups is an excellent way for learners to explore, develop mathematical arguments, conjecture, validate possible solutions, and identify connections among mathematical ideas" (NCTM 1991, p. 128). Several students also questioned the value of having *all* learners communicate their reasoning. One asked, "After all, wouldn't it be enough to … know that … students got … correct answer[s]?"

As the course progressed, class members became more comfortable with course expectations. They discussed not only *how* they were reasoning but also *why*. They looked for alternate ways to represent solutions and attended to how others' views resembled and differed from their own.

Carrie Bottomley/iStockphoto.com

The exploration offers some exciting possibilities for children to reason mathematically in their elementary classrooms.

Moreover, as preservice teachers, they were learning to be mindful of how their own students might respond in given mathematical situations. Class members gradually transitioned to a norm of thinking, sharing, challenging, and negotiating meanings to mathematical investigations. In this sense, the class was designed so that these preservice teachers had an opportunity to learn in an environment that reflected the type of experience they could foster for their own students (NCTM 1991).

However, we noticed some areas within our class learning environment that could be improved. For example, it became increasingly difficult to "replay" what people were talking about from one day to the next when we were using the overhead projector or the document camera (which projects an image of a written page of paper onto a screen) or when we wanted to simply demonstrate something visually in a presentation or via a poster. Because we often referred to earlier work in making new mathematical connections, having a convenient way to recall that work, as well as a way to prompt our memories of the accompanying discussion, became more important as our discourse became richer and our explorations more complex. We also struggled to find means of sharing our thinking about some problems that involved visual models with multiple pieces. Often, entire tables would be covered with collections of pattern blocks, tiles, or other manipulatives that a group of teacher candidates was using to demonstrate the students' thinking. As these collections became too unwieldy to physically present in front of the class, we tried having class members huddle around a given table to see and hear about a certain group's thinking process. This approach also became cumbersome.

"Reflection and communication are intertwined processes in mathematics learning. With explicit attention and planning by teachers, communication for the purposes of reflection can become a natural part of mathematics learning" (NCTM 2000, p. 61). We lacked an efficient method of documenting our in-class learning that allowed for prompt recall and reflection on different aspects of our discussion—as well as multiple commentaries made regarding a particular solution—with minimal confusion about whose contributions were whose. What we really needed was a way to capture small-group investigations at our tables so that we could spread out whatever manipulatives we wanted and still be able to adequately show our solution processes to the class and annotate our classmates' remarks.

Launching the Exploration

In an earlier demonstration of a SMART Board's basic operation, we had noted that the system consists of an electronic whiteboard connected to a computer and an LCD projector. The computer projects images of files onto the board, and any writing produced by a SMART Board "pen," acting as an electronic stylus, can be saved to the computer. We had also recognized the value of the Notebook feature, which let us copy and save multiple "pages" for future reference.

The plan for our exploration was multilayered. First, we used digital cameras to take photographs of our in-progress work, creating slide shows of student solution strategies. Then the photographs were imported via a laptop computer into a Notebook presentation as a set of separate pages, which students could annotate with the SMART Board pens. Fortuitously, because any given Notebook presentation could be saved without annotations, classmates could write their own comments on a copy of the same set of pages without disturbing the set created by another user. What follows is a detailed example of how we implemented our plan.

Each small group (three or four students) received a digital camera. The instructor provided a menu of fraction-related word problems from which the groups could choose. Students first explored the problems visually; the written record and symbolic representations would come later. Students were to determine how to represent their thinking photographically and decide at what point in their visual enactments to take a digital photo of their problem-solving approach. They were to ask themselves which pictures would help them explain their reasoning to peers.

Cameras, manipulatives, and creativity in tow, the groups set out to solve their respective fraction tasks. A bit of controlled chaos ensued as students dispersed to enact problem-solving strategies, discuss among themselves how best to capture their thinking on camera, and import correctly sequenced photographs into the SMART Notebook via a laptop at the front of the classroom.

A group consisting of Sarah, Tina, and Greg chose the single task that this article profiles:

> A normal recipe feeds twelve people. But I only want to make enough to feed five people. The normal recipe uses 1 2/5 cup of flour; how much flour should I use?

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Sarah's group members imported six photographs, using six pages in the Notebook to create a permanent record of their problem-solving approach.

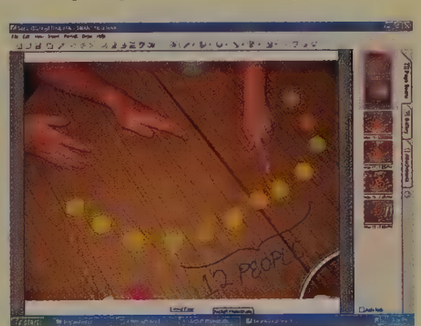
Presenting the Findings

Sarah, Tina, and Greg began their presentation with a photograph. For readers unfamiliar with the Notebook, **figure 1** shows the actual view that the class saw from the laptop projector. On the far right of the figure, a panel called *Page Sorter* shows all six photos or pages. (A Notebook page does not have to consist of a photograph, but for this class episode, the terms are synonymous.) Sarah began by explaining that her teammates had wanted to "see" the twelve people of the problem statement, so they had set out twelve yellow hexagons to represent the people. While speaking, Sarah added the notation *12 people* with the blue SMART Board pen.

Greg tapped on the Page Sorter to advance to the next shot and related how his group eventually decided on "what a cup of flour should look like." He used a white pen to point out the decision. The cup looked like five different-colored strips of

Figure 1

Wanting to "see" the twelve people in the problem statement, this team represents the people with hexagons.



Photograph by Daniel L. Canada; all rights reserved

equal-sized cubes (see **fig. 2**). As he spoke, Greg circled one of the strips and added a comment in green pen to point out the *1/5 cup*. Class members could easily see that Sarah's group members needed a cup for which fifths could be obtained so that they could get *2/5 cup* (shown in black pen). Less obvious was why each strip contained twelve cubes.

Tina explained that—beyond knowing that they wanted to partition a cup into fifths—her group did not start off recognizing what a cup should look like. As Tina advanced to the next page (see **fig. 3**), it became clear that the students wanted to distribute, or deal out, the flour to the twelve people. The act of dealing inspired them to make a connection to division. They had determined that they needed a cup that could also be split into twelfths so that they could deal the flour to twelve people. They concluded that their cups needed to show both fifths and twelfths, resulting in a cup that held sixty total cubes. The group disassembled all the cubes for 1 2/5 cups of flour and put them in a bucket (note the blue *1 2/5 cups flour* annotation in **fig. 3**).

Figure 2

Class members could easily follow the small group members' thinking as they spoke and then annotated their slides.

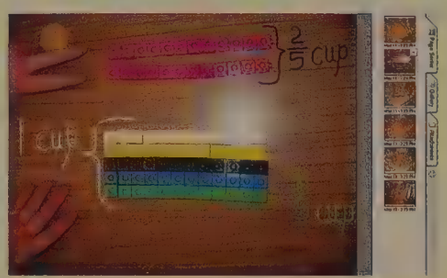


Figure 3

The group concludes that the cup must show twelfths as well as fifths, resulting in a cup that holds a total of sixty cubes.

With **figure 4**, Sarah resumed what her group members called their *action shot*. They planned to model the division process by dealing out all the cubes in the bucket (1 2/5 cups flour) equally among the twelve people. As Greg narrated **figure 5**, the rest of the class saw that each person had seven cubes' worth of flour, but how much was that? How much of a cup? How much of the total? These questions loomed large as Greg showed how he, Tina, and Sarah sought to determine the amount of flour required by five people (note the white annotation).

The SMART Board technology allowed Tina to quickly toggle back and forth between pages using the Page Sorter. As Tina showed the sixth and final page, she explained how her group took all the cubes shown on page 5 and combined them into one model (see **fig. 6**). Her teammates added that they "wanted to see how all that flour [for five people] would compare to a cup." Building on estimation techniques we had practiced in class, other students commented on how they could see that the answer is more than 1/2 cup but also a bit less than 3/5 cup. Tina noted that her small group concluded that five people would need exactly 35/60 cup, pointing with the blue pen to the referent of *1 cup* on the lower right of **figure 6** (which held five strips of twelve cubes, totaling sixty cubes).

Carrying Through

In response to the small-group presentation, some class members were quick to challenge the answer of 35/60 cup. To facilitate discussion, we saved the group's presentation as a computer file. As class members added to the Notebook presentation, they saved the file again each time under the name of the student who commented.

For example, James added to the discussion by toggling back to the second page (see **fig. 2**), and making

Figure 4

Sarah, Tina, and Greg model the division process by dealing sixty cubes (1 2/5 cups of flour) equally among twelve people.

Photographs by Daniel L. Canada; all rights reserved

a note about the eighty-four cubes in the digital picture: "So," he wrote, "there would be 84 cubes in the bucket [on the third page]." (See **fig. 3**.) He then argued that because the five people represented in the sixth slide (see **fig. 6**) collectively had 35 cubes (again affirmed by others in class), the answer should therefore be 35/84. Eventually, fellow class members reminded him of the "class rule" whereby we never utter a fraction without following it with "... of [what?]" Is it 35/84 cup of flour or 35/84 of everything they started with?

In the course of unpacking his thinking, James himself realized that 35/84 was the same as 5/12, in effect claiming that five of twelve people got 5/12 of the total. This exchange helped refocus attention on what the initial task was asking, which was how many cups of flour are needed to feed five people.

In subsequent class sessions, we found it easy to recall the Notebook presentations and the accompanying class discussions. For instance, in the third and fourth slides (see **figs. 3** and **4**), we could again see a great example of a partitive approach to division, which helped give meaning to what an operation such as 1 2/5 ÷ 12 might mean. Being able to replay these examples helped us add meaning as we made connections to symbolic representations.

Figure 5

Each person obviously has seven cubes' worth of flour, but how much of a cup and how much of the total is that?

Figure 6

The answer is more than 1/2 cup but less than 3/5 cup. The small group concludes that five people need exactly 35/60 cup.

Conclusion

Our exploration with the digital cameras and the SMART Board helped us share our thinking on some problems for which demonstrating solution strategies to the whole class had seemed too cumbersome. The immediacy of capturing photographs and importing them directly into the computer to project in front of the class impressed us. We also found the SMART Notebook particularly convenient for writing and displaying our comments on the pages and saving the results under the authors' names.

Ultimately, our experience with incorporating technology helped lead us not only to a deeper understanding of mathematics but also to a richer environment for communicating our reasoning about mathematics. "The term *representation* refers both to process and to product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself" (NCTM 2000, p. 67). In this sense, our class found a new way of using technology to represent mathematics, not simply using technology for the sake of using technology but to allow for better sharing in the class.

An issue that we discussed in class concerned the difficulties in getting all children to share their mathematical thinking. Some children are shy; others may be willing to share, yet they feel constrained in the way they express themselves using either written or spoken language. In some respects, preservice teachers share many of these feelings in their own education classes.

By literally acting out the kinds of problems we did in this exploration and by capturing the photographic documentation, we found roles for all students. For example, some preservice teachers felt comfortable staging a particular photograph; others felt comfortable talking about the photographs they had taken. We could appreciate how children in an elementary classroom might embrace, just as the preservice teachers had, the chance to express themselves in new and dynamic ways. Our experience should encourage others to incorporate technology to clarify students' thinking and enhance their understanding of important mathematical ideas.

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———. *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000. ▲

The mathematics standards in our state (*North Carolina Standard Course of Study* 2003) call for third-grade students to develop fluency in solving problems that involve multiplication and division. Released test items from the statewide standards-based assessment include multiplication and division tasks that require students to read the problem, identify the correct operation, and correctly calculate their solution. We strive to make our mathematics instruction problem oriented, requiring that students read the task, choose among the four operations on the basis of the task's context, and then accurately compute a solution. In this article, we describe a lesson where students wrote story problems for the division task $10 \div 5$ and what we, as educators, learned from the experience.

The Division Unit

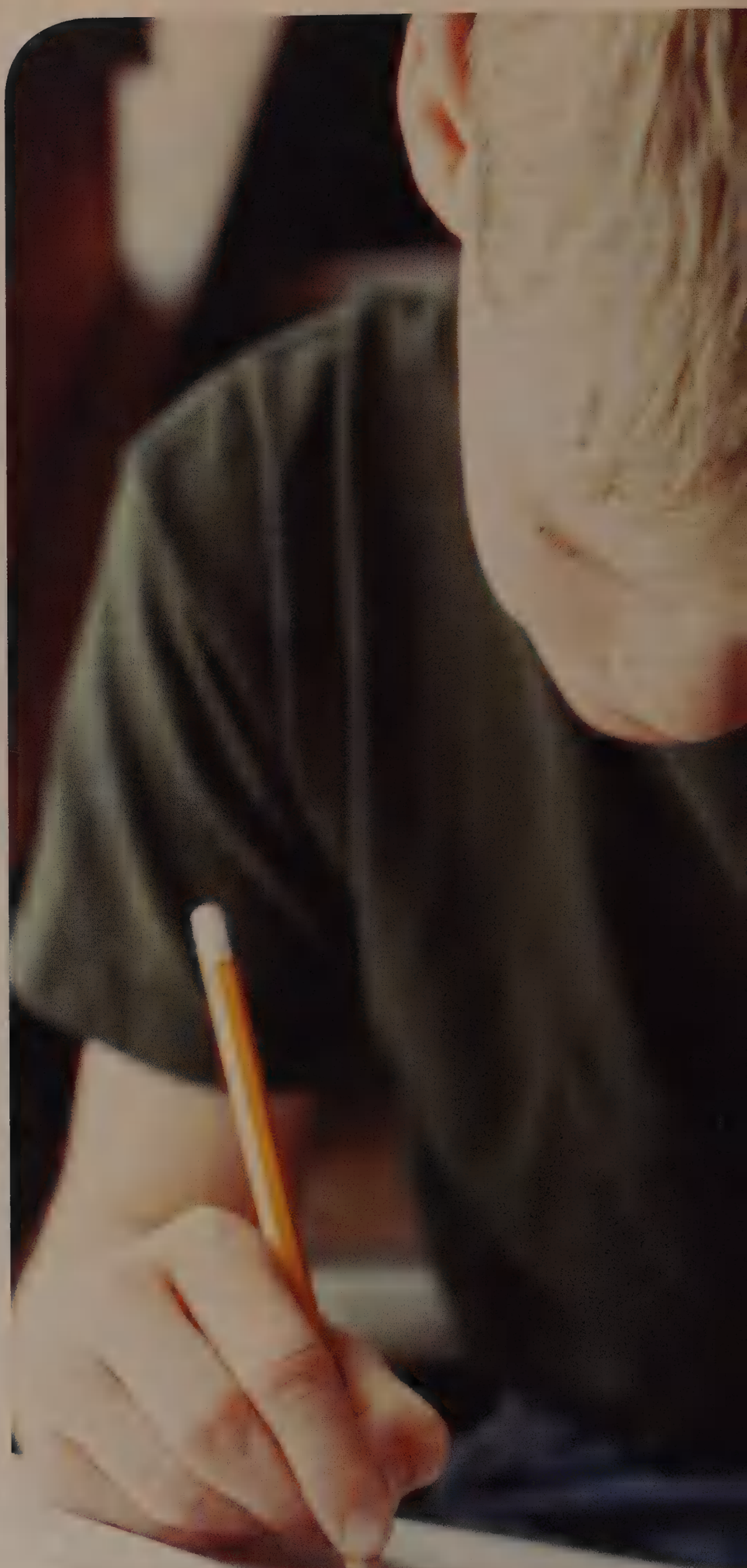
During our five-week unit on division, most of the tasks were based on real-life situations that involved sorting objects into equal groups. This focus started on the first day of the unit with the entire class considering the following task:

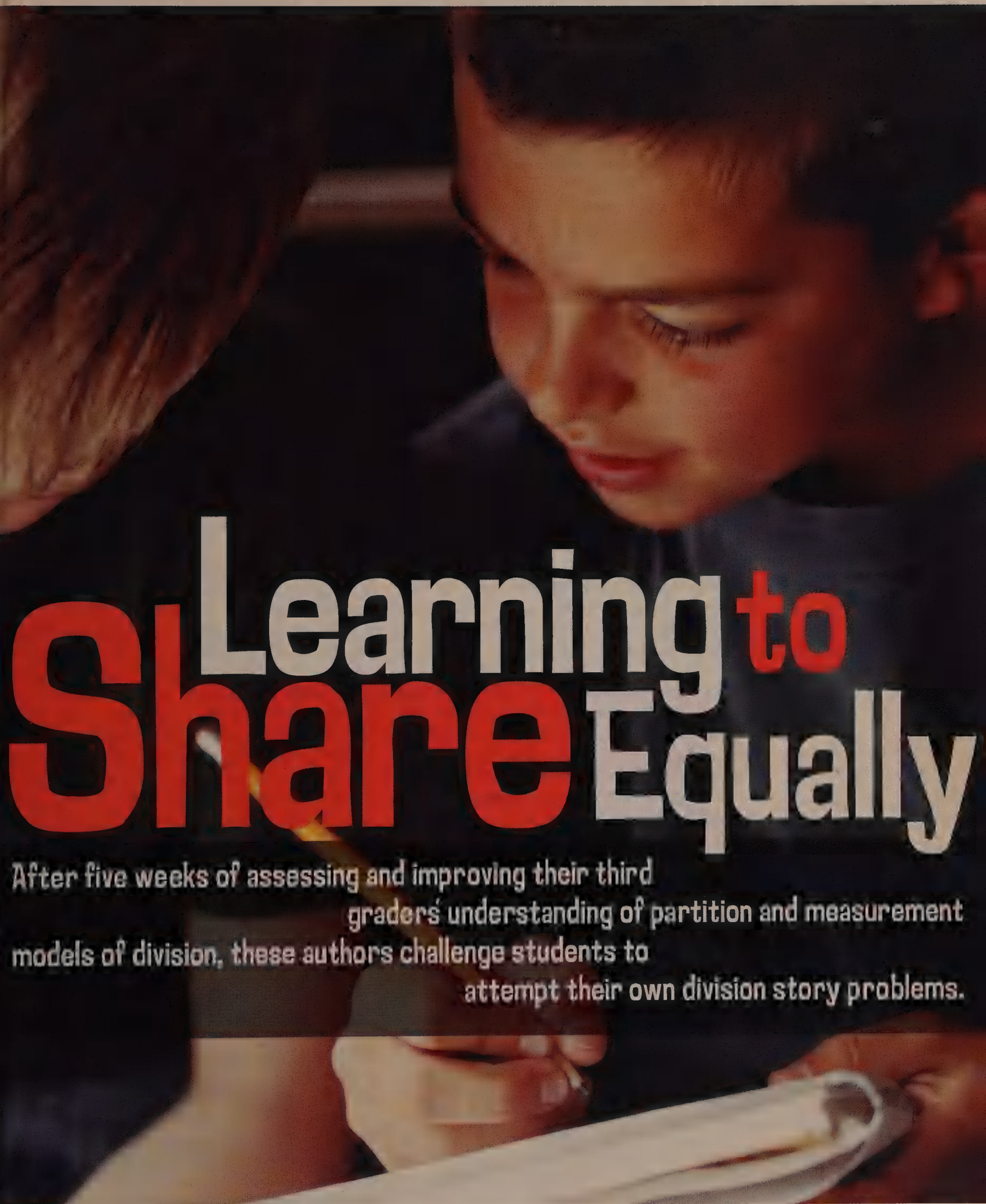
> I have 8 stickers. If I give the same number of stickers to 4 of my friends, how many stickers do I give each friend?

We provided students with numerous opportunities to complete both partition and measurement models of division tasks, during which they used manipulatives such as counters, plastic square tiles, and arrays. The partition model involved splitting a set of objects into a specific number of groups as in the sticker task described above. The measurement model of division involved dividing a specific number of objects into groups that each had a specific number in them:

By Drew Polly and Laura Ruble

Drew Polly, abpolly@uncc.edu, is an assistant professor at the University of North Carolina at Charlotte. His work involves supporting teachers' implementation of learner-centered and standards-based mathematics lessons. Laura Ruble, lauraw.ruble@cms.k12.nc.us, teaches at University Meadows Elementary School in Charlotte-Mecklenburg Schools. As part of a professional development school project, Polly co-teaches weekly in Ruble's third-grade classroom.





Learning to Share Equally

After five weeks of assessing and improving their third graders' understanding of partition and measurement models of division, these authors challenge students to attempt their own division story problems.

If I have twenty-four cans of soda, and there are six cans of soda in a pack, how many packs do I have?

As the unit drew to a close, students worked on tasks that included both multiplication and division tasks within the same class period. Most students struggled to independently determine if a story problem needed to be solved using multiplication or division, but with time, they were able to recognize how to solve various problems. However, we wanted to extend their work and assess whether students were able to create and write their own real-life contexts for a division problem. Therefore, during the last week of the unit, we posed this task:

Write a division problem for $10 \div 5$.

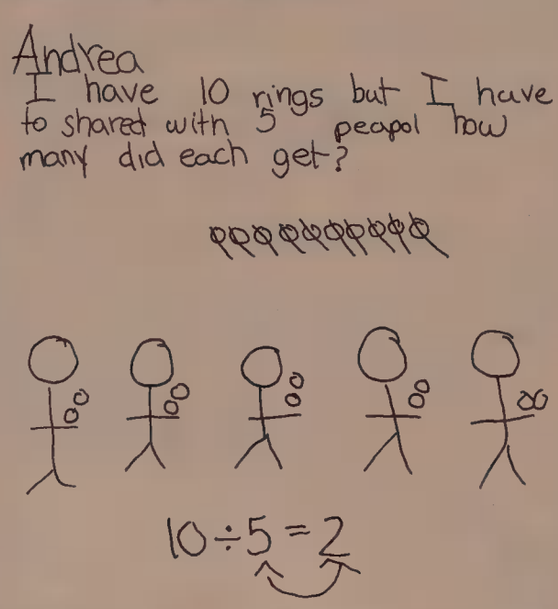
Students' initial work

As students began the task, more than half of the eighteen students immediately wrote a subtraction problem. One student wrote this:

There are 10 markers and 5 people. How many more markers are there?

Figure 1

Students learned that division problems are about items being shared.



Although they used contexts similar to those they had seen during the division unit, the question reflected a desire to take away or remove objects rather than separate objects into groups of equal size. The rest of the students were still thinking about a response when we decided to adjust the lesson by providing an example of a division story problem. About ten minutes into the lesson, we posed the following task:

If I have ten pens, and I want to give an equal number to five students, how many pens would each student get?

Five students were called to the front of the class to act out the task. Students were able to recognize that the task could be written as the equation $10 \div 5$ and also that each student should receive two pens. After briefly discussing the task, students once again tried to write a story problem for $10 \div 5$.

A second attempt

During students' second attempt, three of the eighteen students wrote a correct division story problem without any help. All three story problems included the word *share* and had a context of sharing with five people (see **fig. 1**). The objects being shared included soft drinks, rings, and cookies. The other fifteen students were unable to write a correct story problem. Three types of errors occurred: (1) including the solution in the task, (2) omitting the concept of equal groups (see **fig. 2**), and (3) writing a subtraction story problem instead of a division task. Details on each error are below.

1. Including the solution. Four students included all three quantities (e.g., the divisor, the dividend, and the quotient) in their task. Despite this, in all four tasks, students still phrased their question in a way that asked for the answer that was already given earlier in the problem. An example of a task that a student wrote was, "Kayla had ten cookies. She shared two cookies with her five friends. How many cookies did each friend get?" Although the context was correct, students did not realize that they had provided the solution in their task. We scaffolded students' work by posing a series of questions such as, "What do we want students to try to find?" and "Read the task to yourself. Do you see the solution anywhere?" Three of the four students corrected their mistake and removed the quotient from their task.

2. Omitting the concept of equal groups. Nine students wrote tasks that involved splitting objects among people but did not explicitly include any information about dividing or sharing the objects equally (see **fig. 3**). One student wrote, "There were ten markers and five people. How many markers did each person get?" All these students were able to correct their work when we asked questions such as, "Does each person need to get the same number [of markers]?" In one instance, one student asked another, "Could students get different numbers of markers? What if this student got six markers [*she moved six counters into a pile*], this student got four markers [*she moved four counters into a pile*]. … Is this a correct answer?"

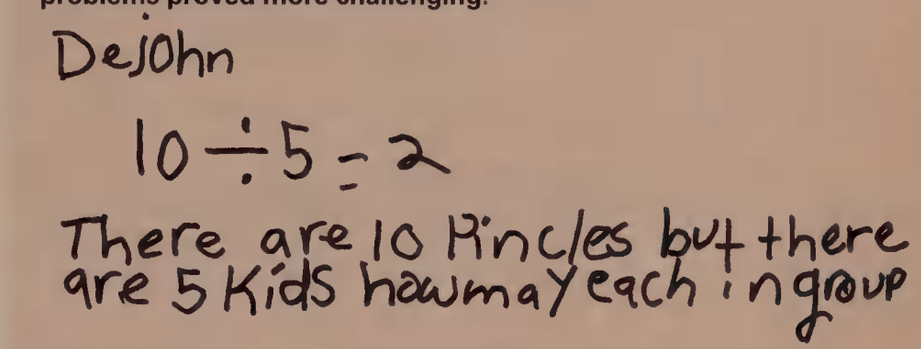
The author then concluded that the answer was not correct. After reviewing the task she had written, she said, "I probably need to include the words *equal number of markers*." She then edited her problem.

3. Writing a subtraction story problem. Despite acting out a sample problem as a whole class, two students still wrote a subtraction story problem instead of a division task. In one case, the student wrote, "Ms. Ruble has ten pens. She gives five of them away. How many does she have left?" Division can be thought of as repeated subtraction of equal groups; in this case, two groups of five could be subtracted from ten. However, the teacher was uncertain of the student's understanding and decided to probe further. The teacher provided counters and had the student act out her problem. While completing the task, the student started with ten counters and took five away. She then independently recognized that her task actually was ten minus five, and commented, "I wrote a subtraction problem." The dialogue continued:

Teacher: How could you make this a division problem?
Student: Use the word *share*: "Ms. Ruble has ten pens. She shares five of them. How many does she have left?"
Teacher: Let's act that out.
Student: Here [*taking ten counters and moving five of them into a different pile*] are the five that are left. [*Student then looks at the counters, puzzled.*]
Teacher: Let's go back to the beginning. What was the division problem we were representing?
Student: Ten divided by five is two.
Teacher: You mentioned the word *share* before. What does that mean?
Student: We want to share the pens and split them up.

Figure 2

With time and practice, most students independently determined when to use division to solve a story problem, but writing division problems proved more challenging.



Teacher: When we split them up, how should the counters be arranged?
Student: Two groups of five [*pointing to the two piles of five that she had already created*].
Teacher: Good. If I were going to share the ten pens and end up with two groups of five pens, how would I explain that?
Student: I think I could say that Ms. Ruble has ten pens. She wants to share them with her friends. She splits them into groups of five pens each. How many groups does she make?
Teacher: And what is the answer to the question you asked?
Student: Two groups.
Teacher: How do you know that your story problem is a division problem?
Student: The pens are being shared, and they are being divided into equal groups.
Teacher: Great work.

What We Learned

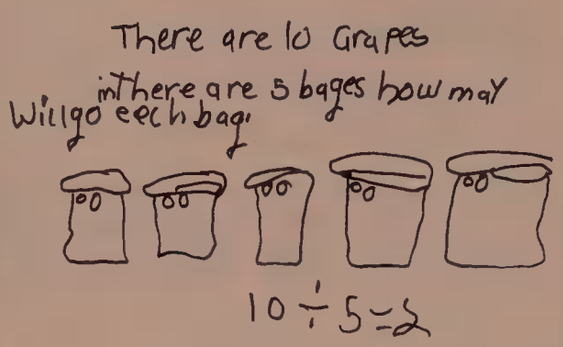
Upon reflection, we learned a number of things from the lesson, including modeling, observing student work, posing effective questions, providing extensive experiences, and implementing curriculum. We detail each of them in the next section.

Modeling

Because we had spent five weeks working on multiplication and division story problems, we immediately began the lesson by posing the task and letting students attempt to write their own division problem. When we started, we observed half of the class writing a subtraction task instead of a division task while the other half of the class was still trying to determine what to write. We decided to stop and

Figure 3

In division problems, items must be shared in *equal* groups.



provide a model and an example of a division story problem. Providing a real-life context using division and acting out the story engaged students and gave them an example that they could refer to as they started to write their own problems. Our first inclination was that students were ready to proceed, but our observations proved otherwise.

In our case, students solving word problems proved to be a task different from students taking a number sentence and writing a word problem for that number sentence. The skill was new to the students, so it was necessary to model it and provide a lot of support at the beginning of the lesson and along the way.

Observing

Our decision to provide an example and model the process was based solely on our observations of students. As we described earlier, even after our modeling, some students struggled with the task and demonstrated a variety of errors. By focusing on students' work while they were writing their story problems, we were able to evaluate how they were thinking about division and determine what assistance they needed to be successful.

Our students, like most third graders, do not like to show their work. They tend to naturally fall into the habit of solving problems in their heads and writing answers without showing all their steps on paper. To support them in recording their work, we give constant reminders and frequently have students discuss their steps and share their written evidence. On the basis of NCTM's Communication Principle (2000), we want students to become effective at writing and sharing their ideas on paper and with their classmates.

Posing questions

Throughout the unit on multiplication and division, we deliberately tried to pose higher-level questions while students were completing tasks. Whereas these students might have finished tasks faster if we had told them how to solve the problems, we wanted to pose questions that encouraged students to think deeply about the tasks.

Typical concerns about a question-focused approach to mathematics teaching include that the approach requires more time for teaching lessons and for learning the concepts that are being taught. Although a questioning approach uses valuable instructional time, we decided on such an approach because we believe that students are more likely to gain a long-term grasp of concepts (e.g., What does division mean?) if they are in situations where they feel comfortable exploring and where their learning is supported by teachers who guide and facilitate their exploration with questions about their work and their thinking.

Providing extensions

As we indicated earlier, students' creation of their own division story problems occurred toward the end of the multiplication and division unit. After several weeks of experiencing multiplication and division in real-world contexts, we felt that they were ready to try writing their own problems. Observing how they still struggled with this assignment, we could not imagine trying it without the multitude of story problems that they had previously solved.

For students to be successful problem creators and solvers, they need ample experiences solving a variety of problems for an extensive amount of time (NCTM 2000). Through our experiences, we found that one-shot problem solving (i.e., solving word problems only one day per week) is not enough time for students to become proficient and competent problem solvers. Furthermore, when mathematics concepts are introduced in isolation without a real-world context, students have no life experiences to link to the concepts.

Implementing curriculum

The task of students writing their own story problems came from a standards-based curriculum that a few teachers have started implementing. Throughout the unit on multiplication and division,

we taught lessons from this curriculum along with similar activities that we found on the Internet or in other curricular resources. A mathematics curriculum should be a coherent set of activities (Reys and Reys 2006), but no one set of educational resources completely aligns with and meets our state standards.

Our school district's traditional mathematics curriculum includes primarily computational activities that do not meet the rigor of our state test. The standards-based resources available have some lessons that address our state standards, but some lessons do not match. This lack of alignment has forced us to create our own curriculum or set of instructional activities based on the state standards, the available resources, and our belief about how mathematics should be taught. Although this is not the ideal curriculum for students, the selection of worthwhile mathematical tasks and activities meets the recommendations of *Principles and Standards for School Mathematics* (NCTM 2000) and our state standards in a way that challenges students to think conceptually about the mathematics we teach.

Concluding Thoughts

Providing students with the opportunity to write a division problem for 10 ÷ 5 was a valuable experience for the students and for us. Through this process, students could think about and apply their understanding of division in real-world contexts. We teachers could gather formative data—about which students understood the concept of division and which ones were still struggling as well as information about their misconceptions. Moreover, we learned that we must scaffold students' new mathematical experiences.

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Summer Reading

Books

For Students

The Writing on the Wall, *Wendy Lichtman, 2008. 240 pp., $16.99 cloth. ISBN 978-0-06-122958-9. Ages 10+. Greenwillow Books; (800) 242-7737; www.harperteen.com.*

This mystery novel was written for middle school students. Tess and her eighth-grade girlfriends notice graffiti painted on the back wall of the church near their school. Sammy believes the numbers are a code and suggests that Tess, a member of the school mathematics team, try to solve the puzzle. Tess immediately recognizes that the numbers are similar to the Four Fours problem (you can make nearly any number using exactly four fours, for example, $4 \times 4 - 4/4 = 15$). Tess identifies the code being used and begins to communicate with the messenger using numerous Four Fours equations. Tess soon discovers that the recent fire in the computer lab might not have been an accident!

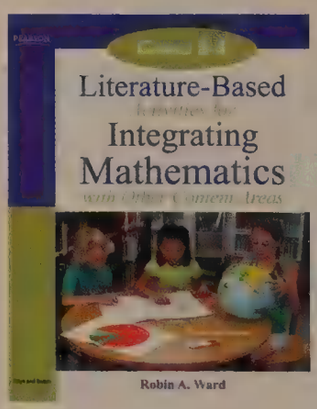
Throughout the story, Tess and her friends encounter typical teen issues—retaliation, bullying, cheating, and shoplifting. They deal with relationships—with boys they like, parents, classmates, and school teams. They face difficult decisions: Do you tell on someone? Do you break the law? While working on the school newspaper, Tess learns about investigative reporting and begins to understand that if you look at something from a different perspective, you might gain a different understanding.

The author weaves many mathematical concepts besides the Four Fours problems throughout the story: graphs, patterns, tessellations, collecting data, liner equations, factorials, tables, and powers. Her explanations are brief, and she uses vocabulary suitable for middle school students.

I highly recommend this book for teachers wishing to integrate language arts and mathematics in their middle school classroom.—*Karen Draper, Portland, OR 97223.*

For Teachers

Literature-Based Activities for Integrating Mathematics with Other Content Areas, Grades 3–5, *Robin A. Ward, 2009. ix + 228 pp., $25.99 paper. ISBN 978-0-205-51409-0. Pearson; www.mypearsonstore.com.*



This research-based handbook for teachers of students in grades 3–5 provides opportunities to incorporate literature-based mathematics with science, social studies, and visual arts activities. The author uses a variety of picture books and poems to help teach these lessons and provides overviews of the literature, descriptions of activities, assessment opportunities, activity extensions, and cross-curricular connections. National mathematics content standards (drawn from NCTM, NSES, NCSS, and NAEA) are cited with each lesson, as are standards for other appropriate content area in the lessons.

The book is well organized. Each chapter begins with a matrix of activities including book title; mathematical concepts and skills; and science, social studies, and visual arts concepts and skills. The end of each lesson provides related children's literature, instructional resources, and Web sites.

Prices on software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

Individual NCTM members receive a 20 percent discount on NCTM publications. To order, visit the NCTM online catalog at www.nctm.org/catalog, or call toll free (800) 235-7566. Free print catalogs of NCTM publications also are available by writing to NCTM.

A very complete list of children's literature references, assessment tools, and rubrics round out this instructional resource. Many of the lessons include cooperative learning group work and can be modified to meet the needs of higher-level thinkers.

Students who struggle with bridging concepts may have some difficulty with the activities. Teachers will need to take time in advance of presenting each lesson to choose the activities most appropriate for their students and to read through all the material in each lesson. However, this will be time well spent as students are sure to enjoy the variety of activities they complete while learning and reviewing skills.—*Wendy Schudmak, Alexandria City Public Schools, Alexandria, VA 22314.*

TAG—Math Is It! Grades 3–5, *Regina Harwood Gresham, Douglas K. Brumbaugh, and Enrique Ortiz, 2007. 112 pp., $10.52 paper. ID no. 1198204. Lulu Marketplace; www.lulu.com.*

The introduction to this book reads, "Mathematics can be fun and exciting if we as teachers make it exciting and fun for our students. Our goal, as authors of this book, is to help you find creative ways to bring enjoyable mathematics material into your classroom."

Based on the NCTM Standards and *Curriculum Focal Points* and compiled by three University of Southern Florida professors, the learning experiences are designed to engage, to challenge, "to help students become powerful mathematics thinkers and to help them make sense out of mathematics." The materials are organized by strand—Number and Operations, Algebra, Geometry, Measurement, and Data Analysis and Probability—with tricks, activities, and games in each. Ideas encompass a broad range, from simple games to complicated teacher directions:

- Divided Products, a blackline master with directions to "do the problems in order. Then, connect the answer dots to complete the picture";
- Algebra Concentration with instructions to copy each of a list of eight terms and definitions on index cards before placing the cards face down to play a matching game; and
- teacher directions for an exploration to construct square regions using Cuisenaire® rods.

At each extreme, a brief reference makes the connection to the NCTM Standards or *Curriculum Focal Points*; for example, "Possible Connections within [the] Focal Points: Number and Operations, Algebra, and Geometry" for the Divided Products activity sheet.

Some material seems more appropriate for students in grades 5–8 than the younger students it supposedly targets: Geometric Representations of Variable Relationships and Fraction Problem Solving.

Although the book contains some unique ideas, it does not communicate a strong enough sense of the content and pedagogy necessary to engage students in just-right tasks to develop their mathematical thinking skills. I was particularly disappointed in the focus on tricks without providing notes on the mathematics that make the solutions predictable, casting the teacher in the role of entertainer rather than educator.—*Kari Augustine, CESA 5 Curriculum Specialist, Portage, WI 53901.*

Teachers Engaging Parents and Children in Mathematical Learning: Nurturing Productive Collaboration, *Regina M. Mistretta, 2008. 128 pp., $19.95 paper. ISBN 978-1-772-1 (1-57886-772-X). Rowman & Littlefield Publishing Group; (800) 462-6420; www.rowmaneducation.com.*

Written for teachers and educators to use as a resource, this book states that its purpose is to involve parents as a crucial element of student success in elementary school through middle school mathematics. Chapters 3–5 focus on three activities, called *initiatives* (e.g., manipulatives, games, and Web-based resources), to encourage parent engagement and provide home activities for students in grades pre-K–8.

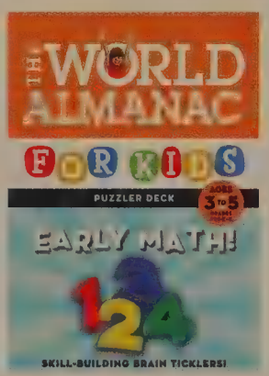
The strength of this book is its stand-alone, nearly identical initiative chapter format, such as (a) letters of inquiry to parents, (b) teacher response forms, (c) a letter of announcement, (d) initial meeting agenda and information, (e) follow-up session agenda, (f) maintaining a connection to parents, and (g) classroom follow-up techniques. Engagement workshop materials are unique to each initiative and include practical advice for the first-time teacher.

Unfortunately, entire paragraphs are repeated from chapter to chapter; an appendix would have permitted the inclusion of more initiatives. I would not recommend this book for a methods course but would suggest that a teacher consider it for ideas for homework collaboration with parents.—*Brenda F. Tiefenbruck, University of St. Thomas, St. Paul, MN 55105.*

Et Cetera

The World Almanac for Kids Puzzler Deck, Early Math! *2008, $9.99. ISBN 978-0-8118-6163-2. Ages 3–5. Chronicle Books; (800) 759-0190; www.chroniclekids.com.*

Thirty-six sturdy cards with a mathematics activity on both sides are for children ages 3–5 and their parents to do together. Moreover, two accordion-style sets of cards with answers to the activities are provided. Because the activities range from basic shape recognition and counting to more challenging activities, parents can select cards that best meet a child's needs.



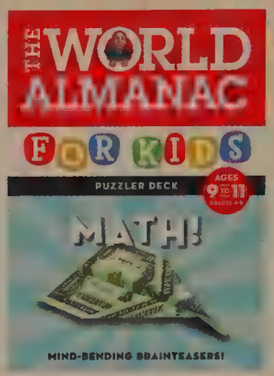
The puzzles cover a wide range of mathematical skills such as counting, comparing, sizes, and basic adding. The Early Math deck activities encourage comprehension of mathematical vocabulary, numbers, and shapes.

I appreciate how the activities provide parents with opportunities to discuss and develop mathematical concepts with young children. For children to solve some of the tasks, parents must provide additional information. One weakness of the card set is that young children might find it difficult to keep track of their responses because of the format: The pictures are small, and the activities are not paper-pencil activities.

I recommend *The World Almanac for Kids Puzzler Deck: Early Math!* to parents. The card activities would also be a good center activity in a first-grade classroom.—*Diana Duffey, Willow Glen Primary, St. Francis, WI 53235.*

The World Almanac for Kids Puzzler Deck, Math! *2008, $9.99. ISBN 978-0-8118-6159-5. Ages 9–11. Chronicle Books; (800) 759-0190; www.chroniclekids.com.*

What do calendars, permutations, combinations, logic, and clocks have in common? Students use them to solve problems in *The World Almanac for Kids Puzzler Deck: Math!* This version of the puzzler deck is mainly for nine- to eleven-year-olds, but my thirty-six-year-old daughter and her nineteen-year-old also had fun solving them! Pictures, graphs, and more accompany each problem



to help students figure out the answers. (Yes, answer sheets for all the problems are included.)

My students and their parents enjoyed solving these problems on my blog and sharing their solutions. I also used them in class when I had a few minutes to fill. Students worked alone or as teams.

I recommend these to any teacher, parent, and student who enjoys applying the concepts learned in class to real-life situations.—*Susan Snyder, Christ Our King–Stella Maris School, Mt. Pleasant, SC 29464.* ▲

TCM NEEDS YOU!
If you would like to referee manuscripts or review books and products for Teaching Children Mathematics, visit www.nctm.org/tcm or e-mail tcm@nctm.org for information.

Thanks from the Editorial Panel

The quality of *Teaching Children Mathematics* (*TCM*) can be maintained only by the voluntary efforts of many mathematics educators. The journal benefits from the experience of those who have served as members of the Editorial Panel, department editors, members of Reader Advisory Panels, reviewers, and referees.

Editorial Panel members are appointed by the NCTM President and are approved by the Board of Directors. In April 2009, Robert Q. Berry III and Maryann Wickett completed their terms. Berry deserves special commendation for his leadership as Panel Chair from April 2008 to April 2009. Both Berry and Wickett are producing the forthcoming October 2009 Focus issue, titled "Equity: Teaching, Learning, and Assessing Mathematics for Diverse Populations."

The importance of department editors in the development of content for this journal is impossible to overstate. "Contemporary Curriculum Issues" debuted in volume 15 under the guidance of Barbara J. Reys. After a five-year hiatus, the "Links to Literature" department returned to the journal under the direction of Andrea S. Holmes. Dana Islas led the "Math by the Month" department writing team in developing a rich set of problems around a unifying theme. Two teams shared the responsibilities of the popular "Problem Solvers" department: Brian Schad, Sarah Bunten, and Joseph Georgeson posed new problems, while Mark Ellis and Cathery Yeh reviewed solutions to previously published problems. As editors of the "Investigations" department, Sue McMillen and Jodelle S. W. Magner provided exploration activities for direct use with students. Cindy Langrall gathered a collection of "Research, Reflection, Practice" articles that were well received by classroom teachers as well as by the research community. The popular "Early Childhood Corner" department benefited from the leadership of Andrew M. Tyminski and Signe E. Kastberg. Last but certainly not least, Kate Kline and Alfreda Jernigan gave us insight into classroom practice in the "From the Classroom" department.

Each feature article that appears in *TCM* is judged for content and style by at least three referees. Reviewers are volunteers who agree to evaluate books and other instructional materials for the journal. Both referees and reviewers are volunteers who serve open-ended terms. The names of NCTM members who were actively reviewing manuscripts or educational materials manuscripts in the eighteen months preceding January 1, 2009, appear on the following list.

NCTM members who are interested in refereeing two or three manuscripts each year or in serving as a reviewer of instructional materials should visit **www.nctm.org/tcm**, click on "Get Involved," and then follow the instructions for "Referee or Review for NCTM." The efforts of all these individuals in maintaining the quality and vitality of *Teaching Children Mathematics* are very much appreciated.

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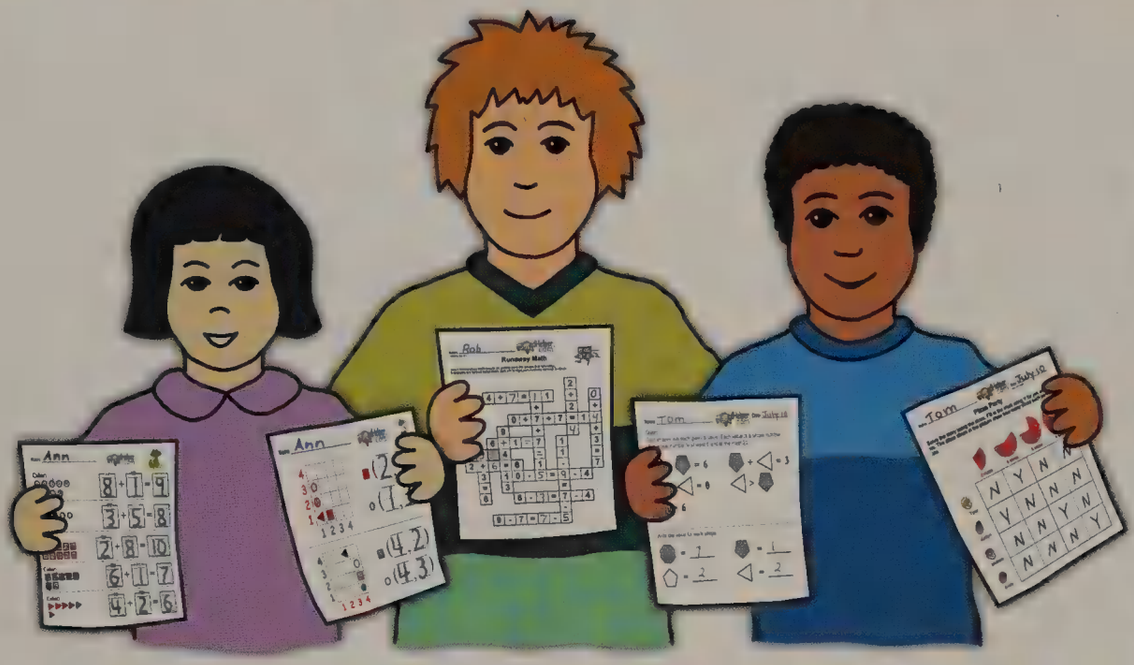

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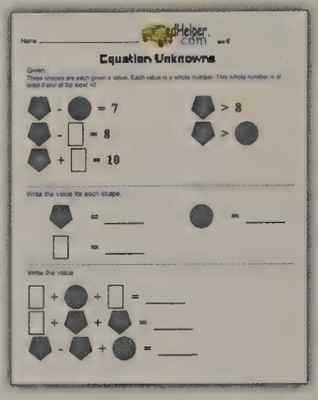

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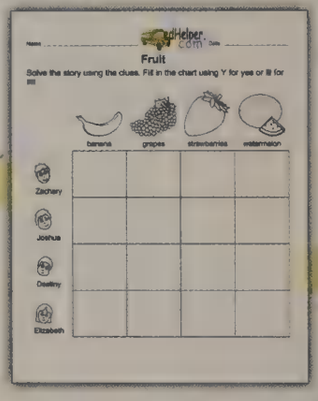
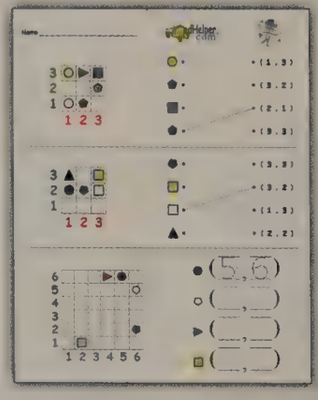
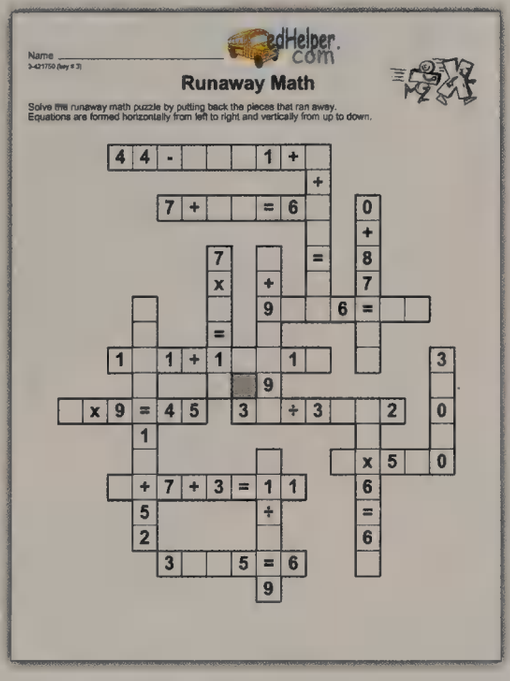


Daily Math



New Runaway Math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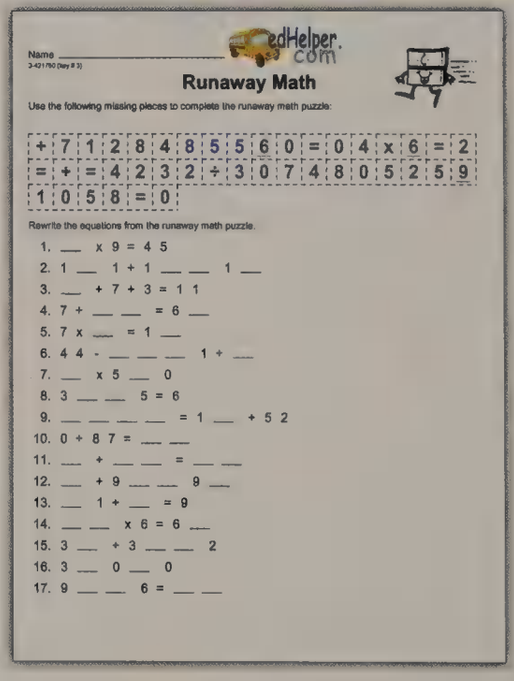
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